On the problem of computing the conglomerable natural extension

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Abstract

Embedding conglomerability as a rationality requirement in probability was among the aims of Walley's behavioural theory of coherent lower previsions. However, recent work has shown that this attempt has only been partly successful. If we focus in particular on the extension of given assessments to a rational and conglomerable model (in the least-committal way), we have that the procedure used in Walley's theory, the *natural extension*, provides only an approximation to the model that is actually sought for: the so-called *conglomerable natural extension*. In this paper we consider probabilistic assessments in the form of a coherent lower prevision $\underline{P}$, which is another name for a lower expectation functional, and make an in-depth mathematical study of the problem of computing the conglomerable natural extension for this case: that is, where it is defined as the smallest coherent lower prevision $\underline{F} \geq \underline{P}$ that is conglomerable, in case it exists. Past work has shown that $\underline{F}$ can be approximated by an increasing sequence $(\underline{E}_n)_{n\in\mathbb{N}}$ of coherent lower previsions. We solve an open problem by showing that this sequence can consist of infinitely many distinct elements. Moreover, we give sufficient conditions, of quite broad applicability, to make sure that the point-wise limit of the sequence is $\underline{F}$ in case $\underline{P}$ is the lower envelope of finitely many linear previsions. In addition, we study the question of the existence of $\underline{F}$ and its relationship with the notion of marginal extension.

Keywords: Coherent lower previsions, conglomerability, conglomerable natural extension, natural extension, marginal extension.

1. Introduction

Within subjective probability, one of the most influential approaches is Bruno de Finetti's theory [4, 7]. It considers probability as derived from expectation, which de Finetti calls *prevision*. The prevision of a gamble (that is, a bounded real-valued variable) is behaviourally interpreted as a fair price for buying or selling the gamble itself.

De Finetti's work on subjective probability has been extended to the imprecise case to deal with situations where we are not capable of assessing a fair price for a gamble. In that case, instead of previsions we use *lower* and *upper* previsions, which can be interpreted as supremum acceptable buying prices and infimum acceptable selling prices, respectively. However, the extension to the imprecise case is not straightforward; we distinguish between two main approaches: that of Peter Williams [18] and Peter Walley's [17]. The main difference between the two approaches, in addition to some structural requirements in Walley's case, has to do with the notion of *conglomerability*.

Conglomerability was first discussed by de Finetti himself in 1930 [3] as a property that a finitely additive probability that is not countably additive may not satisfy. Roughly speaking, a finitely additive probability is conglomerable with respect to a partition $\mathcal{B}$ of the possibility space, when it holds that

$$P(A) \in \left[\inf_{B\in\mathcal{B}} P(A|B), \sup_{B\in\mathcal{B}} P(A|B)\right]$$

for all events A. Although the lack of conglomerability leads to some counterintuitive properties [8], de Finetti argued [6, 7] that conglomerability should not be imposed as a rationality requirement in the subjective approach to probability

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given by the theory of previsions. Let us stress that this question is relevant only for the case of infinite partitions, because in the remaining case conglomerability is actually secured by de Finetti's axioms. The issue has remained a controversial one in time (again, only for the case of infinite partitions) and there has been quite some literature devoted to this subject (e.g., [1, 9, 16]). Taking this into account, it is not surprising that conglomerability has also been a subject of discussion when extending de Finetti's theory to the case where we consider lower and upper previsions instead of linear ones.

In that case, conglomerability is usually formulated in an alternative manner: if we consider a partition $\mathcal{B}$ of the set of outcomes (that is, the possibility space) and a gamble that we are disposed to accept once we know the element of the partition that includes the outcome, *irrespectively of which set this is*, then conglomerability means that we should also accept the gamble without knowing which of the sets in the partition includes the outcome (this is equivalent to assuming that a particular infinite sum of acceptable gambles should be acceptable).

Using this idea, we have on the one hand Williams' approach [14, 18], which is close to de Finetti's, and which is purely finitary in its formulation: conglomerability is considered as a rationality requirement only if the partition is finite. One argument against the extension of such a requirement to the case of infinite partitions is that many useful, and somewhat intuitive, properties of the class of coherent lower previsions do not hold if we impose conglomerability. This is detailed in Section 2.5 and in Appendix A of this paper: the class of conglomerable and coherent lower previsions is not closed under convex combinations or point-wise limits.

On the other hand, Walley has argued at some length in [17, Section 6.8] that conglomerability should be imposed. The main motivation for such a standpoint seems his support to the idea that the sum of infinitely many acceptable gambles should be acceptable; conglomerability follows from this as an implication.[1] For these reasons, Walley's theory of imprecise previsions imposes a *conglomerative axiom* when dealing with the conditional case.

The two pillars of Walley's theory of lower previsions are the notion of *coherence*, which determines whether or not the assessments modeled by the lower previsions are consistent with each other, and that of *natural extension*, which allows us to correct incoherent assessments into coherent ones in a least-committal way (these and other concepts of Walley's theory are summarised in Section 2). The extent to which conglomerability is taken into account in Walley's procedures of coherence and natural extension has been the subject of some recent work [12, 13].

It has turned out [12] that the definition of coherence does not always consider all the implications of conglomerability and that to solve this problem ones needs to employ an alternative definition of coherence. The two definitions are equivalent in the special case where we consider just one conditional and one unconditional lower prevision [12, 13].

When we focus the attention on the notion of the natural extension, the problem comes up even in the simplest case. If we use conglomerability as a rationality requirement, then the correction of a coherent lower prevision $\underline{P}$ that is not conglomerable should provide us with the weakest conglomerable and coherent lower prevision $\underline{F}$ that extends $\underline{P}$. However, such a model does not necessarily exist, and in Section 3 we give necessary and sufficient conditions for the correction to be feasible.

Even if the conglomerable natural extension exists, its construction can well be problematic; in fact, Walley's notion of natural extension does not provide the closest conglomerable model, but only a conservative (i.e., an outer) approximation. This approximation is iterated in [13] so as to approximate $\underline{F}$ better and better through a sequence of coherent lower previsions $(\underline{E}_n)_{n\in\mathbb{N}}$ such that $\underline{P} \leq \underline{E}_1 \leq \underline{E}_2 \leq \cdots \leq \underline{E}_i \leq \cdots \leq \underline{F}$. What was known so far is that if the sequence becomes stable, that is, if $\underline{E}_i = \underline{E}_{i+1}$ for some i, then $\underline{E}_i = \underline{F}$; and, conversely, if the sequence breaks down, which means that $\underline{E}_{i+1}$ cannot be produced for some i, then $\underline{F}$ does not exist.

However, some fundamental questions were left open with regard to the sequence $(\underline{E}_n)_{n\in\mathbb{N}}$. One of them was whether or not it may be made of infinitely many distinct elements; in this case we also say that the sequence does not stabilise. If that is the case, then the next question would be whether or not the point-wise limit $\underline{Q}$ of the sequence equals $\underline{F}$. In fact, in principle it could be the case that $\underline{Q}$ is not conglomerable while $\underline{F}$ exists; this would mean that we should restart a new sequence from $\underline{Q}$ in order to get to $\underline{F}$ (and possibly another, and another, and another, etc.).

In this paper we answer the first question mentioned above: we construct in Example 4 in Appendix A a model $\underline{P}$ whose related sequence $(\underline{E}_n)_{n\in\mathbb{N}}$ does not stabilise. In this specific case the limit $\underline{Q}$ of the sequence equals $\underline{F}$; this

[1] Walley tried to support conglomerability also by giving examples that aimed to show that the failure of conglomerability may have us subject to a Dutch book. It has been argued recently that for this to be the case one has to consider a dynamic interpretation of probability that explicitly deals with both present and future probability models, and moreover that these have to be established by a subject at the same time [20]. This indeed gives support to conglomerability, but to a lesser extent than Walley seemed to be aiming at.

does not allow us to address the second question, which remains open.

In Section 5 we deepen the study, started in [13], on the relationship between *marginal extension* and conglomerable natural extension. We consider in particular the relationship between $(\underline{E}_n)_{n\in\mathbb{N}}$ and the sequence $(\underline{M}_n)_{n\in\mathbb{N}}$, where $\underline{M}_n := \underline{E}_{n-1}(\underline{E}_{n-1}(\cdot|\mathcal{B}))$ is the marginal extension of $\underline{E}_{n-1}$ and its conditional natural extension $\underline{E}_{n-1}(\cdot|\mathcal{B})$ (which is the least-committal coherent extension of $\underline{E}_{n-1}$ to the conditional case, see Eq. (4) in Section 2.4). It turns out that $(\underline{M}_n)_{n\in\mathbb{N}}$ is also an increasing sequence of coherent lower previsions that is dominated by $\underline{F}$; however we show in Example 5 in Appendix A that the point-wise limit $\underline{Q}'$ of the sequence $(\underline{M}_n)_{n\in\mathbb{N}}$ may differ from $\underline{F}$. In addition, by detailing the relationships between $\underline{P}$, $\underline{Q}$, $\underline{Q}'$ and $\underline{F}$ we deduce in Proposition 9 that if $(\underline{E}_n(\cdot|\mathcal{B}))_{n\in\mathbb{N}}$ converges uniformly to the conditional natural extension $\underline{Q}(\cdot|\mathcal{B})$ of $\underline{Q}$, then $\underline{Q} = \underline{F}$.

In Section 6 we focus on the special case where $\underline{P}$ is the lower envelope of finitely many linear previsions. This allows us to deduce two new simple conditions, which seem to be quite broadly applicable, that make sure that $(\underline{E}_n(\cdot|\mathcal{B}))_{n\in\mathbb{N}}$ converges uniformly to $\underline{Q}(\cdot|\mathcal{B})$, and hence, through Proposition 9, that $\underline{Q} = \underline{F}$. This analysis shows in particular that $\underline{Q}$ always equals $\underline{F}$ when $\underline{P}$ is the lower envelope of two linear previsions.

We report a summary of our views in Section 7. To ease the reading, all the counterexamples have been gathered in Appendix A.

2. Introduction to imprecise probabilities

Let us introduce the basics of the theory of coherent lower previsions that we use in this paper. We refer to [17] for an in-depth study, and to [11] for a survey.

2.1. Unconditional coherent lower previsions

Consider a possibility space Ω (very often in the examples of this paper Ω will be equal, or related, to the set of positive natural numbers, which we denote by $\mathbb{N}$). A *gamble* is a bounded map $f\colon \Omega \to \mathbb{R}$. The set of all gambles is denoted by $\mathcal{L}(\Omega)$, or simply by $\mathcal{L}$ when there is no ambiguity about the possibility space we are working with. In particular, we use $f \lneq 0$ to denote a gamble $f \leq 0, f \neq 0$ (and we will refer to this as a *negative* gamble), and $f \gneq 0$ to denote a gamble $f \geq 0, f \neq 0$ (this will be called a *positive* gamble). We use the notation $\mathcal{L}^+(\Omega)$, or simply $\mathcal{L}^+$, to refer to the set of positive gambles.

A *lower prevision* $\underline{P}$ is a real-valued functional defined on some set of gambles $\mathcal{K} \subseteq \mathcal{L}$. It determines a conjugate *upper prevision* $\overline{P}$ on $-\mathcal{K} := \{-f\colon f \in \mathcal{K}\}$ by $\overline{P}(-f) := -\underline{P}(f)$. When the domain $\mathcal{K}$ of $\underline{P}$ is a linear space—closed under point-wise addition and multiplication by real numbers—$\underline{P}$ is called *coherent* when it satisfies the following conditions:

C1. $\underline{P}(f) \geq \inf f$ for all gambles $f \in \mathcal{K}$;

C2. $\underline{P}(\lambda f) = \lambda \underline{P}(f)$ for all gambles $f \in \mathcal{K}$ and all positive real λ;

C3. $\underline{P}(f + g) \geq \underline{P}(f) + \underline{P}(g)$ for all gambles $f, g \in \mathcal{K}$.

We shall let

$$\mathbb{P} := \{\underline{P} : \mathcal{L} \to \mathbb{R} \text{ coherent}\} \tag{1}$$

denote the class of coherent lower previsions defined on the set $\mathcal{L}$ of all gambles.

A particular case of coherent lower previsions is that of *linear* previsions. A linear prevision is a functional $P\colon \mathcal{L} \to \mathbb{R}$ satisfying conditions C1 and C2, and

$$P(f + g) = P(f) + P(g) \text{ for all } f, g \in \mathcal{L}.$$

In this paper we identify an event $B \subseteq \Omega$ with its corresponding indicator function $\mathbb{I}_B$—and in fact we shall often use B to denote the indicator $\mathbb{I}_B$. With this in mind, we can consider the restriction of P to $\mathcal{P}(\Omega)$, the powerset of Ω: it turns out that such a restriction is a finitely additive probability and that P is the corresponding expectation operator.[2]

[2]The expectation is obtained by taking the Dunford integral [2].

The set of all linear previsions is denoted by $\mathbb{P}$. Let $\underline{P}$ be a coherent lower prevision on $\mathcal{K}$. We define its associated *credal set* as

$$\mathcal{M}(\underline{P}) \coloneqq \{P \in \mathbb{P} \colon (\forall f \in \mathcal{K}) P(f) \geq \underline{P}(f)\},$$

and each P in $\mathcal{M}(\underline{P})$ is said to *dominate* $\underline{P}$.

We say that a lower prevision $\underline{P}$ *avoids sure loss* when its associated credal set $\mathcal{M}(\underline{P})$ is not empty. In that case, its *natural extension* is simply the lower envelope of $\mathcal{M}(\underline{P})$, and it corresponds to the smallest coherent lower prevision on $\mathcal{L}$ that dominates $\underline{P}$ on its domain. It represents the least-committal correction of the assessments present in $\underline{P}$ that is necessary to satisfy the property of coherence.

The same notion can be used to extend a coherent lower prevision from its domain $\mathcal{K}$ to the set $\mathcal{L}$ of all gambles: the lower envelope of $\mathcal{M}(\underline{P})$ is again the smallest coherent lower prevision on $\mathcal{L}$ that dominates $\underline{P}$ on its domain.

2.2. *Separately coherent conditional lower previsions*

Similar concepts arise in the conditional case. Consider a partition $\mathcal{B}$ of Ω and a lower prevision $\underline{P}(\cdot|B)$ on $\mathcal{L}(\Omega)$ for each event $B \in \mathcal{B}$. The collection of all these functionals summarised by $\underline{P}(\cdot|\mathcal{B}) \coloneqq \sum_{B\in\mathcal{B}} B\underline{P}(\cdot|B)$ is called a *conditional lower prevision* on $\mathcal{L}(\Omega)$. $\underline{P}(\cdot|\mathcal{B})$ is said *separately coherent* when $\underline{P}(\cdot|B)$ is coherent and $\underline{P}(B|B) = 1$ for every $B \in \mathcal{B}$. For every gamble f, $\underline{P}(f|\mathcal{B})$ is a gamble on Ω that is constant on the elements of $\mathcal{B}$; such gambles are called *$\mathcal{B}$-measurable*. It may be convenient to regard a separately coherent conditional lower prevision $\underline{P}(\cdot|\mathcal{B})$ also as the functional[3] from $\mathcal{L}(\Omega) \times \Omega$ to $\mathbb{R}$ such that $\underline{P}(f|\mathcal{B})(\omega) = \underline{P}(f|B)$, with $B \ni \omega$.

In an analogous way, a *conditional linear prevision* is a functional $P(\cdot|\mathcal{B})$ from $\mathcal{L}(\Omega) \times \Omega$ to $\mathbb{R}$ such that $P(B|B) = 1$ and $P(\cdot|B)$ is a linear prevision for every $B \in \mathcal{B}$.

Similarly to the unconditional case, a conditional lower prevision $\underline{P}(\cdot|\mathcal{B})$ is separately coherent if and only if $\underline{P}(\cdot|B)$ is, for all $B \in \mathcal{B}$, the lower envelope of a family of linear previsions (we shall sometimes abuse terminology by saying that $\underline{P}(\cdot|\mathcal{B})$ is the lower envelope of a family of conditional linear previsions). As a consequence, the theory can be given a Bayesian sensitivity analysis interpretation. In this regard, it is important to keep in mind that the precise models correspond to expectation functionals with respect to finitely additive probabilities that need not be σ-additive.

2.3. *The behavioural interpretation*

The concepts above can be given a behavioural interpretation, in terms of buying and selling prices [7, 17]. Given a gamble f, its lower prevision $\underline{P}(f)$ can be seen as the supremum acceptable *buying* price for f, in the sense that for every $\mu < \underline{P}(f)$, and for no $\mu > \underline{P}(f)$, we would accept the gamble $f - \mu$ in case it was offered to us (sometimes this is also referred to as an acceptable transaction).

When this supremum acceptable buying price coincides with the infimum acceptable selling price for f, which is equal to $-\underline{P}(-f)$, this common value can be seen as a fair price for f, and if we can establish fair prices for all gambles, we determine a linear prevision. A similar interpretation can be provided for the conditional lower previsions: $\underline{P}(f|B)$ is the supremum price we would (currently) give for f, if we observed the event B.

The rationality of our buying and selling prices can be verified by means of a number of axioms: for instance, we may require that a transaction that can never make us lose utiles, and possibly make us gain some, should be acceptable; that one that can never make us win utiles should *not* be acceptable; and that a positive linear combination of acceptable gambles should again be acceptable. All these axioms together imply that by combining a finite number of acceptable transactions a Dutch book cannot be built against us, and moreover that our supremum buying prices are the result of some thorough reflection, in the sense that one cannot force a change in our prices by taking into account the implications of any *finite* number of our acceptable gambles. These ideas lie behind the definition of natural extension we have given in Section 2.1: it is the lower prevision associated to the buying prices whose rationality is guaranteed by the assessments that are implicit in $\underline{P}$.

[3] We are abusing terminology here as, strictly speaking, $\underline{P}(\cdot|\mathcal{B})$ is a collection of functionals.

2.4. Coherence of conditional and unconditional models

The behavioral interpretation allows us to determine when the assessments present in a (conditional or an unconditional) lower prevision are compatible with each other, in the sense that they are mutually consistent, which also implies that they cannot be exploited to make a Dutch book against us. When we have a conditional *and* an unconditional lower prevision, we need to verify that the assessments present in these two models are also compatible when taken together. This gives rise to a joint notion of coherence.

For every lower prevision $\underline{P}$ and every conditional lower prevision $\underline{P}(\cdot|\mathcal{B})$, we use the notations:

$$G_{\underline{P}}(f) \coloneqq f - \underline{P}(f),\ G_{\underline{P}}(f|B) \coloneqq B(f - \underline{P}(f|B)),$$
$$G_{\underline{P}}(f|\mathcal{B}) \coloneqq f - \underline{P}(f|\mathcal{B}) = \sum_{B\in\mathcal{B}} G_{\underline{P}}(f|B).$$

Definition 1 (**Coherence of conditional and unconditional lower previsions**). If we consider a coherent lower prevision $\underline{P}$ on $\mathcal{L}$ and a separately coherent conditional lower prevision $\underline{P}(\cdot|\mathcal{B})$ on $\mathcal{L}$, they are called *coherent*[4] if and only if for every gamble f and every $B \in \mathcal{B}$,

$$\underline{P}(G_{\underline{P}}(f|\mathcal{B})) \geq 0 \tag{CNG}$$

and

$$\underline{P}(G_{\underline{P}}(f|B)) = 0. \tag{GBR}$$

The first is a condition of conglomerability, which will be discussed at some length later in this section; for the time being, it is enough to notice that conglomerability is always relative to a given partition $\mathcal{B}$. The second condition is called the Generalised Bayes Rule, and if $\underline{P}(B) > 0$ it can be used to uniquely determine the value $\underline{P}(f|B)$: in that case there is only one value satisfying (GBR) with respect to $\underline{P}$. If $\underline{P}$ and $\underline{P}(\cdot|\mathcal{B})$ satisfy (GBR), we also say that they are *Williams coherent* [18].[5]

One particular case of coherent $\underline{P}$, $\underline{P}(\cdot|\mathcal{B})$ are the *vacuous* unconditional and conditional lower previsions, given by

$$\underline{P}(f) \coloneqq \inf_{\omega\in\Omega} f(\omega) \text{ and } \underline{P}(f|B) \coloneqq \inf_{\omega\in B} f(\omega) \text{ for all } f \in \mathcal{L} \text{ and all } B \in \mathcal{B}.$$

Another instance can be constructed by means of *marginal extension*: given a coherent lower prevision $\underline{P}$ and a separately coherent conditional lower prevision $\underline{P}(\cdot|\mathcal{B})$ on $\mathcal{L}$, their marginal extension is the coherent lower prevision

$$\underline{P}(\underline{P}(\cdot|\mathcal{B})). \tag{2}$$

Then $\underline{P}(\underline{P}(\cdot|\mathcal{B}))$ and $\underline{P}(\cdot|\mathcal{B})$ are coherent.

A weaker requirement of coherence is avoiding partial loss.

Definition 2 (**Avoiding partial loss of conditional and unconditional coherent lower previsions**). Given a coherent lower prevision $\underline{P}$ and a separately coherent conditional lower prevision $\underline{P}(\cdot|\mathcal{B})$ on $\mathcal{L}$, they are said to *avoid partial loss* when

$$\sup\left[G_{\underline{P}}(f) + G_{\underline{P}}(g|\mathcal{B})\right] \geq 0 \tag{3}$$

for every pair of gambles $f, g \in \mathcal{L}$.

Eq. (3) holds whenever $\underline{P}(\cdot|\mathcal{B})$ is the vacuous conditional lower prevision irrespective of the coherent lower prevision $\underline{P}$, because in that case $G_{\underline{P}}(f|\mathcal{B}) \geq 0$ for any gamble f.

Similarly to the unconditional case, we can also determine the conditional behavioural implications of a coherent lower prevision. Given a coherent lower prevision $\underline{P}$, its *conditional natural extension* is given by

$$\underline{P}(f|B) = \begin{cases} \inf_{\omega\in B} f(\omega) & \text{if } \underline{P}(B) = 0 \\ \min\{P(f|B) : P \geq \underline{P}\} & \text{otherwise} \end{cases} \tag{4}$$

for every $B \in \mathcal{B}$ and every $f \in \mathcal{L}$. It follows from this definition that if $\underline{P}, \underline{Q}$ are coherent lower previsions and $\underline{Q} \geq \underline{P}$, then the conditional natural extension of $\underline{Q}$ must dominate that of $\underline{P}$.

[4] See [17, Section 6.3.2] for a definition of coherence on more general domains.

[5] Williams coherence is not constrained to gambles whose conditioning events form a partition of the sure event; this is one of the reasons why it does not necessarily satisfy conglomerability, as we shall see later on.

Proposition 1. *[17, Section 6.8] Let $\underline{P}$ be a coherent lower prevision, and let $\underline{P}(\cdot|\mathcal{B})$ be its conditional natural extension, given by Eq.* (4)*.*

1. *$\underline{P}(\cdot|\mathcal{B})$ is a separately coherent conditional lower prevision.*
2. *$\underline{P}, \underline{P}(\cdot|\mathcal{B})$ always satisfy* (GBR). *Thus, they are coherent if and only if* (CNG) *holds.*
3. *The following are equivalent:*
   (a) *$\underline{P}$ is coherent with some conditional lower prevision $\underline{Q}(\cdot|\mathcal{B})$.*
   (b) *$\underline{P}$ is coherent with its conditional natural extension $\underline{P}(\cdot|\mathcal{B})$.*

Taking this into account, we consider the following definition:

Definition 3 (**Conglomerably coherent lower prevision**). We say that a coherent lower prevision $\underline{P}$ is a *conglomerably coherent lower prevision* relative to $\mathcal{B}$, or simply that it is $\mathcal{B}$*-conglomerable*, when it is coherent with its conditional natural extension $\underline{P}(\cdot|\mathcal{B})$.

Similarly to Eq. (1), we denote by

$$\underline{\mathbb{F}} \coloneqq \{\underline{P} : \mathcal{L} \to \mathbb{R} \text{ conglomerably coherent relative to } \mathcal{B}\}$$

the class of $\mathcal{B}$-conglomerable coherent lower previsions. This class includes for instance the lower previsions that are constructed by means of marginal extension as in Eq. (2). Conglomerably coherent lower previsions relative to $\mathcal{B}$ are those that can be coherently updated into a lower prevision conditional on $\mathcal{B}$. In that case, the conditional natural extension is the smallest conditional lower prevision that is coherent with them. Let us recall that in this paper we restrict the attention to the case of one partition only. For this reason, we shall often drop the reference to $\mathcal{B}$ and more simply say that a coherent lower prevision is conglomerable.[6]

Conglomerability holds trivially whenever $\underline{P}(B) = 0$ for all but a finite number of conditioning events $B \in \mathcal{B}$, as it is often the case when the partition $\mathcal{B}$ is uncountable. In particular, (CNG) always holds whenever the *support* of the gamble f,

$$S(f) \coloneqq \{B \in \mathcal{B} : Bf \neq 0\},$$

is finite. This means that conglomerability holds trivially for finite partitions. In fact, in that case condition (CNG) is a consequence of (GBR), which means that Walley's and Williams' notions of coherence are equivalent.

Remark 1. Let us recall that given a separately coherent conditional lower prevision $\underline{P}(\cdot|\mathcal{B})$, by letting $C \subseteq \Omega$ be the union of some arbitrary elements of the partition $\mathcal{B}$, it holds for all $f \in \mathcal{L}$ that

$$\begin{aligned} G_{\underline{P}}(f|\mathcal{B}) &= f - \underline{P}(f|\mathcal{B}) = Cf + C^c f - \sum_{B\in\mathcal{B}} B\underline{P}(Cf + C^c f|B) \\ &= \left[Cf - \sum_{B\in\mathcal{B}:B\subseteq C} B\underline{P}(Cf|B)\right] + \left[C^c f - \sum_{B\in\mathcal{B}:B\subseteq C^c} B\underline{P}(C^c f|B)\right] \\ &= G_{\underline{P}}(Cf|\mathcal{B}) + G_{\underline{P}}(C^c f|\mathcal{B}). \end{aligned}$$

If $\underline{P}(\cdot|\mathcal{B})$ is given by (4), i.e., if it is the conditional natural extension of a coherent lower prevision $\underline{P}$, then if we let $C \coloneqq \cup_{B\in\mathcal{B}:\underline{P}(B)>0} B$, we have in addition that $G_{\underline{P}}(C^c f|\mathcal{B}) \geq 0$, taking into account that $\underline{P}(\cdot|B)$ is vacuous for all $B \subseteq C^c$, whence, applying the super-additivity (due to coherence) of $\underline{P}$,

$$\underline{P}(G_{\underline{P}}(f|\mathcal{B})) \geq \underline{P}(G_{\underline{P}}(Cf|\mathcal{B})) + \underline{P}(G_{\underline{P}}(C^c f|\mathcal{B})) \geq \underline{P}(G_{\underline{P}}(Cf|\mathcal{B})). \blacklozenge$$

[6] It is also possible to study the other extreme called *full conglomerability*, in which a coherent lower prevision is conglomerable with respect to all partitions of the possibility space (see [17, Sections 6.8 and 6.9] for a discussion).

This remark shows that we may as well restrict the attention to countable partitions to the extent of studying conglomerability, and more specifically to the partition $\{B \in \mathcal{B} : B \subseteq C\} \bigcup \{C^c\}$.

In the particular case of linear previsions, P is conglomerable if and only if it is coherent with its conditional natural extension $\underline{P}(\cdot|\mathcal{B})$; note however that this conditional prevision may not be precise (as we have seen, $\underline{P}(\cdot|B)$ is vacuous if $P(B) = 0$). Indeed, there are conglomerable linear previsions P that are not coherent with any conditional linear prevision $P(\cdot|\mathcal{B})$ (see [17, Example 6.6.10] for an example). A sufficient condition for the conglomerability of a linear prevision P is that it is countably additive on $\mathcal{B}$, in the sense that $\sum_{B \in \mathcal{B}} P(B) = 1$. However, this condition is not necessary, and there are interesting situations that can be modeled by means of conglomerable linear previsions that are not countably additive [17, Examples 6.6.4 and 6.6.5]. If instead we require that the linear prevision be conglomerable with respect to all partitions, then there is a very strong relationship to countable additivity,[7] and de Finetti and others have argued [3] that in some cases countable additivity can give rise to unreasonable conclusions.

The reason why conglomerability is controversial is because, unlike coherence, it involves the combination of an infinite number of transactions:[8] it means that the infinite sum of acceptable gambles that depend on different elements of a partition should be acceptable. This is called the *conglomerative principle* in [17] and implies, for instance, that the gamble $G_{\underline{P}}(f|\mathcal{B}) + \varepsilon$ should be acceptable for all $\varepsilon > 0$. This assumption is not made by authors such as Williams, for whom the gamble $G_{\underline{P}}(f|\mathcal{B}) + \varepsilon$ is acceptable for all $\varepsilon > 0$ only when f has a finite support in $\mathcal{B}$, i.e., when there is only a finite number of elements of $\mathcal{B}$ on which f is non-zero. Walley's position on the other hand is to support conglomerability, and for this reason his definition of coherence for conditional and unconditional lower previsions is based on the conglomerative principle.

In this paper, we study this property in detail and we investigate to which extent conglomerability can be fully incorporated into the theory of coherent lower previsions as a rationality assessment, in the sense that we can extend a coherent, but non-conglomerable, model into a conglomerably coherent one in the least-committal way.

2.5. *Basic properties of conglomerability*

A useful feature of coherence, as discussed by Walley in [17, Section 2.6], is that the class $\underline{\mathbb{P}}$ of coherent lower previsions is closed under lower envelopes, point-wise limits, and convex combinations.

The situation is not so straightforward if we restrict our attention to the subclass $\underline{\mathbb{F}}$ of the conglomerably coherent lower previsions: for instance, there are coherent conditional and unconditional lower previsions that are not necessarily the lower envelopes of a set of coherent conditional and unconditional linear previsions (see [17, Examples 6.6.9, 6.6.10]). In other words, the notion of conglomerable coherence is only partially compatible with the Bayesian sensitivity analysis interpretation: even though the lower envelope of a family of conglomerably coherent lower previsions is again conglomerable by [17, Theorem 6.9.3], it may even happen that a conglomerably coherent lower prevision is not dominated by any conglomerably coherent linear prevision.

On the other hand, we show in Example 1 in Appendix A that conglomerability is not preserved by point-wise limits (note that a similar observation was made by Walley in [17, Section 6.6.7]); moreover, in [15, Example 4.1] it is showed that conglomerability is not preserved by convex combinations. These two properties differentiate conglomerably coherent lower previsions from those that are simply coherent. Indeed, the second example shows perhaps better than others the counterintuitive nature of conglomerability: in fact, "one would expect convex combinations of reasonable models to be reasonable", quoting Walley from note 4 to [17, Section 6.9.2]—where he made a point similar to that of [15, Example 4.1] while focusing on the case of fully conglomerable models.[9]

3. On the existence of the conglomerable natural extension

The preliminary results reported above illustrate the fact that some of the properties of coherent lower previsions do not extend themselves to the more stringent framework originated by the additional requirement of conglomerability.

[7] Specifically, countable additivity is necessary for *full* conglomerability (that is, conglomerability with respect to all partitions) when the linear prevision takes infinitely many values on events. See [15] and [17, Section 6.9] for details.

[8] But see also [20] for a recent finitary interpretation.

[9] However Walley used that to argue that some fully conglomerable models could be unreasonable and one should rather consider countably additive models.

Another example of this is that a lower prevision $\underline{P}$ that avoids sure loss has always a dominating coherent lower prevision (its natural extension), but it may not have a dominating conglomerably coherent lower prevision.

As discussed at the beginning of Section 2.5, $\underline{\mathbb{F}}$ is closed under taking lower envelopes. Hence, if $\underline{P}$ has a dominating conglomerable model, then there is also a smallest dominating conglomerable model. We shall refer to it as the conglomerable natural extension of $\underline{P}$.

Definition 4 (**Conglomerable natural extension of a coherent lower prevision**). Let $\underline{P}$ be a coherent lower prevision on $\mathcal{L}(\Omega)$ and let $\mathcal{B}$ be a partition of Ω. If it exists, the smallest coherent lower prevision $\underline{F}$ on $\mathcal{L}(\Omega)$ that dominates $\underline{P}$ and is conglomerable with respect to $\mathcal{B}$, is called the *$\mathcal{B}$-conglomerable natural extension* of $\underline{P}$. (In the following we shall refer to $\underline{F}$ more simply as the conglomerable natural extension, given that we always focus on the single partition $\mathcal{B}$.)

In other words, the conglomerable natural extension of $\underline{P}$ is the lower envelope of the set $\{\underline{Q} \in \underline{\mathbb{F}} : \underline{Q} \geq \underline{P}\}$. However, this set may be empty, and in that case the conglomerable natural extension of a lower prevision $\underline{P}$ does not exist. An instance is given by [15, Example 4.1], where a linear prevision P is not conglomerable: since $\mathcal{M}(P) = \{P\}$, any dominating coherent lower prevision $\underline{P}$ must coincide with P, and therefore there are no dominating conglomerably coherent models.[10] Note that it follows from the above definition that if $\underline{P}, \underline{Q}$ are coherent lower previsions and $\underline{Q} \geq \underline{P}$ then the conglomerable natural extension of $\underline{Q}$ must dominate that of $\underline{P}$.

We see then that the existence of the conglomerable natural extension is not a trivial matter. Next, we provide a number of necessary and sufficient conditions:

Proposition 2. *Let $\underline{P}$ be a coherent lower prevision on $\mathcal{L}(\Omega)$, $\mathcal{B}$ a partition of Ω, and $\underline{P}(\cdot|\mathcal{B})$ a separately coherent lower prevision. Consider the following statements:*

(a) *$\underline{P}, \underline{P}(\cdot|\mathcal{B})$ are coherent.*

(b) *$\underline{P}, \underline{P}(\cdot|\mathcal{B})$ are dominated by coherent $\underline{Q}, \underline{Q}(\cdot|\mathcal{B})$.*

(c) *The conglomerable natural extension of $\underline{P}$ exists.*

(d) *$\underline{P}, \underline{P}(\cdot|\mathcal{B})$ are dominated by $\underline{Q}, \underline{Q}(\cdot|\mathcal{B})$ that avoid partial loss.*

(e) *$\underline{P}, \underline{P}(\cdot|\mathcal{B})$ avoid partial loss.*

Then the following implications hold:

$$\begin{array}{ccc} & \text{(a)} & \\ & \Downarrow & \\ & \text{(b)} & \Rightarrow \ \text{(c)} \\ & \Downarrow & \\ \text{(d)} & \Leftrightarrow & \text{(e).} \end{array}$$

If, in addition, $\underline{P}(\cdot|\mathcal{B})$ is the conditional natural extension of $\underline{P}$, then (c) $\Rightarrow$ (b) *holds as well, and if in particular $\underline{P}$ is linear then we have also that* (b) $\Rightarrow$ (a) *and* (d) $\Rightarrow$ (b)*, so all of them are equivalent conditions:*

$$\text{(a)} \Leftrightarrow \text{(b)} \Leftrightarrow \text{(c)} \Leftrightarrow \text{(d)} \Leftrightarrow \text{(e)}.$$

Proof. Let us prove the different implications in the case of generic $\underline{P}, \underline{P}(\cdot|\mathcal{B})$.

(a) $\Rightarrow$ (b) Trivial.

(b) $\Rightarrow$ (c) If $\underline{Q}, \underline{Q}(\cdot|\mathcal{B})$ are coherent, then $\underline{Q}$ is conglomerable and dominates $\underline{P}$, so the conglomerable natural extension of $\underline{P}$ exists.

(b) $\Rightarrow$ (d) Trivial.

[10] Indeed, for any linear prevision P there are only two possible scenarios: either P is conglomerable (and then it coincides with its conglomerable natural extension) or the conglomerable natural extension does not exist. This is because the set $\{\underline{Q} \in \underline{\mathbb{F}} : \underline{Q} \geq P\}$ is equal to either $\{P\}$ or the empty set.

(d) $\Rightarrow$ (e) Given gambles f, g it holds that

$$\sup_{\Omega}[G_{\underline{P}}(f) + G_{\underline{P}}(g|\mathcal{B})] \geq \sup_{\Omega}[G_{\underline{Q}}(f) + G_{\underline{Q}}(g|\mathcal{B})] \overset{\text{(d)}}{\geq} 0,$$

so Eq. (3) holds and therefore $\underline{P}, \underline{P}(\cdot|\mathcal{B})$ avoid partial loss.

(e) $\Rightarrow$ (d) Trivial.

In case $\underline{P}(\cdot|\mathcal{B})$ is the conditional natural extension of $\underline{P}$, then:

(c) $\Rightarrow$ (b) If $\underline{F}$ exists, then $\underline{P}, \underline{P}(\cdot|\mathcal{B})$ are dominated by the coherent $\underline{F}, \underline{F}(\cdot|\mathcal{B})$, where $\underline{F}(\cdot|\mathcal{B})$ is the conditional natural extension of $\underline{F}$.

Finally, if in addition $\underline{P}$ is linear, then:

(b) $\Rightarrow$ (a) If $\underline{P}, \underline{P}(\cdot|\mathcal{B})$ are dominated by coherent $\underline{Q}, \underline{Q}(\cdot|\mathcal{B})$, then it must be $\underline{Q} = P$. Thus, $P, \underline{Q}(\cdot|\mathcal{B})$ are coherent, whence P is conglomerable and as a consequence $P, \underline{P}(\cdot|\mathcal{B})$ are coherent. The last implication holds also because $\underline{P}(\cdot|\mathcal{B})$ is the conditional natural extension of P, so that $P, \underline{P}(\cdot|\mathcal{B})$ satisfy (GBR).

(d) $\Rightarrow$ (b) If $P, \underline{P}(\cdot|\mathcal{B})$ are dominated by $\underline{Q}, \underline{Q}(\cdot|\mathcal{B})$ that avoid partial loss, then $P, \underline{P}(\cdot|\mathcal{B})$ avoid partial loss; applying the second part of the consequences of avoiding partial loss from [17, Theorem 6.3.5(3)] together with the linearity of P, it follows that $P(G_{\underline{P}}(f|\mathcal{B})) \geq 0$ for every gamble f. Since a coherent lower prevision always satisfies (GBR) with its conditional natural extension, we obtain that $P, \underline{P}(\cdot|\mathcal{B})$ are coherent. □

Example 2 in Appendix A shows that there is no additional implication.

Now, if we consider a coherent lower prevision $\underline{P}$, it follows that its conglomerable natural extension exists if and only if there is a coherent lower prevision $\underline{F} \geq \underline{P}$ that is conglomerable. Since conglomerability is equivalent to the coherence with the conditional natural extension, it follows that the conglomerable natural extension of $\underline{P}$ exists if and only if $\underline{P}, \underline{P}(\cdot|\mathcal{B})$ are dominated by coherent $\underline{Q}, \underline{Q}(\cdot|\mathcal{B})$, where $\underline{P}(\cdot|\mathcal{B})$ denotes the conditional natural extension of $\underline{P}$. We deduce from Proposition 2 that the following implications hold:

$$\underline{P}, \underline{P}(\cdot|\mathcal{B}) \text{ coherent } \Rightarrow \text{ the conglomerable natural extension of } \underline{P} \text{ exists} \Rightarrow \underline{P}, \underline{P}(\cdot|\mathcal{B}) \text{ avoid partial loss,} \quad (5)$$

and moreover the conglomerable natural extension of $\underline{P}$ exists if and only if $\underline{P}, \underline{P}(\cdot|\mathcal{B})$ *avoid conglomerable partial loss*, in the sense of [12, Definition 21]. The converses of the implications in (5) do not hold in general: on the one hand, there are previsions $\underline{P}$ that are not conglomerable but whose conglomerable natural extension exists (for instance the one in Example 4 in Appendix A). We also show in Example 3 in Appendix A that the converse of the second implication does not hold either. In other words, the conditions of avoiding partial loss and avoiding conglomerable partial loss are not equivalent in general. This second finding is important because it shows that the notion of logical consistency in the conditional case, which means that our assessments can be corrected into coherent ones, is not the notion of avoiding partial loss proposed by Walley in [17, Section 7.1]; we should instead consider the notion of avoiding conglomerable partial loss, which is equivalent to the existence of the conglomerable natural extension.

We can get more, and different, results in the special case where the conditional natural extension of $\underline{P}$ is linear. In order to establish this result, we need to define the notion of unconditional natural extension:

Definition 5 (**Unconditional natural extension of** $\underline{P}, \underline{P}(\cdot|\mathcal{B})$). Let $\underline{P}$ be a coherent lower prevision and $\underline{P}(\cdot|\mathcal{B})$ be a separately coherent conditional lower prevision on $\mathcal{L}$. Their *unconditional natural extension* $\underline{E}$ is

$$\underline{E}(f) := \sup\{\alpha : f - \alpha \geq G_{\underline{P}}(g) + G_{\underline{P}}(h|\mathcal{B}) \text{ for some } g, h \in \mathcal{L}\}. \quad (6)$$

$\underline{E}$ expresses the coherent implications of $\underline{P}, \underline{P}(\cdot|\mathcal{B})$ on an unconditional model, and it is itself a coherent lower prevision, on $\mathcal{L}$, if and only if $\underline{P}, \underline{P}(\cdot|\mathcal{B})$ avoid partial loss. Moreover, if $\underline{P}(\cdot|\mathcal{B})$ is the conditional natural extension of $\underline{P}$ and $\underline{E}(\cdot|\mathcal{B})$ is that of $\underline{E}$, then any coherent $\underline{Q}, \underline{Q}(\cdot|\mathcal{B})$ that dominate $\underline{P}, \underline{P}(\cdot|\mathcal{B})$ must also dominate $\underline{E}, \underline{E}(\cdot|\mathcal{B})$. In other words, $\{\underline{Q} \in \underline{\mathbb{F}} : \underline{Q} \geq \underline{P}\} = \{\underline{Q} \in \underline{\mathbb{F}} : \underline{Q} \geq \underline{E}\}$, and therefore the conglomerable natural extensions of $\underline{P}$ and $\underline{E}$ coincide.

Proposition 3. *Let $\underline{P}$ be a coherent lower prevision on $\mathcal{L}$ and assume that its conditional natural extension is a linear prevision $P(\cdot|\mathcal{B})$. Then:*

(a) *$\underline{P}, P(\cdot|\mathcal{B})$ avoid partial loss if and only if $\underline{P}, P(\cdot|\mathcal{B})$ avoid conglomerable partial loss.*[11]

(b) *$\underline{P}$ is conglomerable if and only if it is a lower envelope of conglomerable linear models.*

Proof.

(a) The converse implication is trivial. To show the direct implication, let us assume that $\underline{P}, P(\cdot|\mathcal{B})$ avoid partial loss. Then we can build their natural extension $\underline{E}$, which is a coherent lower prevision. Its conditional natural extension $\underline{E}(\cdot|\mathcal{B})$ must dominate that of $\underline{P}$, so $\underline{E}(\cdot|\mathcal{B}) = P(\cdot|\mathcal{B})$. But by construction of $\underline{E}$, we have that $\underline{E}(G_P(f|\mathcal{B})) \geq 0$ for all $f \in \mathcal{L}$, whence $\underline{E}, P(\cdot|\mathcal{B})$ are coherent and as a consequence $\underline{E}$ is conglomerable. Hence, the conglomerable natural extension of $\underline{P}$ exists.

(b) The converse implication is trivial. To show the direct implication, note that if $\underline{P}$ is conglomerable then it is coherent with its conditional natural extension $P(\cdot|\mathcal{B})$. From the results on conditional coherence in [17, Section 6.5.5], a conditional linear prevision is coherent with an unconditional lower prevision if and only if $\underline{P} = \underline{P}(P(\cdot|\mathcal{B}))$. The envelope theorem for marginal extension models established in [17, Theorem 6.7.4] implies then that $\underline{P}(P(\cdot|\mathcal{B}))$ is the lower envelope of the set $\{P(P(\cdot|\mathcal{B})) : P \geq \underline{P}\}$. Thus, if $\underline{P}$ is conglomerable then it is a lower envelope of conglomerable linear models. $\square$

As we mentioned in Section 2.5, the set of conglomerably coherent lower previsions is closed under lower envelopes, but not every conglomerably coherent lower prevision is the lower envelope of a family of conglomerably coherent linear previsions. Hence, the Bayesian sensitivity analysis interpretation is not fully compatible with conglomerable coherence. Proposition 3 provides a particular situation when the set of conglomerably coherent lower previsions is necessarily the lower envelope of a set of conglomerably coherent linear previsions: when the conditional natural extension of $\underline{P}$ is linear.

Note that in that case the equivalence in Proposition 3(b) is not trivial, in the sense that even if the conditional natural extension of $\underline{P}$ is linear the coherent lower prevision $\underline{P}$ may not be conglomerable. To see this, we refer to Example 1 in Appendix A.

4. Approximation by a sequence

In [13], a procedure was devised to approximate the conglomerable natural extension (if it exists) of a coherent lower prevision $\underline{P}$: we consider the sequence of coherent lower previsions $(\underline{E}_n)_{n\in\mathbb{N}}$, where $\underline{E}_0 := \underline{P}$ and for every $n \geq 1$:

$$\underline{E}_n \text{ is the natural extension of } \underline{E}_{n-1}, \underline{E}_{n-1}(\cdot|\mathcal{B}), \text{ given by Eq. (6);} \tag{7}$$

$$\underline{E}_{n-1}(\cdot|\mathcal{B}) \text{ is the conditional natural extension of } \underline{E}_{n-1}, \text{ given by Eq. (4).} \tag{8}$$

Then $\{\underline{Q} \in \underline{\mathbb{F}} : \underline{Q} \geq \underline{E}_n\} = \{\underline{Q} \in \underline{\mathbb{F}} : \underline{Q} \geq \underline{P}\}$ for every n, so the conglomerable natural extension of $\underline{P}$ coincides with that of $\underline{E}_n$ for every n. It can be also checked that:

Proposition 4 (See [13]). *Assume that the conglomerable natural extension $\underline{F}$ of $\underline{P}$ exists. Then:*

1. *$(\underline{E}_n)_{n\in\mathbb{N}}$ is an increasing sequence of coherent lower previsions, and $(\underline{E}_n(\cdot|\mathcal{B}))_{n\in\mathbb{N}}$ is an increasing sequence of separately coherent conditional lower previsions.*

2. *Given their point-wise limits $\underline{Q}, \underline{Q}(\cdot|\mathcal{B})$, it holds that $\underline{Q}(\cdot|\mathcal{B})$ is the conditional natural extension of $\underline{Q}$.*

3. *$\underline{Q} \leq \underline{F}$, and they coincide if and only if $\underline{Q}$ is conglomerable.*

[11] This has essentially been shown already in [13, Proposition 15].

Moreover, it was shown in [13, Example 5] that the sequence does not necessarily stabilise in the first step, or, in other words, that the natural extension of $\underline{P}, \underline{P}(\cdot|\mathcal{B})$ does not always coincide with the conglomerable natural extension.

In terms of credal sets, we have the following:

Proposition 5 (Propositions 13 and 14 in [13])**.** *Let $\underline{P}$ be a coherent lower prevision on $\mathcal{L}(\Omega)$, $\mathcal{B}$ a partition of Ω and $\underline{P}(\cdot|\mathcal{B})$ its conditional natural extension. Let $\underline{E}$ be the unconditional natural extension of $\underline{P}, \underline{P}(\cdot|\mathcal{B})$. Then*

$$\mathcal{M}(\underline{E}) = \{P \in \mathcal{M}(\underline{P}) : (\forall f \in \mathcal{L})P(G_{\underline{P}}(f|\mathcal{B})) \geq 0\} = \mathcal{M}(\underline{P}) \cap \mathcal{M}(\underline{M}),$$

where $\underline{M} = \underline{P}(\underline{P}(\cdot|\mathcal{B}))$.

In addition, it was shown in [13] that if the sequence stabilises in a finite number of steps, i.e., if $\underline{Q} = \underline{E}_n$ for some n, then $\underline{Q}$ is the conglomerable natural extension of $\underline{P}$. However, the question of whether or not the sequence always stabilises in a finite number of steps was left open. We solve this problem by means of Proposition 21 and Example 4 in Appendix A: the example shows that it may happen that the sequence is made of infinitely many distinct elements. This is the most important result in this paper, for it shows that in general the computation of the consequences of the conglomerative axiom cannot be made in a finitary manner. The idea underlying the example is to consider the lower envelope $\underline{P}$ of two non-conglomerable linear previsions, of which only one convex combination is conglomerable (and this is the conglomerable natural extension of $\underline{P}$); then the sequence $(\underline{E}_n)_{n\in\mathbb{N}}$ defined by Eq. (6) is in a one-to-one correspondence with a strictly decreasing sequence of subintervals of $[0,1]$, whose intersection determines the conglomerable natural extension.

5. Conglomerability and marginal extension

When $\underline{Q}$ does not coincide with $\underline{E}_n$ for any n (as in Example 4 in Appendix A), it is an open problem whether $\underline{Q}$ always coincides with the conglomerable natural extension or not. Here, we give a number of sufficient conditions for the equality $\underline{Q} = \underline{F}$. We show that one particular case of interest is that where $\underline{Q}$ is a marginal extension model (see Eq. (2)) and we are going to explore in more detail the connection between conglomerably coherent lower previsions and marginal extensions.

We begin by proving an elementary, and yet interesting, result. The interest arises in particular from the following reasoning: given a coherent lower prevision $\underline{P}$, we can deduce from it both its $\mathcal{B}$-marginal model, which is $\underline{P}$ itself restricted to the subset of $\mathcal{B}$-measurable gambles, and the weakest (i.e., more conservative) conditional model that is compatible with it, that is, its conditional natural extension $\underline{P}(\cdot|\mathcal{B})$; then we can re-agglomerate these two pieces of information in what appears to be the weakest possible way, that is, by creating their marginal extension $\underline{M} := \underline{P}(\underline{P}(\cdot|\mathcal{B}))$ (remember that the marginal extension is the generalisation of the law of iterated expectation to the case of coherent lower previsions). So it is tempting to somewhat take for granted that $\underline{M}$ cannot be more informative than $\underline{P}$, in the sense that it should be dominated by $\underline{P}$. But the following result shows that the reasoning so far is correct if and only if $\underline{P}$ is conglomerable:

Proposition 6. *Let $\underline{P}$ be a coherent lower prevision on $\mathcal{L}$, $\mathcal{B}$ a partition of Ω and $\underline{P}(\cdot|\mathcal{B})$ the conditional natural extension of $\underline{P}$. Let $\underline{M} := \underline{P}(\underline{P}(\cdot|\mathcal{B}))$. Then*

$$\underline{M} \leq \underline{P} \Leftrightarrow \underline{P} \text{ conglomerable}. \tag{9}$$

Proof. ($\Rightarrow$) If $\underline{M} \leq \underline{P}$, then $\mathcal{M}(\underline{M}) \cap \mathcal{M}(\underline{P}) = \mathcal{M}(\underline{P})$. By Proposition 5, the natural extension $\underline{E}$ of $\underline{P}, \underline{P}(\cdot|\mathcal{B})$ coincides with $\underline{P}$. But this can only hold (that the sequence stabilises) when $\underline{P}$ is conglomerable, by [13, Proposition 16].

($\Leftarrow$) By definition, if $\underline{P}$ is conglomerable then $\underline{P}, \underline{P}(\cdot|\mathcal{B})$ are coherent, and by [17, Theorem 6.3.5(5)] a consequence of coherence is that $\underline{P} \geq \underline{P}(\underline{P}(\cdot|\mathcal{B})) = \underline{M}$. ☐

Example 5 in Appendix A shows that we do not necessarily have the equality $\underline{P} = \underline{M}$ in Eq. (9).

Next, drawing inspiration from Proposition 5, we investigate the properties of the sequence of marginal extensions $(\underline{M}_n)_{n\in\mathbb{N}}$ associated to $(\underline{E}_n)_{n\in\mathbb{N}}$, where $\underline{M}_n := \underline{E}_{n-1}(\underline{E}_{n-1}(\cdot|\mathcal{B}))$ for every $n > 1$ and $\underline{M}_1 = \underline{P}(\underline{P}(\cdot|\mathcal{B}))$, recalling that $\underline{E}_{n-1}(\cdot|\mathcal{B})$ and $\underline{P}(\cdot|\mathcal{B})$ are the conditional natural extensions of $\underline{E}_{n-1}$ and $\underline{P}$, respectively. Remember that $\underline{M}_n$ is conglomerable for all n because all marginal extension models are, by [17, Theorem 6.7.2]. It follows from Proposition 5 that $\mathcal{M}(\underline{E}_n) = \mathcal{M}(\underline{E}_{n-1}) \cap \mathcal{M}(\underline{M}_n)$, so $\underline{M}_n \leq \underline{E}_n$ for all n. Since the sequence $(\underline{E}_n(\cdot|\mathcal{B}))_{n\in\mathbb{N}}$ is also increasing, we deduce that so is the sequence $(\underline{M}_n)_{n\in\mathbb{N}}$. Thus, $(\underline{M}_n)_{n\in\mathbb{N}}$ is an increasing sequence of conglomerable and coherent lower previsions that is dominated by $\underline{F}$, the conglomerable natural extension of $\underline{P}$. Moreover, if $\underline{E}_n$ is not conglomerable, then it cannot be $\underline{M}_n \geq \underline{P}$, because, since $\underline{M}_n$ is conglomerable, it would be $\underline{P} \leq \underline{M}_n = \underline{F} \leq \underline{E}_n \leq \underline{F}$, and therefore also $\underline{E}_n = \underline{F}$ would be conglomerable. In other words, if the sequence does not stabilise in a finite number of steps (and therefore $\underline{E}_n$ is not conglomerable for any n), then

$$\forall n \in \mathbb{N}, \exists f \in \mathcal{L} \text{ such that } \underline{M}_n(f) = \underline{E}_{n-1}(\underline{E}_{n-1}(f|\mathcal{B})) < \underline{P}(f) \leq \underline{E}_{n-1}(f).$$

This notwithstanding, the sequence of marginal extensions can actually converge to the conglomerable natural extension in the limit; it can be checked that this is in fact the case of the model in Example 4 in Appendix A; an easier (trivial) example can be made by considering a lower prevision $\underline{P}$ satisfying $\underline{P} = \underline{P}(\underline{P}(\cdot|\mathcal{B}))$, for which $\underline{F} = \underline{M} = \underline{P}$.

However, it may be that the conglomerable natural extension is not a marginal extension model, and therefore that the increasing sequence of marginal extensions stabilises on a model that is not the conglomerable natural extension. This is showed in Example 5 in Appendix A.

Let us study in more detail the sequence $(\underline{M}_n)_{n\in\mathbb{N}}$. We begin by characterising the relationship between these marginal extensions and $\underline{Q}$ in terms of credal sets.

Proposition 7. *Let $\underline{P}$ be a coherent lower prevision and $(\underline{E}_n)_{n\in\mathbb{N}}$ the sequence of coherent lower previsions it originates by Eq. (7). Assume that the limit of the sequence exists and let $\underline{Q} := \lim_{n\to\infty} \underline{E}_n$, $\underline{Q}' := \lim_{n\to\infty} \underline{Q}(\underline{E}_n(\cdot|\mathcal{B}))$ and $\underline{M}_n := \underline{E}_{n-1}(\underline{E}_{n-1}(\cdot|\mathcal{B}))$ for $n > 1$, $\underline{M}_1 := \underline{P}(\underline{P}(\cdot|\mathcal{B}))$. Then:*

(a) $\mathcal{M}(\underline{Q}) = \{P \geq \underline{P} : (\forall f \in \mathcal{L})(\forall n \in \mathbb{N})P(G_{\underline{E}_n}(f|\mathcal{B})) \geq 0\}$.

(b) $\mathcal{M}(\underline{Q}) = \mathcal{M}(\underline{P}) \cap (\cap_{n\in\mathbb{N}}\mathcal{M}(\underline{M}_n))$.

(c) $\mathcal{M}(\underline{Q}) = \mathcal{M}(\underline{P}) \cap \mathcal{M}(\underline{Q}')$ *since* $\mathcal{M}(\underline{Q}') = \cap_{n\in\mathbb{N}}\mathcal{M}(\underline{M}_n)$.

Proof.

(a) To prove the direct inclusion, consider that $\underline{Q} \geq \underline{E}_n \geq \underline{P}$, and also that by construction $\underline{Q}(G_{\underline{E}_n}(f|\mathcal{B})) \geq \underline{E}_{n+1}(G_{\underline{E}_n}(f|\mathcal{B})) \geq 0$ for all $f \in \mathcal{L}$ and $n \in \mathbb{N}$, where the last inequality follows from Proposition 5. Hence, any $P \in \mathcal{M}(\underline{Q})$ also satisfies these two conditions. For the converse inclusion, if $P(G_{\underline{E}_n}(f|\mathcal{B})) \geq 0$ for all $f \in \mathcal{L}$, $n \in \mathbb{N}$ and $P \geq \underline{P}$, it follows by Eqs. (7), (8) that $P \geq \underline{E}_n$ for all $n \in \mathbb{N}$, whence $P \geq \underline{Q}$.

(b) For the direct inclusion, given $P \in \mathcal{M}(\underline{Q})$, it holds that $P \geq \underline{E}_n \geq \underline{M}_n$ for all n, where the last inequality follows from Proposition 5. Hence, $P \in \cap_{n\in\mathbb{N}}\mathcal{M}(\underline{M}_n)$; and by construction $P \in \mathcal{M}(\underline{P})$. Conversely, if $P \in \cap_{n\in\mathbb{N}}\mathcal{M}(\underline{M}_n)$ for all n, then for any gamble f and any $n > 1$ it holds that $P(f) \geq \underline{M}_n(f) = \underline{E}_{n-1}(\underline{E}_{n-1}(f|\mathcal{B}))$. In particular, $P(G_{\underline{E}_{n-1}}(f|\mathcal{B})) \geq \underline{E}_{n-1}(\underline{E}_{n-1}(G_{\underline{E}_{n-1}}(f|\mathcal{B})|\mathcal{B})) = \underline{E}_{n-1}(0) = 0$, because for any conditioning event B, $\underline{E}_{n-1}(G_{\underline{E}_{n-1}}(f|\mathcal{B})|B) = \underline{E}_{n-1}(f|B) - \underline{E}_{n-1}(f|B) = 0$. Applying (a), we conclude that $P \in \mathcal{M}(\underline{Q})$.

(c) The statement follows from (b) if we show that $\mathcal{M}(\underline{Q}') = \cap_{n\in\mathbb{N}}\mathcal{M}(\underline{M}_n)$. If $P \geq \underline{Q}' = \lim_{n\to\infty} \underline{Q}(\underline{E}_n(\cdot|\mathcal{B}))$, then in particular for any fixed n it holds that $P(f) \geq \underline{E}_{n-1}(\underline{E}_{n-1}(f|\mathcal{B})) = \underline{M}_n(f)$. Thus $P \in \cap_{n\in\mathbb{N}}\mathcal{M}(\underline{M}_n)$. Conversely, if $P \in \cap_{n\in\mathbb{N}}\mathcal{M}(\underline{M}_n)$, then for any f, $P(f) \geq \underline{E}_n(\underline{E}_n(f|\mathcal{B}))$. Therefore $P(f) \geq \underline{E}_n(\underline{E}_{n-1}(f|\mathcal{B}))$ and hence, for any fixed n, $P(f) \geq \underline{E}_m(\underline{E}_n(f|\mathcal{B}))$ for all $m \geq n$. Thus, $P(f) \geq \lim_{m\to\infty} \underline{E}_m(\underline{E}_n(f|\mathcal{B})) = \underline{Q}(\underline{E}_n(f|\mathcal{B}))$, and since this happens for every n, $P(f) \geq \lim_{n\to\infty} \underline{Q}(\underline{E}_n(f|\mathcal{B})) = \underline{Q}'$. ☐

So we see that $\underline{Q}'$ actually defines the limit of the sequence of marginal extensions $(\underline{M}_n)_{n\in\mathbb{N}}$ and that there is a tight relationship between this limit and the limit $\underline{Q}$ of the sequence of natural extensions $(\underline{E}_n)_{n\in\mathbb{N}}$. We can moreover use these results to characterise when $\underline{Q}'$ is conglomerable:

Corollary 8. *Under the previous conditions,*

$$\underline{Q}' \text{ conglomerable } \Leftrightarrow \underline{Q}' = \underline{Q}(\underline{Q}(\cdot|\mathcal{B})), \tag{10}$$

where $\underline{Q}(\cdot|\mathcal{B})$ is the conditional natural extension of $\underline{Q}$.

Proof. Since $\underline{Q}' \geq \underline{E}_n(\underline{E}_n(\cdot|\mathcal{B}))$ for every n, its conditional natural extension $\underline{Q}'(\cdot|\mathcal{B})$ dominates the conditional natural extension of $\underline{M}_{n+1}$, that in turn dominates $\underline{E}_n(\cdot|\mathcal{B})$ because $\underline{M}_{n+1}, \underline{E}_n(\cdot|\mathcal{B})$ are coherent. On the other hand, $\underline{Q}' \leq \underline{Q}$ by Proposition 7(c), and therefore its conditional natural extension satisfies $\underline{Q}'(\cdot|\mathcal{B}) \leq \underline{Q}(\cdot|\mathcal{B})$. Thus,

$$\underline{E}_n(\cdot|\mathcal{B}) \leq \underline{Q}'(\cdot|\mathcal{B}) \leq \underline{Q}(\cdot|\mathcal{B})$$

for every n. Since by Proposition 4 $\underline{Q}(\cdot|\mathcal{B})$ is the point-wise limit of $(\underline{E}_n(\cdot|\mathcal{B}))_{n\in\mathbb{N}}$, we deduce that $\underline{Q}'(\cdot|\mathcal{B}) = \underline{Q}(\cdot|\mathcal{B})$.

By Proposition 5, the conglomerable natural extension of $\underline{Q}'$ must dominate $\underline{Q}'(\underline{Q}'(\cdot|\mathcal{B}))$. Since for any $\mathcal{B}$-measurable gamble g it holds that $\underline{Q}'(g) = \lim_{n\to\infty} \underline{Q}(\underline{E}_n(g|\mathcal{B})) = \lim_{n\to\infty} \underline{Q}(g) = \underline{Q}(g)$, we have $\underline{Q}'(\underline{Q}'(\cdot|\mathcal{B})) = \underline{Q}(\underline{Q}(\cdot|\mathcal{B}))$, and therefore the conglomerable natural extension of $\underline{Q}'$ dominates $\underline{Q}(\underline{Q}(\cdot|\mathcal{B}))$.

On the other hand, it follows from monotonicity that $\underline{Q}' = \lim_{n\to\infty} \underline{Q}(\underline{E}_n(\cdot|\mathcal{B})) \leq \underline{Q}(\underline{Q}(\cdot|\mathcal{B}))$; a marginal extension (i.e., the concatenation of a conditional and a marginal model) is always a conglomerable model because of [17, Theorem 6.7.2]. We deduce that the conglomerable natural extension of $\underline{Q}'$ is given by $\underline{Q}(\underline{Q}(\cdot|\mathcal{B}))$. Since a coherent lower prevision is conglomerable if and only if it coincides with its conglomerable natural extension, we conclude that Eq. (10) holds. □

Taking all this into account, we can establish a sufficient condition for the limit of the sequence of marginal extensions to be conglomerable, and use this to show that the limit of the natural extensions has then to be the conglomerable natural extension.

Proposition 9. *Let $\underline{P}$ be a coherent lower prevision on $\mathcal{L}$, $\mathcal{B}$ a partition of Ω and $(\underline{E}_n)_{n\in\mathbb{N}}, (\underline{E}_n(\cdot|\mathcal{B}))_{n\in\mathbb{N}}$ the sequences it determines by means of Eqs. (7) and (8). Let $\underline{Q} := \lim_{n\to\infty} \underline{E}_n$, $\underline{Q}(\cdot|\mathcal{B}) := \lim_{n\to\infty} \underline{E}_n(\cdot|\mathcal{B})$, and $\underline{Q}' := \lim_{n\to\infty} \underline{Q}(\underline{E}_n(\cdot|\mathcal{B}))$. Consider the following statements:*

(a) *$\underline{Q}(f|\mathcal{B})$ is the uniform limit of the sequence $(\underline{E}_n(f|\mathcal{B}))_{n\in\mathbb{N}}$ for all $f \in \Omega$.*[12]

(b) *$\underline{Q} = \underline{Q}' = \underline{F}$.*

(c) *$\underline{Q}'$ is conglomerable.*

(d) *$\underline{Q}$ is conglomerable.*

(e) *$\underline{Q} = \underline{F}$.*

Then the following implications hold:

$$\begin{array}{ccccc} \text{(a)} & \Rightarrow & \text{(c)} & \Leftarrow & \text{(b)} \\ & & \Downarrow & & \\ & & \text{(d)} \Leftrightarrow \text{(e)}. & & \end{array}$$

If in addition $\underline{Q}' \geq \underline{P}$, then:

1. (b) $\Leftrightarrow$ (c) $\Leftrightarrow$ $\underline{Q}' = \underline{Q}(\underline{Q}(\cdot|\mathcal{B}))$.
2. (d) $\Leftrightarrow$ (e) $\Leftrightarrow$ $\underline{Q} = \underline{Q}(\underline{Q}(\cdot|\mathcal{B}))$.
3. (a) $\Rightarrow$ (b) $\Leftrightarrow$ (c) $\Rightarrow$ (d) $\Leftrightarrow$ (e).

Proof.

[12] By an abuse of terminology, in the following we shall also say that $\underline{Q}(\cdot|\mathcal{B})$ is the uniform limit of the sequence $(\underline{E}_n(\cdot|\mathcal{B}))_{n\in\mathbb{N}}$.

(a) ⇒ (c) If $\underline{Q}(\cdot|\mathcal{B})$ is the uniform limit of $(\underline{E}_n(\cdot|\mathcal{B}))_{n\in\mathbb{N}}$, then, since coherent lower previsions are continuous under uniform convergence (this is established in [17, Theorem 2.6.1(ℓ)]), it follows that given $f \in \mathcal{L}$, $\underline{Q}'(f) = \lim_{n\to\infty} \underline{Q}(\underline{E}_n(f|\mathcal{B})) = \underline{Q}(\underline{Q}(f|\mathcal{B}))$. Applying Corollary 8, we deduce that $\underline{Q}'$ is conglomerable.

(b) ⇒ (c) If $\underline{Q} = \underline{Q}' = \underline{F}$, then trivially $\underline{Q}'$ is conglomerable.

(c) ⇒ (d) If $\underline{Q}'$ is conglomerable, then it follows from Corollary 8 that $\underline{Q}' = \underline{Q}(\underline{Q}(\cdot|\mathcal{B}))$. Since by Proposition 7(c) $\underline{Q} \geq \underline{Q}'$, we conclude that $\underline{Q} \geq \underline{Q}(\underline{Q}(\cdot|\mathcal{B}))$ and, applying Proposition 6, $\underline{Q}$ is conglomerable.

(d) ⇔ (e) Since $\underline{Q} \leq \underline{F}$, it follows that $\underline{Q}$ is conglomerable if and only if it agrees with the conglomerable natural extension, that is, if and only if $\underline{Q} = \underline{F}$.

Assume next that $\underline{Q}' \geq \underline{P}$.

1. By Corollary 8, (c) ⇔ $\underline{Q}' = \underline{Q}(\underline{Q}(\cdot|\mathcal{B}))$, so it suffices to show that (c) implies (b). Now, if $\underline{Q}' \geq \underline{P}$, then $\underline{P} \leq \underline{Q}' \leq \underline{Q} \leq \underline{F}$. Hence, if $\underline{Q}'$ is conglomerable, then we have $\underline{Q}' = \underline{Q} = \underline{F}$.

2. The equivalence between (d) and (e) has already been established in the first part of the proposition. To see they are moreover equivalent to $\underline{Q} = \underline{Q}(\underline{Q}(\cdot|\mathcal{B}))$, note that if $\underline{Q}$ is conglomerable, then $\underline{Q} \geq \underline{Q}(\underline{Q}(\cdot|\mathcal{B})) \geq \underline{Q}' \geq \underline{P}$. But since $\underline{Q}(\underline{Q}(\cdot|\mathcal{B}))$ is a conglomerable model that dominates $\underline{P}$, we must have $\underline{Q} \leq \underline{F} \leq \underline{Q}(\underline{Q}(\cdot|\mathcal{B}))$, whence $\underline{Q} = \underline{Q}(\underline{Q}(\cdot|\mathcal{B}))$. For the converse implication, note that if $\underline{Q} = \underline{Q}(\underline{Q}(\cdot|\mathcal{B}))$ then it is trivially a conglomerable model, because all marginal extensions are.

3. The last statement is a consequence of the previous two and the first part of the proposition. □

The importance of Proposition 9 lies especially in the link that it establishes between the condition of uniform convergence of the sequence of conditionals $(\underline{E}_n(\cdot|\mathcal{B}))_{n\in\mathbb{N}}$ to $\underline{Q}(\cdot|\mathcal{B})$ and the fact that the sequence $(\underline{E}_n)_{n\in\mathbb{N}}$ attains the conglomerable natural extension in the limit. We shall exploit this connection in Section 6 to give relatively simple sufficient conditions for the sequence $(\underline{E}_n)_{n\in\mathbb{N}}$ to get to the conglomerable natural extension when $\underline{P}$ is a finitary model.

Remark 2. For any $B \in \mathcal{B}$ such that $\underline{Q}(B) = 0$, it holds that $\underline{E}_n(B) = 0$ for every n and then $\underline{Q}(\cdot|B), \underline{E}_n(\cdot|B)$ are vacuous for every n. This means that condition (a) in Proposition 9 can be simplified as follows: for any gamble f, $(\underline{E}_n(f|\mathcal{B}))_{n\in\mathbb{N}}$ converges uniformly to $\underline{Q}(f|\mathcal{B})$ if and only if $(\underline{E}_n(Cf|\mathcal{B}))_{n\in\mathbb{N}}$ converges uniformly to $\underline{Q}(Cf|\mathcal{B})$, where $C := \cup\{B : \underline{Q}(B) > 0\}$.

In particular, $\overline{P}(B) = 0 \Rightarrow \overline{Q}(B) = 0 \Rightarrow \underline{Q}(B) = 0$, so this holds for the conditioning events that have zero upper probability with respect to the initial model. ⧫

6. Sufficient conditions for getting to the conglomerable natural extension in the limit

As we show in Example 4 in Appendix A, the sequence $(\underline{E}_n)_{n\in\mathbb{N}}$ of coherent lower previsions that provides a lower bound for the conglomerable natural extension may not stabilise in a finite number of steps. On the other hand, in Proposition 9 we have shown that a sufficient condition for $\underline{E}_n$ to converge towards the conglomerable natural extension is the uniform convergence of the sequence of conditional lower previsions given by Eq. (8). In this section, we give two sufficient conditions for this uniform convergence. Taking into account Remark 2, we are going to assume without loss of generality that $\overline{P}(B) > 0$ for every $B \in \mathcal{B}$.

We focus on the case of an initial lower prevision $\underline{P}$ characterised by an associated credal set $\mathcal{M}(\underline{P})$ that contains *finitely many* extreme points. We call this a *finitary* model, or a finitary lower prevision. In other words, we consider finitely many linear previsions $P_1, \dots, P_k$ on $\mathcal{L}$ and let $\underline{P} := \min\{P_1, \dots, P_k\}$. We have that

$$\mathcal{M}(\underline{P}) = \{\alpha_1 P_1 + \cdots + \alpha_k P_k : (\alpha_1, \dots, \alpha_k) \in \Delta\} = \{P_{\bar{\alpha}} : \bar{\alpha} \in \Delta\},$$

where $\Delta := \{(\alpha_1, \dots, \alpha_k) : (\forall i)\alpha_i \geq 0, \sum_{i=1}^k \alpha_i = 1\}$ is the $(k-1)$-dimensional simplex, and where we simplify the notation by letting $P_{\bar{\alpha}} := \alpha_1 P_1 + \cdots + \alpha_k P_k$, with $\bar{\alpha} := (\alpha_1, \dots, \alpha_k)$. We consider as usual a partition $\mathcal{B}$ of Ω and the sequence $(\underline{E}_n)_{n\in\mathbb{N}}$ of coherent lower previsions that we use to approximate the conglomerable natural extension

$\underline{F}$ of $\underline{P}$ (provided that it exists). Let $\underline{Q}$ in particular be the point-wise limit of the coherent lower previsions in the sequence $(\underline{E}_n)_{n\in\mathbb{N}}$. We aim at giving conditions on $\underline{P}$ to make sure that $\underline{Q}$ is the conglomerable natural extension of $\underline{P}$.

If there is $m \in \mathbb{N}$ such that $\underline{E}_m = \underline{E}_{m-1}$, then $\lim_{n\to\infty} \underline{E}_n = \underline{E}_m = \underline{F}$ and in particular $Q = \underline{F}$. Otherwise, if the sequence never stabilises, then $\underline{E}_n \lneq \underline{E}_{n+1}$ for all n, whence $\mathcal{M}(\underline{E}_n) \supsetneq \mathcal{M}(\underline{E}_{n+1})$. Moreover, given that $\underline{E}_n$ dominates $\underline{P}$ for all $n \in \mathbb{N}$, we can represent its corresponding credal set as

$$\mathcal{M}(\underline{E}_n) = \{P_{\bar{\alpha}} : \bar{\alpha} \in \Delta_n\},$$

for some $\Delta_n \subseteq \Delta$. Let us show that Δ_n is in addition closed and convex as a consequence of the fact that $\mathcal{M}(\underline{E}_n)$ is a closed and convex set of linear previsions: this follows taking into account that

$$P_{\lambda\bar{\alpha}_1+(1-\lambda)\bar{\alpha}_2} = \lambda P_{\bar{\alpha}_1} + (1-\lambda)P_{\bar{\alpha}_2}$$

for every $\lambda \in [0, 1]$ and $\bar{\alpha}_1, \bar{\alpha}_2 \in \Delta_n$, and moreover

$$(\bar{\alpha}_m)_m \to \bar{\alpha} \Rightarrow P_{\bar{\alpha}_m} \to P_{\bar{\alpha}}$$

for every $(\alpha_m)_m, \alpha \in \Delta_n$, where the convergence is in the metric associated with the Euclidean distance.

Hence $(\Delta_n)_{n\in\mathbb{N}}$ is a strictly decreasing sequence of closed and convex subsets of Δ; since Δ is a compact subset of $\mathbb{R}^k$, we deduce that $\lim_{n\to\infty} \Delta_n =: \Delta'$ is a compact subset of Δ. By construction, Δ' must determine a coherent lower prevision that dominates all $\underline{E}_n$ for all $n \in \mathbb{N}$, and in particular their limit $\underline{Q} = \lim_{n\to\infty} \underline{E}_n$; conversely, $\underline{Q}$ is the smallest coherent lower prevision that dominates $\underline{E}_n$ for all $n \in \mathbb{N}$, so it must be associated to the intersection of the credal sets. This means that Δ' determines the lower prevision $\underline{Q}$.

We are going to use these sets to give a sufficient condition for the uniform convergence of the sequence of conditional natural extensions. For this, it will be important the characterisation of compactness by means of the *finite intersection property*: a decreasing sequence of compact sets has empty intersection if and only if there is a finite n such that the intersection of the first n elements of the sequence is empty. In particular, we have the following:

Lemma 10. *Given a compact set X and a decreasing sequence $(B_n)_{n\in\mathbb{N}}$ of non-empty closed sets of X, it holds that $\cap_{n\in\mathbb{N}} B_n \neq \emptyset$.*

Lemma 11. *Let $(\Delta_n)_{n\in\mathbb{N}}$ be a decreasing sequence of closed convex subsets of the $(k-1)$-dimensional simplex Δ, and let $\Delta' := \cap_{n\in\mathbb{N}} \Delta_n$. Let $\mathrm{d}(\bar{\gamma}, \bar{\gamma}')$ denote the Euclidean distance between vectors $\bar{\gamma}$ and $\bar{\gamma}'$, and let $\mathrm{d}(\bar{\gamma}, \Delta') := \inf\{\mathrm{d}(\bar{\gamma}, \bar{\gamma}') : \bar{\gamma}' \in \Delta'\}$. Then given $d_n := \sup\{\mathrm{d}(\bar{\gamma}, \Delta') : \bar{\gamma} \in \Delta_n\}$, $\lim_{n\to\infty} d_n = 0$.*

Proof. Since $(\Delta_n)_{n\in\mathbb{N}}$ is a decreasing sequence, $(d_n)_{n\in\mathbb{N}}$ is a decreasing sequence of non-negative numbers, which therefore has a non-negative limit. Assume ex-absurdo that $\lim_{n\to\infty} d_n = \varepsilon > 0$. Consider the set $B := \{\bar{\gamma} \in \Delta : \mathrm{d}(\bar{\gamma}, \Delta') \geq \varepsilon\}$. Then its complementary $B^c := \{\bar{\gamma} \in \Delta : \mathrm{d}(\bar{\gamma}, \Delta') < \varepsilon\}$ is an open set, using that $\mathrm{d}(\bar{\gamma}', \Delta') \leq \mathrm{d}(\bar{\gamma}', \bar{\gamma}) + \mathrm{d}(\bar{\gamma}, \Delta')$, and thus B is closed. As a consequence, $(\Delta_n \cap B)_{n\in\mathbb{N}}$ is a decreasing sequence of non-empty compact subsets of the simplex Δ. Applying Lemma 10, we deduce that $\emptyset \neq \cap_{n\in\mathbb{N}}(\Delta_n \cap B) = \Delta' \cap B = \emptyset$, a contradiction. □

One important issue when studying the uniform convergence of $(\underline{E}_n(f|\mathcal{B}))_{n\in\mathbb{N}}$ towards $\underline{Q}(f|\mathcal{B})$ is that of the positivity of the lower probabilities of the conditioning events: as we have shown in (4), $\underline{Q}(f|B)$ can only be non-vacuous when $\underline{Q}(B) > 0$, and similarly for $\underline{E}_n$. Then it may be that $\underline{Q}(B) > 0$ for all B in $\mathcal{B}$ while for every n there is an infinity of B for which $\underline{E}_n(B) = 0$, preventing the uniform convergence. Our next result shows that for finitary models this situation cannot arise:

Lemma 12. *Let $(\Delta_n)_{n\in\mathbb{N}}$ be a decreasing sequence of non-empty closed subsets of the simplex, and let $\Delta' := \cap_{n\in\mathbb{N}} \Delta_n$. Consider $\underline{Q} := \min\{P_{\bar{\alpha}} : \bar{\alpha} \in \Delta'\}$ and let $\mathcal{B}$ be a partition of Ω. Then there is some natural number n such that, for all $B \in \mathcal{B}$,*

$$\underline{Q}(B) > 0 \Rightarrow \underline{E}_n(B) > 0, \tag{11}$$

where $\underline{E}_n := \min\{P_{\bar{\alpha}} : \bar{\alpha} \in \Delta_n\}$.

Proof. Consider $B \in \mathcal{B}$. If $P_i(B) > 0$ for all $i = 1, \dots, k$, then $\underline{E}_n(B) \geq \min_{i \in \{1,\dots,k\}} P_i(B) > 0$ for all $n \in \mathbb{N}$. On the other hand, if $C_B := \{i \in \{1, \dots, k\} : P_i(B) = 0\}\} \neq \emptyset$, then $\underline{E}_n(B) = 0$ if and only if there is some $\bar{\alpha} \in \Delta_n$ such that $\sum_{i \in C_B} \alpha_i = 1$.

Now, $\underline{Q}(B) > 0$ implies that there is no $\bar{\alpha} \in \Delta'$ such that $\sum_{i \in C_B} \alpha_i = 1$. Since $\Delta' = \cap_{n \in \mathbb{N}} \Delta_n$, this means that if we let $D_B := \{\bar{\alpha} \in \Delta : \sum_{i \in C_B} \alpha_i = 1\}$, then there is some $n_B \in \mathbb{N}$ such that $\Delta_n \cap D_B = \emptyset$ for all $n \geq n_B$; otherwise we would contradict Lemma 10.

Since the sets C_B, for $B \in \mathcal{B}$, are subsets of $\{1, \dots, k\}$, there are at most a finite number of them. Each of them has an associated n_B such that $\Delta_n \cap D_B = \emptyset$ for all $n \geq n_B$, and if we let $n^\star := \max\{n_B : \underline{Q}(B) > 0\}$, we obtain that

$$\underline{E}_n(B) > 0 \text{ for every } n \geq n^\star \text{ and every } B \in \mathcal{B} \text{ with } \underline{Q}(B) > 0. \qquad \square$$

Recall that the conglomerable natural extension of $\underline{P}$ coincides with that of $\underline{E}_n$ for every $n \in \mathbb{N}$. Taking this into account, Lemma 12 allows us to assume that $\underline{P}(B) > 0$ whenever $\underline{Q}(B) > 0$; otherwise, it suffices to start the sequence at the n for which Eq. (11) holds.

Next, we shall establish in Propositions 15 and 18 two sufficient conditions for the uniform convergence of the sequence of conditional lower previsions. The first of these two conditions depends on the ratios between the upper and the lower probabilities of the conditioning events, and is a consequence of the following two lemmas:

Lemma 13. *Consider two vectors $\bar{\alpha}, \bar{\beta} \in \Delta$ such that $\|\bar{\alpha} - \bar{\beta}\| < \delta$ for some $\delta > 0$, where $\|\cdot\|$ denotes the Euclidean distance, and $B \in \mathcal{B}$ such that $\underline{P}(B) > 0$. Then*

$$\frac{P_{\bar{\alpha}}(B)}{P_{\bar{\beta}}(B)} \in \left[1 - k\delta \frac{\overline{P}(B)}{\underline{P}(B)}, 1 + k\delta \frac{\overline{P}(B)}{\underline{P}(B)}\right].$$

Proof. Since $|\alpha_i - \beta_i| \leq \|\bar{\alpha} - \bar{\beta}\| < \delta$, we deduce that $\alpha_i P_i(B) \leq (\beta_i + \delta) P_i(B) \leq \beta_i P_i(B) + \delta \overline{P}(B)$ for every $i = 1, \dots, k$, whence

$$\begin{aligned}\frac{P_{\bar{\alpha}}(B)}{P_{\bar{\beta}}(B)} &= \frac{\alpha_1 P_1(B) + \dots \alpha_k P_k(B)}{\beta_1 P_1(B) + \dots \beta_k P_k(B)} \leq \frac{\beta_1 P_1(B) + \dots \beta_k P_k(B) + k\delta \overline{P}(B)}{\beta_1 P_1(B) + \dots \beta_k P_k(B)} \\ &= 1 + k\delta \frac{\overline{P}(B)}{P_{\bar{\beta}}(B)} \leq 1 + k\delta \frac{\overline{P}(B)}{\underline{P}(B)}.\end{aligned}$$

Similarly, since $\alpha_i P_i(B) \geq (\beta_i - \delta) P_i(B) \geq \beta_i P_i(B) - \delta \overline{P}(B)$,

$$\frac{P_{\bar{\alpha}}(B)}{P_{\bar{\beta}}(B)} \geq \frac{\beta_1 P_1(B) + \dots \beta_k P_k(B) - k\delta \overline{P}(B)}{\beta_1 P_1(B) + \dots \beta_k P_k(B)} = 1 - k\delta \frac{\overline{P}(B)}{P_{\bar{\beta}}(B)} \geq 1 - k\delta \frac{\overline{P}(B)}{\underline{P}(B)}. \qquad \square$$

As a consequence of Lemma 13, we have the following:

Lemma 14. *Consider two vectors $\bar{\alpha}, \bar{\beta} \in \Delta$ such that $\|\bar{\alpha} - \bar{\beta}\| < \delta$ for some $\delta > 0$, where $\|\cdot\|$ denotes the Euclidean distance, and take $B \in \mathcal{B}$ such that $\underline{P}(B) > 0$. Given $f \in \mathcal{L}$ and the linear previsions $P_{\bar{\alpha}}, P_{\bar{\beta}}$, it holds that*

$$|P_{\bar{\alpha}}(f|B) - P_{\bar{\beta}}(f|B)| \leq k\delta \frac{\overline{P}(B)}{\underline{P}(B)} \left[k \frac{\overline{P}(B)}{\underline{P}(B)} + 1\right] \sup_B |f|.$$

Proof. Since $P_{\bar{\alpha}}(B), P_{\bar{\beta}}(B) > 0$, it follows from the definition of conditional linear previsions that

$$|P_{\bar{\alpha}}(f|B) - P_{\bar{\beta}}(f|B)| = \left| \frac{\sum_{i=1}^k P_i(Bf)[\alpha_i P_{\bar{\beta}}(B) - \beta_i P_{\bar{\alpha}}(B)]}{P_{\bar{\alpha}}(B) P_{\bar{\beta}}(B)} \right| \leq \sum_{i=1}^k \left| \frac{P_i(Bf)}{P_{\bar{\alpha}}(B)} \right| \left| \alpha_i - \beta_i \frac{P_{\bar{\alpha}}(B)}{P_{\bar{\beta}}(B)} \right|.$$

Now,

$$\frac{P_i(Bf)}{P_{\bar{\alpha}}(B)} = \frac{P_i(f|B) P_i(B)}{P_{\bar{\alpha}}(B)} \leq \frac{\sup_B |f| \, \overline{P}(B)}{\underline{P}(B)}.$$

It also follows from Lemma 13 that

$$\left|\frac{P_{\bar{\alpha}}(B)}{P_{\bar{\beta}}(B)}-1\right| \leq k\delta\frac{\overline{P}(B)}{\underline{P}(B)},$$

whence

$$\left|\beta_i\frac{P_{\bar{\alpha}}(B)}{P_{\bar{\beta}}(B)}-\beta_i\right| \leq \beta_i k\delta\frac{\overline{P}(B)}{\underline{P}(B)}$$

and since $|\alpha_i-\beta_i| \leq \|\bar{\alpha}-\bar{\beta}\| < \delta$, we deduce that

$$\left|\beta_i\frac{P_{\bar{\alpha}}(B)}{P_{\bar{\beta}}(B)}-\alpha_i\right| \leq \beta_i k\delta\frac{\overline{P}(B)}{\underline{P}(B)}+\delta \leq k\delta\frac{\overline{P}(B)}{\underline{P}(B)}+\delta = \delta\left[k\frac{\overline{P}(B)}{\underline{P}(B)}+1\right],$$

taking into account that $\beta_i \leq 1$. Hence,

$$|P_{\bar{\alpha}}(f|B)-P_{\bar{\beta}}(f|B)| \leq k\delta\frac{\overline{P}(B)}{\underline{P}(B)}\left[k\frac{\overline{P}(B)}{\underline{P}(B)}+1\right]\sup_B|f|. \qquad \square$$

Proposition 15. *Consider any gamble $f \in \mathcal{L}$. Then $\underline{Q}(f|\mathcal{B})$ is the uniform limit of $(\underline{E}_n(f|\mathcal{B}))_{n\in\mathbb{N}}$ provided that there is some $N>0$ such that $\frac{\overline{P}(B)}{\underline{P}(B)} < N$ for all $B \in \mathcal{B}$ such that $\underline{P}(B)>0$.*

Proof. Consider $B \in \mathcal{B}$. If $\underline{Q}(B)=0$, then $\underline{E}_n(B)=0$ for all n, and $\underline{Q}(f|B)=\inf_B f=\underline{E}_n(f|B)$ for all $n\in\mathbb{N}$.

On the other hand, from Lemma 12 there is some $n_1\in\mathbb{N}$ such that Eq. (11) holds for every $n\geq n_1$. Fix $B\in\mathcal{B}$ such that $\underline{Q}(B)>0$ (and also $\underline{E}_n(B)>0$). Then for every $n\geq n_1$, it follows from [17, Theorem 6.4.2] that the conditional natural extensions can be computed by

$$\begin{aligned}\underline{Q}(f|B) &= \min\{P_{\bar{\alpha}}(f|B):\bar{\alpha}\in\Delta'\},\\ \underline{E}_n(f|B) &= \min\{P_{\bar{\alpha}}(f|B):\bar{\alpha}\in\Delta_n\}.\end{aligned}$$

Fix $\varepsilon>0$. From Lemma 14, there is $\delta>0$ such that if $\|\bar{\alpha}-\bar{\beta}\|<\delta$, then $|P_{\bar{\alpha}}(f|B)-P_{\bar{\beta}}(f|B)|<\varepsilon$: it suffices to consider

$$\delta < \frac{\varepsilon}{k\sup_B|f|N(kN+1)},$$

where N is the uniform bound on $\frac{\overline{P}(B)}{\underline{P}(B)}$ that exists by hypothesis.

On the other hand, Lemma 11 implies that there is some $n_2\in\mathbb{N}$ such that $\mathrm{d}(\bar{\gamma},\Delta')\leq\delta$ for all $\bar{\gamma}\in\Delta_n$, $n\geq n_2$. This means that for every $\bar{\alpha}\in\Delta_n$ there exists $\bar{\beta}\in\Delta'$ such that $\|\bar{\alpha}-\bar{\beta}\|\leq\delta$. Consider then $n^\star:=\max\{n_1,n_2\}$. For every B such that $\underline{Q}(B)>0$, there is some $\bar{\alpha}_B\in\Delta_{n^\star}$ such that $\underline{E}_{n^\star}(f|B)=P_{\bar{\alpha}_B}(f|B)$. Given $\bar{\beta}_B\in\Delta'$ such that $\|\bar{\alpha}_B-\bar{\beta}_B\|\leq\delta$, it follows from the property established above that $|P_{\bar{\alpha}_B}(f|B)-P_{\bar{\beta}_B}(f|B)|\leq\varepsilon$. Therefore,

$$\begin{aligned}|\underline{Q}(f|B)-\underline{E}_{n^\star}(f|B)| &= \underline{Q}(f|B)-\underline{E}_{n^\star}(f|B) \leq P_{\bar{\beta}_B}(f|B)-\underline{E}_{n^\star}(f|B)\\ &= |P_{\bar{\beta}_B}(f|B)-\underline{E}_{n^\star}(f|B)| = |P_{\bar{\beta}_B}(f|B)-P_{\bar{\alpha}_B}(f|B)|\leq\varepsilon.\end{aligned}$$

Since this holds irrespectively of B, we conclude that $\|\underline{Q}(f|\mathcal{B})-\underline{E}_{n^\star}(f|\mathcal{B})\|\leq\varepsilon$, and this implies that $(\underline{E}_n(f|\mathcal{B}))_{n\in\mathbb{N}}$ converges uniformly to $\underline{Q}(f|\mathcal{B})$. $\square$

As we shall see later, this sufficient condition is not necessary. Next, we give another condition, which depends on the features of the subset of Δ we have in the limit:

Lemma 16. *Consider two vectors $\bar{\alpha},\bar{\beta}\in\Delta$ such that $\|\bar{\alpha}-\bar{\beta}\|<\delta$ and $\min_{i=1}^k\alpha_i>\delta>0$. Given $B\subseteq\Omega$ such that $P_{\bar{\alpha}}(B)>0, P_{\bar{\beta}}(B)>0$, then for all $f\in\mathcal{L}$ it holds that*

$$\|P_{\bar{\alpha}}(f|B)-P_{\bar{\beta}}(f|B)\| \leq \delta k(k-1)\frac{1}{(\min_{i=1}^k\alpha_i-\delta)^2}\sup_B|f|.$$

Proof.

$$\begin{aligned}
|P_{\bar{\alpha}}(f|B) - P_{\bar{\beta}}(f|B)| &= \left| \frac{P_{\bar{\alpha}}(Bf)P_{\bar{\beta}}(B) - P_{\bar{\beta}}(Bf)P_{\bar{\alpha}}(B)}{P_{\bar{\alpha}}(B)P_{\bar{\beta}}(B)} \right| \\
&= \left| \frac{\sum_{i \neq j}(\alpha_i\beta_j - \alpha_j\beta_i)(P_i(Bf)P_j(B))}{P_{\bar{\alpha}}(B)P_{\bar{\beta}}(B)} \right| \\
&\leq \delta \left| \frac{\sum_{i \neq j} P_i(Bf)P_j(B)}{P_{\bar{\alpha}}(B)P_{\bar{\beta}}(B)} \right| \\
&\leq \delta \sup_B |f| \sum_{i \neq j} \frac{P_i(B)P_j(B)}{P_{\bar{\alpha}}(B)P_{\bar{\beta}}(B)}.
\end{aligned}$$

To prove the first inequality, note that

$$|\alpha_i\beta_j - \alpha_j\beta_i| = |\alpha_i(\beta_j - \alpha_j) - \alpha_j(\beta_i - \alpha_i)| \leq \alpha_i|\beta_j - \alpha_j| + \alpha_j|\alpha_i - \beta_i| \leq \max_{\ell=1}^{k}\{|\beta_\ell - \alpha_\ell|\} \leq \delta,$$

because $\alpha_i, \alpha_j \in [0, 1]$, and moreover $\|\bar{\alpha} - \bar{\beta}\| \geq |\alpha_i - \beta_i|$ for $i = 1, \ldots, k$.

Now, given $i \neq j$ in $\{1, \ldots, k\}$ such that $P_i(B) \neq 0 \neq P_j(B)$,

$$\frac{P_i(B)P_j(B)}{P_{\bar{\alpha}}(B)P_{\bar{\beta}}(B)} \leq \frac{P_i(B)P_j(B)}{\alpha_i\beta_j P_i(B)P_j(B)} \leq \frac{1}{\alpha_i(\alpha_j - \delta)} \leq \frac{1}{(\min_{i=1}^k \alpha_i)(\min_{i=1}^k \alpha_i - \delta)} \leq \frac{1}{(\min_{i=1}^k \alpha_i - \delta)^2},$$

and therefore

$$|P_{\bar{\alpha}}(f|B) - P_{\bar{\beta}}(f|B)| \leq \delta k(k-1) \frac{1}{(\min_{i=1}^k \alpha_i - \delta)^2} \sup_B |f|. \qquad \square$$

From this we deduce the following:

Corollary 17. *Consider $(\bar{\alpha}_n)_{n\in\mathbb{N}}, \bar{\alpha} \in \Delta$ such that $\min_{i=1}^k \alpha_i > \delta > 0$ and $(\bar{\alpha}_n)_{n\in\mathbb{N}}$ converges to $\bar{\alpha}$. Then for every gamble f the sequence $(P_{\bar{\alpha}_n}(f|\mathcal{B}))_{n\in\mathbb{N}}$ converges uniformly to $P_{\bar{\alpha}}(f|\mathcal{B})$.*

Proof. Consider $B \in \mathcal{B}$. Since by Remark 2 we can assume without loss of generality that $\overline{P}(B) > 0$ and we have that $\min_{i=1}^k \alpha_i > \delta > 0$, it follows that $P_{\bar{\alpha}}(B) > 0$: the inequality $\overline{P}(B) > 0$ means that $P_i(B) > 0$ for some i and since $\alpha_i > 0$ by assumption, then $P_{\bar{\alpha}}(B) > 0$.

Moreover, reasoning as in Lemma 12, if $(\bar{\alpha}_n)_{n\in\mathbb{N}} \to \bar{\alpha}$ then there is some natural number n_1 such that for all $n \geq n_1$,

$$P_{\bar{\alpha}}(B) > 0 \Rightarrow P_{\bar{\alpha}_n}(B) > 0.$$

This means that we can assume without loss of generality that $P_{\bar{\alpha}}(B) > 0, P_{\bar{\alpha}_n}(B) > 0$ for all B. Now, if we apply Lemma 16, we deduce that

$$|P_{\bar{\alpha}}(f|B) - P_{\bar{\alpha}_n}(f|B)| \leq \|\bar{\alpha} - \bar{\alpha}_n\| k(k-1) \frac{1}{(\min_i \alpha_i - \delta)^2} \sup_B |f|.$$

If we now consider n_2 such that

$$\|\bar{\alpha} - \bar{\alpha}_n\| \leq \frac{\varepsilon(\min_i \alpha_i - \delta)^2}{k(k-1)\sup|f|}$$

for all $n \geq n_2$, we deduce that for every $n \geq n_2$ and every $B \in \mathcal{B}$ it holds that $|P_{\bar{\alpha}}(f|B) - P_{\bar{\alpha}_n}(f|B)| \leq \varepsilon$. As a consequence, we have uniform convergence. $\square$

Proposition 18. *If there is some $\nu > 0$ such that $\min_{i=1}^k \alpha_i \geq \nu$ for all $\bar{\alpha} \in \Delta'$, then $(\underline{E}_n(f|\mathcal{B}))_{n\in\mathbb{N}}$ converges uniformly towards $\underline{Q}(f|\mathcal{B})$ for every $f \in \mathcal{L}$.*

Proof. Consider $B \in \mathcal{B}$. If $\underline{Q}(B) = 0$, then $\underline{E}_n(B) = 0$ for all n, and $\underline{Q}(f|B) = \inf_B f = \underline{E}_n(f|B)$ for all $n \in \mathbb{N}$.

On the other hand, from Lemma 12 there is some $n_1 \in \mathbb{N}$ such that Eq. (11) holds for all $n \geq n_1$. Fix $B \in \mathcal{B}$ such that $\underline{Q}(B) > 0$. Then for every $n \geq n_1$,

$$\begin{aligned} \underline{Q}(f|B) &= \inf\{P_{\bar{\alpha}}(f|B) : \bar{\alpha} \in \Delta'\}, \\ \underline{E}_n(f|B) &= \inf\{P_{\bar{\alpha}}(f|B) : \bar{\alpha} \in \Delta_n\}. \end{aligned}$$

Fix $\varepsilon > 0$, and consider $0 < \delta \leq \min\{\frac{\nu}{2}, \frac{\varepsilon \frac{\nu^2}{4}}{k(k-1)\sup|f|}\}$. It follows from Lemma 16 that if $\|\bar{\alpha} - \bar{\beta}\| < \delta$ and $P_\alpha(B) > 0, P_\beta(B) > 0$, then

$$|P_{\bar{\alpha}}(f|B) - P_{\bar{\beta}}(f|B)| \leq \delta k(k-1) \frac{1}{(\min_{i=1}^k \alpha_i - \delta)^2} \sup|f| \leq \delta k(k-1) \frac{1}{(\frac{\nu}{2})^2} \sup|f| \leq \varepsilon; \tag{12}$$

to prove the second inequality, note that since $\min_{i=1}^k \alpha_i \geq \nu$ and we have selected $\delta \leq \frac{\nu}{2}$, it follows that $\min_{i=1}^k \alpha_i - \delta \geq \frac{\nu}{2}$; the third inequality holds because $\delta \leq \frac{\varepsilon \frac{\nu^2}{4}}{k(k-1)\sup|f|}$.

On the other hand, Lemma 11 implies that there is some $n_2 \in \mathbb{N}$ such that $\mathrm{d}(\bar{\gamma}, \Delta') \leq \delta$ for all $\bar{\gamma} \in \Delta_n$, $n \geq n_2$. This means that for every $\bar{\alpha} \in \Delta_n$, there exists $\bar{\beta} \in \Delta'$ such that $\|\bar{\alpha} - \bar{\beta}\| \leq \delta$.

Consider then $n^\star := \max\{n_1, n_2\}$. For every B such that $\underline{E}_n(B) > 0$, there is some $\bar{\alpha} \in \Delta_{n^\star}$ such that $\underline{E}_{n^\star}(f|B) = P_{\bar{\alpha}}(f|B)$. Given $\bar{\beta} \in \Delta'$ such that $\|\bar{\alpha} - \bar{\beta}\| \leq \delta$, it follows from Eq. (12) that $|P_{\bar{\alpha}}(f|B) - P_{\bar{\beta}}(f|B)| \leq \varepsilon$. Therefore,

$$|\underline{Q}(f|B) - \underline{E}_{n^\star}(f|B)| \leq |P_{\bar{\beta}}(f|B) - \underline{E}_{n^\star}(f|B)| = |P_{\bar{\beta}}(f|B) - P_{\bar{\alpha}}(f|B)| \leq \varepsilon.$$

Since this holds irrespectively of B, we conclude that $\|\underline{Q}(f|\mathcal{B}) - \underline{E}_{n^\star}(f|\mathcal{B})\| \leq \varepsilon$, and this implies that $(\underline{E}_n(f|\mathcal{B}))_{n\in\mathbb{N}}$ converges uniformly to $\underline{Q}(f|\mathcal{B})$. □

This result is particularly revealing in the binary case, that is, where we consider the lower envelope of two linear previsions, $\underline{P} := \min\{P_1, P_2\}$. If we let $P_\alpha := \alpha P_1 + (1-\alpha)P_2$, then we can identify each Δ_n with a subset of $[0, 1]$:

$$\begin{aligned} \mathcal{M}(\underline{P}) &:= \{P_\alpha : \alpha \in [0,1]\}, \\ \mathcal{M}(\underline{E}_n) &:= \{P_\alpha : \alpha \in [a_n, b_n]\}, \\ \mathcal{M}(\underline{Q}) &:= \{P_\alpha : \alpha \in [a, b]\}, \end{aligned}$$

where $0 \leq a_n \leq b_n \leq 1$ for all n, and $(a_n)_{n\in\mathbb{N}} \uparrow a$, $(b_n)_{n\in\mathbb{N}} \downarrow b$. We obtain that:

Corollary 19. *If $\underline{P}$ is the lower envelope of two linear previsions and the conglomerable natural extension of $\underline{P}$ exists, then it coincides with $\underline{Q}$.*

Proof. If $\underline{Q}(B) = 0$, then we have that $\underline{Q}(f|B) = \underline{E}_n(f|B) = \inf_B f$ for every $n \in \mathbb{N}$. On the other hand, taking into account Lemma 12, we are going to assume that $\underline{P}(B) > 0$ whenever $\underline{Q}(B) > 0$ (otherwise we would start the sequence with the coherent lower prevision $\underline{E}_n$ given by that lemma). It then follows from Eq. (4) that, for every gamble f on Ω and every $B \in \mathcal{B}$ such that $\underline{P}(B) > 0$, it holds that

$$\underline{E}_n(f|B) = \min\{P_{a_n}(f|B), P_{b_n}(f|B)\} \text{ and } \underline{Q}(f|B) = \min\{P_a(f|B), P_b(f|B)\}.$$

Now note that we have the following:

- ▷ If $a = b = 1$, then $\underline{Q} = P_1$, so the conglomerable natural extension $\underline{F} \geq \underline{Q}$ exists if and only if it coincides with $\underline{Q} = P_1$.
- ▷ If $a = b = 0$, then $\underline{Q} = P_2$, so the conglomerable natural extension $\underline{F} \geq \underline{Q}$ exists if and only if it coincides with $\underline{Q} = P_2$.

On the other hand, when $a < b$ or $a = b \in (0, 1)$ we can apply Corollary 17 to deduce that $P_a(f|\mathcal{B})$ is the uniform limit of $(P_{a_n}(f|\mathcal{B}))_{n\in\mathbb{N}}$ (note that the result is trivial for $a = 0$, and we only need to invoke the corollary for $a > 0$); similarly, $P_b(f|\mathcal{B})$ is the uniform limit of $(P_{b_n}(f|\mathcal{B}))_{n\in\mathbb{N}}$. As a consequence, given $\varepsilon > 0$ there are natural numbers n_1, n_2 such that

$$(\forall B \in \mathcal{B})(\forall n \geq n_1)|P_{a_n}(f|B) - P_a(f|B)| < \varepsilon \text{ and } (\forall B \in \mathcal{B})(\forall n \geq n_2)|P_{b_n}(f|B) - P_b(f|B)| < \varepsilon.$$

By taking $n^\star := \max\{n_1, n_2\}$ we deduce that $|\underline{Q}(f|B) - \underline{E}_n(f|B)| < \varepsilon$ for every $B \in \mathcal{B}$ and every $n \geq n^\star$, and as a consequence, $\underline{Q}(f|\mathcal{B})$ is the uniform limit of $(\underline{E}_n(f|\mathcal{B}))_{n\in\mathbb{N}}$. ☐

We summarise the previous findings in the following theorem:

Theorem 20. *The limit $\underline{Q}$ of the sequence $(\underline{E}_n)_{n\in\mathbb{N}}$ is the conglomerable natural extension of $\underline{P}$ provided any of the following conditions holds:*

1. *There is some $N > 0$ such that $\frac{\overline{P}(B)}{\underline{P}(B)} < N$ for all $B \in \mathcal{B}$.*
2. *There is some $\nu > 0$ such that $\min_{i=1}^{k} \alpha_i \geq \nu > 0$ for all $\bar{\alpha} \in \Delta'$.*

Proof. From Propositions 15 and 18, under any of these conditions $\underline{Q}(f|\mathcal{B})$ is the uniform limit of $(\underline{E}_n(f|\mathcal{B}))_{n\in\mathbb{N}}$ for every gamble $f \in \mathcal{L}$. The result follows then from Proposition 9. ☐

However, neither of these sufficient conditions is necessary for the limit to be conglomerable, as Example 6 in Appendix A shows.

7. Conclusions

Conglomerability has been advocated by Walley as a rationality requirement in the theory of coherent lower previsions, when establishing the consistency of the assessments between the unconditional and the conditional models [17]. Even though controversial, the requirement of conglomerability has recently received some renewed support, in a special case, through considerations of dynamic coherence [20]. However, the notion of conglomerability is not fully incorporated within Walley's theory, as recently shown in [12, 13]: the two fundamental procedures of checking the coherence of a number of assessments, and of extending them to coherent ones in case they are not, which is called natural extension, take the requirement of conglomerability only partially into account. In other words, coherence and natural extension may be understood as ways to approximate the actual procedures fully based on conglomerability.

For the case of coherence, the situation is not problematic in the context considered in this paper: if we only deal with one conditional and one unconditional lower prevision, then Walley's coherence is all we need to deal properly with conglomerability. However, the problem remains also in the simplest of the cases when we want to extend some non-conglomerable assessments into the least-committal coherent and conglomerable model: in fact, Walley's procedure of natural extension has been shown to provide only an approximation even in that case.

In this paper, we have studied in which cases it is possible to make the aforementioned correction in the case of an unconditional model; it is called its conglomerable natural extension. The importance of this notion can perhaps be appreciated when one realises that it is the counterpart, for a theory of probability based on conglomerability, of the deductive closure in logic.

One of the main drawbacks of the conglomerable natural extension, as we can see from recent and current results, is the lack of a constructive definition: the most we can do is to approximate it as the limit of an increasing sequence of coherent lower previsions, each of them defined by means of Walley's notion of natural extension. Solving an open problem from [13], we have shown that this sequence may be infinite, meaning that the closure operator represented by the conglomerable natural extension is not finitary. This may be due to the fact that the very notion of conglomerability involves an infinite number of acceptable transactions, and we conjecture that using tools from infinitary logic may prove useful in this context. The infinitary character of conglomerability may also be at the heart of some of the examples mentioned in Section 2.5, showing that the class of conglomerably coherent lower previsions is not closed under convex combinations or point-wise limits.

Taking this into account, we have obtained a number of sufficient conditions for the approximating sequence to converge towards the conglomerable natural extension, when the initial model is the lower envelope of a finite number of linear previsions. This kind of models may be of interest in practice, for instance when aggregating the opinions of several experts. In particular, we have shown that if the unconditional model is the lower envelope of two linear previsions, our sequence always gets to the conglomerable natural extension, when it exists.

The main open problem still pending is whether the sequence of coherent lower previsions always gets to the conglomerable natural extension. Taking into account our results in this paper, a possible approach may be the study of the uniform convergence of the increasing sequence of conditional lower previsions.

Another problem of interest would be the study of a notion of conglomerable natural extension with respect to several partitions simultaneously. In this respect, we think that it will be necessary to deal with the notions of weak and strong coherence in [17], as well as with the results in [12] about the coherence of several conditional lower previsions and their relationship with conglomerability. We conjecture that it should be possible to extend our results to that context for some particular situations, such as the case where the partitions are nested (taking into account the results in [13]) or when the unconditional model is finitary (using our results in Section 6).

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Appendix A. Counterexamples

In this appendix, we have gathered a number of counterexamples related to conglomerability and to the existence and properties of the conglomerable natural extension. We shall denote by $\mathbb{N}$ the set of natural numbers without zero, and consider the possibility space $\Omega := \mathbb{N} \cup -\mathbb{N}$ and conglomerability with respect to the partition $\mathcal{B} := \{B_n : n \in \mathbb{N}\}$, where $B_n := \{n, -n\}$ for every n.

Example 1 (**$\underline{\mathbb{F}}$ is not closed under point-wise limits; $P(\cdot|\mathcal{B})$ linear $\not\Rightarrow$ P conglomerable**). Consider the linear prevision $Q_n := P(P_n(\cdot|\{\mathbb{N}, -\mathbb{N}\}))$, where $P(\mathbb{N}) := P(-\mathbb{N}) := \frac{1}{2}$, $P_n(\cdot| - \mathbb{N})$ is a finitely additive probability that gives probability zero to all the singletons (the same one for all $n \in \mathbb{N}$) and $P_n(\cdot|\mathbb{N})$ is a σ-additive probability characterised by the mass function $P_n(\{m\}|\mathbb{N}) := \frac{1}{2^m}$ for $m = 1, ..., n-1$, $P_n(\{n\}|\mathbb{N}) := \frac{1}{2^{n-1}}$, $P_n(\{m\}|\mathbb{N}) := 0$ for all $m > n$.

Then $Q_n(B_m) > 0$ only for $m \leq n$, so Q_n is conglomerable. But the point-wise limit Q' of Q_n as n goes to infinity becomes the non-conglomerable linear prevision in [17, Example 6.8.5], that is,

$$Q' := P(P'(\cdot|\{\mathbb{N}, -\mathbb{N}\})),$$

where $P'(\cdot| - \mathbb{N})$ is the same finitely additive probability that gives probability zero to all the singletons we had before and $P'(\cdot|\mathbb{N})$ is a σ-additive probability characterised by the mass function $P_n(\{m\}|\mathbb{N}) := \frac{1}{2^m}$ for all $m \in \mathbb{N}$. To prove this, it suffices to consider that

$$P'(f|\mathbb{N}) = \lim_{n\to\infty} P_n(f|\mathbb{N})$$

for any gamble f on Ω: indeed, $|P'(f|\mathbb{N}) - P_n(f|\mathbb{N})| \leq \frac{1}{2^{n-1}} \sup |f|$ for every n.

Since on the other hand we have that $P'(f| - \mathbb{N}) = P_n(f| - \mathbb{N})$ for every gamble f on Ω, we deduce that

$$\begin{aligned} |Q'(f) - Q_n(f)| &= |0.5P'(f|\mathbb{N}) + 0.5P'(f|\mathbb{N}) - 0.5P_n(f|\mathbb{N}) - 0.5P_n(f| - \mathbb{N})| \\ &= 0.5|P'(f|\mathbb{N}) - P_n(f|\mathbb{N})| \leq \frac{1}{2^n} \sup |f|, \end{aligned}$$

whence $\lim_n Q_n(f) = Q'(f)$ for any gamble f on Ω.

Finally, note that $Q'(B_n) = \frac{1}{2^{n-2}} > 0$ for every n. This means that the conditional natural extension $P'(\cdot|\mathcal{B})$ of Q' is uniquely determined by (GBR), and since Q' is a linear prevision this implies that this conditional natural extension is linear: $P'(f|B_n) = \frac{Q'(B_n f)}{Q'(B_n)}$ for any gamble f on Ω and every natural number n. This shows that the equivalence in Proposition 3(b) is not trivial. ♦

Example 2 (**No additional implication between the conditions in Proposition 2 holds in general**). Let us show that there is no additional implication:

(b) $\not\Rightarrow$ (a) Let $\underline{P}$ be non-conglomerable but such that the conglomerable natural extension exists, and let $\underline{P}(\cdot|\mathcal{B})$ be its conditional natural extension. Then $\underline{P}, \underline{P}(\cdot|\mathcal{B})$ are not coherent but they are dominated by coherent models.

(c) $\not\Rightarrow$ (e) Let P be a linear prevision that is conglomerable and $P(\cdot|\mathcal{B})$ a linear conditional prevision that is not coherent with P. It follows that $P, P(\cdot|\mathcal{B})$ do not avoid partial loss, because the two conditions are equivalent under linearity. Nevertheless, the conglomerable natural extension of P exists (it is P itself).

(c) $\not\Rightarrow$ (b) This follows from the previous point and Proposition 2.

(e) $\not\Rightarrow$ (c) Let P be a linear prevision that is not conglomerable, and let $\underline{P}(\cdot|\mathcal{B})$ be the vacuous coherent lower prevision. Since $\underline{P}(\cdot|\mathcal{B})$ is vacuous, it avoids partial loss with any unconditional coherent lower prevision (and in particular with P), so $P, \underline{P}(\cdot|\mathcal{B})$ avoid partial loss. However, the conglomerable natural extension of P does not exist because it should be P itself.

(e) $\not\Rightarrow$ (b) This follows from the previous point and Proposition 2. ♦

Example 3 ($\underline{P}, \underline{P}(\cdot|\mathcal{B})$ **APL** $\not\Rightarrow$ $\{\underline{Q} \in \underline{\mathbb{F}} : \underline{Q} \geq \underline{P}\} \neq \emptyset$). Let us start by defining a few linear previsions.

▷ Let P_1 be a linear prevision on $\mathcal{L}$ given by

$$P_1(f) := \sum_{n\in\mathbb{N}} [f(n) + f(-n)] \frac{1}{2^{n+1}}$$

for every $f \in \mathcal{L}$. Since its restriction to events is a σ-additive probability, P_1 is conglomerable.

▷ Let on the other hand P be a linear prevision on $\mathcal{L}(\mathbb{N})$ whose restriction to events satisfies $P(\{n\}) = 0$ for all n, $P(\{2n+1 : n \in \mathbb{N}\}) = 0$. Then we can use P to define a linear prevision P_2 on $\mathcal{L}$ by

$$P_2(f) := \frac{3}{4}P(f^+) + \frac{1}{4}P(f^-),$$

where f^+, f^- are given by

$$\begin{aligned} f^+ : \mathbb{N} &\to \mathbb{R} \\ n &\mapsto f(n) \end{aligned} \quad \text{and} \quad \begin{aligned} f^- : \mathbb{N} &\to \mathbb{R} \\ n &\mapsto f(-n). \end{aligned} \tag{A.1}$$

From this we derive the linear prevision

$$P_3 := \frac{1}{2}P_1 + \frac{1}{2}P_2.$$

▷ Let now P' be another linear prevision on $\mathcal{L}(\mathbb{N})$ whose restriction to events satisfies $P'(\{n\}) = 0$ for all n, and such that $P'(\{2n-1 : n \in \mathbb{N}\}) = \frac{1}{2} = P'(\{2n : n \in \mathbb{N}\})$, and define the linear prevision P_4 on $\mathcal{L}$ by

$$P_4(g) := \frac{1}{4}\sum_{n\in\mathbb{N}} g(n)\frac{1}{2^n} + \frac{3}{4}P'(g^-).$$

▷ Take $\underline{P} := \min\{P_3, P_4\}$. Given $n \in \mathbb{N}$,

$$\begin{aligned} \underline{P}(B_n) &= \min\{P_3(B_n), P_4(B_n)\} = \min\left\{\frac{1}{2}P_1(B_n) + \frac{1}{2}\left[\frac{3}{4}\underbrace{P(B_n \cap \mathbb{N})}_{=0} + \frac{1}{4}\underbrace{P(-(B_n \cap -\mathbb{N}))}_{=0}\right], \frac{1}{4}\frac{1}{2^n}\right\} \\ &= \min\left\{\frac{1}{2^{n+1}}, \frac{1}{2^{n+2}}\right\} > 0, \end{aligned}$$

whence the conditional natural extension of $\underline{P}$ is

$$\begin{aligned}\underline{P}(f|B_n) &= \min\{P_3(f|B_n), P_4(f|B_n)\} = \min\left\{\frac{P_3(B_nf)}{P_3(B_n)}, \frac{P_4(B_nf)}{P_4(B_n)}\right\} \\ &= \min\left\{\frac{f(n)\frac{1}{2^{n+2}} + f(-n)\frac{1}{2^{n+2}}}{\frac{1}{2^{n+1}}}, \frac{f(n)\frac{1}{2^{n+2}} + f(-n)0}{\frac{1}{2^{n+2}}}\right\} = \min\left\{\frac{f(n)+f(-n)}{2}, f(n)\right\}\end{aligned}$$

for every $f \in \mathcal{L}$ and every $n \in \mathbb{N}$.

Fix a gamble f and let $C := \cup_{n\in\mathbb{N}:f(n)<f(-n)} B_n$, so that $\underline{P}(f|B_n) = f(n)$ if $B_n \subseteq C$ and $\underline{P}(f|B_n) = \frac{f(n)+f(-n)}{2}$ otherwise. Then, using Remark 1, $G_{\underline{P}}(f|\mathcal{B}) = G_{\underline{P}}(Cf|\mathcal{B}) + G_{\underline{P}}(C^c f|\mathcal{B}) \geq G_{\underline{P}}(C^c f|\mathcal{B})$ because

$$\begin{cases} G_{\underline{P}}(f|\mathcal{B})(n) = 0 \\ G_{\underline{P}}(f|\mathcal{B})(-n) = f(-n) - f(n) > 0 \end{cases}$$

if $B_n \subseteq C$.

▷ Let $P_\alpha := \alpha P_3 + (1-\alpha)P_4$.

We are going to determine for which $\alpha \in [0,1]$ it holds that $P_\alpha(G_{\underline{P}}(f|\mathcal{B})) \geq 0$ for all f. Taking into account the previous observation, we can conclude that

$$(\forall f \in \mathcal{L}) P_\alpha(G_{\underline{P}}(f|\mathcal{B})) \geq 0 \Leftrightarrow (\forall f \in \mathcal{L} : (\forall n \in \mathbb{N}) f(n) \geq f(-n)) P_\alpha(G_{\underline{P}}(f|\mathcal{B})) \geq 0.$$

In fact, the direct implication is trivial. For the converse implication it suffices to consider a gamble f for which it does not hold that $f(n) \geq f(-n)$ for all n; then decomposing it as $f = Cf + C^c f$, we see that $P_\alpha(G_{\underline{P}}(Cf|\mathcal{B})) \geq 0$ (because $G_{\underline{P}}(Cf|\mathcal{B}) \geq 0$) while $P_\alpha(G_{\underline{P}}(C^c f|\mathcal{B})) \geq 0$ by assumption, whence $P_\alpha(G_{\underline{P}}(f|\mathcal{B})) \geq 0$.

Take therefore any f such that $f(n) \geq f(-n)$ for all n (in this case C is empty). Then

$$\begin{cases} G_{\underline{P}}(f|B_n)(n) = \frac{f(n)-f(-n)}{2} \geq 0 \\ G_{\underline{P}}(f|B_n)(-n) = \frac{f(-n)-f(n)}{2} \leq 0. \end{cases} \tag{A.2}$$

$$\begin{aligned}P_3(G_{\underline{P}}(f|\mathcal{B})) &= P_3(G_{\underline{P}}(f|\mathcal{B})\mathbb{I}_{\mathbb{N}}) + P_3(G_{\underline{P}}(f|\mathcal{B})\mathbb{I}_{-\mathbb{N}}) \\ &= \frac{1}{2}\left[P_1(G_{\underline{P}}(f|\mathcal{B})\mathbb{I}_{\mathbb{N}}) + P_1(G_{\underline{P}}(f|\mathcal{B})\mathbb{I}_{-\mathbb{N}})\right] + \frac{1}{2}\left[P_2(G_{\underline{P}}(f|\mathcal{B})\mathbb{I}_{\mathbb{N}}) + P_2(G_{\underline{P}}(f|\mathcal{B})\mathbb{I}_{-\mathbb{N}})\right].\end{aligned}$$

If we let $g := G_{\underline{P}}(f|\mathcal{B})$, it holds that $g(n) = -g(-n)$, whence $P_1(g\mathbb{I}_{\mathbb{N}}) + P_1(g\mathbb{I}_{-\mathbb{N}}) = 0$. On the other hand, $P_2(g) = \frac{3}{4}P(g^+) + \frac{1}{4}P(g^-) = \frac{1}{2}P(g^+) \geq 0$, taking into account that $g^+ \geq 0$ by (A.2).

If in particular we fix $n \in \mathbb{N}$ and let $f := 2\mathbb{I}_{\{2n+1,2n+3,\dots\}}$, then, using (A.2) again, $G_{\underline{P}}(f|\mathcal{B}) = \mathbb{I}_{\{2n+1,2n+3,\dots\}} - \mathbb{I}_{\{-2n-1,-2n-3,\dots\}}$ and

$$\begin{aligned}P_1(G_{\underline{P}}(f|\mathcal{B})) &= 0, \\ P_2(G_{\underline{P}}(f|\mathcal{B})) &= \frac{3}{4}P(\{2n+1,2n+3,\dots\}) - \frac{1}{4}P(\{2n+1,2n+3,\dots\}) = 0,\end{aligned}$$

because we have chosen P such that $P(\{2n+1 : n \in \mathbb{N}\}) = 0$. Hence, $P_3(G_{\underline{P}}(f|\mathcal{B})) = 0$.

On the other hand, for this gamble f we obtain that

$$\begin{aligned}P_4(G_{\underline{P}}(f|\mathcal{B})) &= \sum_{k\geq n}\frac{1}{2^{(2k+1)+2}} - \frac{3}{4}P'(\{2n+1,2n+3,\dots\}) \\ &= \sum_{k\geq n}\frac{1}{2^{(2k+1)+2}} - \frac{3}{4}[\underbrace{P'(2n-1 : n\in\mathbb{N})}_{=\frac{1}{2}} - \underbrace{P'(\{2m+1 : m\in\mathbb{N}, m<n\})}_{=0 \text{ (as the set is finite)}}] \\ &= \sum_{k\geq n}\frac{1}{2^{(2k+1)+2}} - \frac{3}{8} < 0\end{aligned}$$

for n large enough.

This implies that $P_\alpha(G_{\underline{P}}(f|\mathcal{B})) < 0$ for all $\alpha \neq 1$. As a consequence $\{P_\alpha : (\forall f \in \mathcal{L})P_\alpha(G_{\underline{P}}(f|\mathcal{B})) \geq 0\} = P_3 = \underline{E}$, taking into account that $\mathcal{M}(\underline{P}) = \{P_\alpha : \alpha \in [0,1]\}$ and using Proposition 5. Since the unconditional natural extension $\underline{E}$ of $\underline{P}, \underline{P}(\cdot|\mathcal{B})$ exists, it follows that $\underline{P}, \underline{P}(\cdot|\mathcal{B})$ avoid partial loss. But P_3 is not conglomerable: given $g := 2\mathbb{I}_{-\mathbb{N}}$, we can use the expression of $P_3(\cdot|B_n)$ (available from that of $\underline{P}(\cdot|B_n)$) to prove that $P_3(g|B_n) = \frac{[2\mathbb{I}_{-\mathbb{N}}](n)+[2\mathbb{I}_{-\mathbb{N}}](-n)}{2} = \frac{2}{2} = 1$, so that $G_{P_3}(g|\mathcal{B}) = -\mathbb{I}_{\mathbb{N}} + \mathbb{I}_{-\mathbb{N}}$ and

$$P_3(G_{P_3}(g|\mathcal{B})) = \frac{1}{2}P_1(G_{P_3}(g|\mathcal{B})) + \frac{1}{2}P_2(G_{P_3}(g|\mathcal{B})) = 0 + \frac{1}{2}\left(-\frac{3}{4}1 + \frac{1}{4}1\right) = -\frac{1}{4} < 0.$$

Thus $P_3, P_3(\cdot|\mathcal{B})$ do not avoid partial loss, and applying (5) we deduce that the conglomerable natural extension of P_3 does not exist. But since P_3 is the unconditional natural extension of $\underline{P}, \underline{P}(\cdot|\mathcal{B})$, the conglomerable natural extension of $\underline{P}$ coincides with that of P_3. This implies that the conglomerable natural extension of $\underline{P}$ does not exist, either. ♦

Now we proceed to show the most important result in this paper: that the sequence $(\underline{E}_n)_{n\in\mathbb{N}}$ may not stabilise in a finite number of steps. To this end, we need a preliminary result, which provides a tool that allows us to build sequences whose limits can be made either conglomerable or non-conglomerable depending on the choice of two parameters.

Proposition 21. *Let P_1, P_2 be two linear previsions on $\mathcal{L}(\mathbb{N})$ characterised by the facts that the restriction to events of P_1 is a σ-additive probability such that $P_1(\{n\}) > 0$ for all $n \in \mathbb{N}$, and that P_2 is a finitely additive probability such that $P_2(\{n\}) = 0$ for all $n \in \mathbb{N}$. We consider $\Omega := \mathbb{N} \cup -\mathbb{N}$ and $\mathcal{B} := \{B_n : n \in \mathbb{N}\}$, with $B_n := \{n, -n\}$. For any gamble f on Ω, let us define the gambles f^+, f^- on $\mathbb{N}$ by Eq.* (A.1).

Consider $\alpha, \beta \in [0,1]$ and let Q_1, Q_2 on $\mathcal{L}$ be

$$Q_1(f) := \alpha P_1(f^+) + (1-\alpha)P_1(f^-) \text{ and } Q_2(f) := \beta P_2(f^+) + (1-\beta)P_2(f^-).$$

Consider also $\gamma \in (0,1)$ and let $Q := \gamma Q_1 + (1-\gamma)Q_2$. Then

$$Q \text{ is conglomerable} \Leftrightarrow \alpha = \beta.$$

Proof. For $n \in \mathbb{N}$, $Q(\{n\}) = \alpha\gamma P_1(\{n\})$ and $Q(\{-n\}) = (1-\alpha)\gamma P_1(\{n\})$, whence $Q(B_n) = \alpha\gamma P_1(\{n\}) + (1-\alpha)\gamma P_1(\{n\}) = \gamma P_1(\{n\}) > 0$. Moreover, $Q(B_n f) = f(n)\alpha\gamma P_1(\{n\}) + f(-n)(1-\alpha)\gamma P_1(\{n\})$, so $Q(f|B_n) = \alpha f(n) + (1-\alpha)f(-n)$. As a consequence, using the conditional natural extension of Q, we get

$$\begin{cases} G_Q(f|\mathcal{B})(n) = (1-\alpha)[f(n) - f(-n)] \\ G_Q(f|\mathcal{B})(-n) = \alpha[f(-n) - f(n)]. \end{cases}$$

Let $g := G_Q(f|\mathcal{B})$. Then $\alpha g(n) + (1-\alpha)g(-n) = 0$ for every n, or, equivalently, $\alpha g^+ + (1-\alpha)g^- = 0$. Hence,

$$\begin{aligned} Q(g) &= \gamma Q_1(g) + (1-\gamma)Q_2(g) = \gamma[\alpha P_1(g^+) + (1-\alpha)P_1(g^-)] + (1-\gamma)[\beta P_2(g^+) + (1-\beta)P_2(g^-)] \\ &= \gamma[P_1(\alpha g^+ + (1-\alpha)g^-)] + (1-\gamma)[P_2(\beta g^+ + (1-\beta)g^-)] \\ &= 0 + (1-\gamma)[P_2(\beta g^+ + (1-\beta)g^-)]. \end{aligned}$$

If $\alpha = 0$, then $g^- = 0$ and $Q(g)$ is equal to $(1-\gamma)\beta P_2(g^+)$. Since we can always find f such that $P_2(g^+) \neq 0$, it follows that $Q(g) = 0 \Leftrightarrow \beta = \alpha = 0$.

Similarly, if $\alpha \neq 0$, the equation above becomes

$$\begin{aligned} Q(g) &= (1-\gamma)[P_2(\beta g^+ + (1-\beta)g^-)] \\ &= (1-\gamma)[-\beta\frac{(1-\alpha)}{\alpha}P_2(g^-) + (1-\beta)P_2(g^-)] \\ &= (1-\gamma)P_2(g^-)(\beta - \beta/\alpha + 1 - \beta) = (1-\gamma)P_2(g^-)(1-\beta/\alpha), \end{aligned}$$

and since we can always find f such that $P_2(g^-) \neq 0$, it follows that $Q(g) = 0 \Leftrightarrow \beta/\alpha = 1 \Leftrightarrow \alpha = \beta$. □

Example 4 (**The sequence $(\underline{E}_n)_{n\in\mathbb{N}}$ may not stabilise in a finite number of steps**). Consider the following linear previsions on $\mathcal{L}$:

$$\begin{aligned} P_1(f) &:= \sum_{n\in\mathbb{N}}(f(n)+f(-n))\frac{1}{2^{n+1}} && \text{(A.3)}\\ P_2(f) &:= \frac{1}{2}\sum_{n\in\mathbb{N}} f(n)\frac{1}{2^n}+\frac{1}{2}P(f^-) && \text{(A.4)}\\ P_3(f) &:= \frac{3}{4}P(f^+)+\frac{1}{4}P(f^-) && \text{(A.5)}\\ P_4(f) &:= \frac{1}{2}P_1(f)+\frac{1}{2}P_3(f), && \text{(A.6)}\end{aligned}$$

where P is a finitely additive probability on $\mathbb{N}$ such that $P(\{n\})=0$ for all $n\in\mathbb{N}$ and f^+,f^- are determined by Eq. (A.1). Given $\alpha\in[0,1]$, we let

$$Q_\alpha := \alpha P_2+(1-\alpha)P_4. \tag{A.7}$$

It follows that

$$\begin{aligned} Q_\alpha(f) &= \frac{\alpha}{2}\sum_{n\in\mathbb{N}} f(n)\frac{1}{2^n}+\frac{\alpha}{2}P(f^-)+(1-\alpha)\left[\frac{1}{4}\sum_{n\in\mathbb{N}} f(n)\frac{1}{2^n}+\frac{1}{4}\sum_{n\in\mathbb{N}} f(-n)\frac{1}{2^n}+\frac{3}{8}P(f^+)+\frac{1}{8}P(f^-)\right]\\ &= \left[\sum_{n\in\mathbb{N}} f(n)\frac{1}{2^n}\right]\left(\frac{\alpha}{2}+\frac{1-\alpha}{4}\right)+\left[\sum_{n\in\mathbb{N}} f(-n)\frac{1}{2^n}\right]\left(\frac{1-\alpha}{4}\right)+(1-\alpha)\frac{3}{8}P(f^+)+\frac{1+3\alpha}{8}P(f^-)\\ &= \frac{1+\alpha}{4}\tilde{P}_1(f^+)+\frac{1-\alpha}{4}\tilde{P}_1(f^-)+\frac{3-3\alpha}{8}P(f^+)+\frac{1+3\alpha}{8}P(f^-)\\ &= \frac{1}{2}\left[\frac{1+\alpha}{2}\tilde{P}_1(f^+)+\frac{1-\alpha}{2}\tilde{P}_1(f^-)\right]+\frac{1}{2}\left[\frac{3-3\alpha}{4}P(f^+)+\frac{1+3\alpha}{4}P(f^-)\right],\end{aligned}$$

where $\tilde{P}_1$ is the linear prevision determined by $\tilde{P}_1(\{n\}):=\frac{1}{2^n}$ for all $n\in\mathbb{N}$. At this point Proposition 21 yields:

$$Q_\alpha \text{ is conglomerable} \Leftrightarrow \frac{1+\alpha}{2}=\frac{3-3\alpha}{4}\Leftrightarrow \alpha=\frac{1}{5}.$$

Let $\underline{P}$ be the lower envelope of the credal set $\{Q_\alpha:\alpha\in[a,b]\}$, for given a,b such that $0<a<\frac{1}{5}<b<1$. The conglomerable natural extension of $\underline{P}$ exists since $\underline{P}\leq Q_{\frac{1}{5}}$. We aim at analysing whether the sequence of coherent lower previsions $\underline{P},\underline{E}_1,\underline{E}_2,\ldots$, originated by $\underline{P}$, yields the conglomerable natural extension in the limit and whether or not the sequence itself stabilises in (i.e., becomes constant after) a finite number of steps.

We start by detailing the form of the conditional natural extension of $\underline{P}$. From

$$\begin{aligned} P_1(\{n\}) &= \frac{1}{2^{n+1}}, & P_1(\{-n\}) &= \frac{1}{2^{n+1}},\\ P_2(\{n\}) &= \frac{1}{2^{n+1}}, & P_2(\{-n\}) &= 0,\\ P_3(\{n\}) &= 0, & P_3(\{-n\}) &= 0,\\ P_4(\{n\}) &= \frac{1}{2^{n+2}}, & P_4(\{-n\}) &= \frac{1}{2^{n+2}},\end{aligned}$$

we obtain, using Eq. (A.7),

$$Q_\alpha(\{n\})=(1+\alpha)\frac{1}{2^{n+2}},\quad Q_\alpha(\{-n\})=(1-\alpha)\frac{1}{2^{n+2}}$$

and hence $Q_\alpha(B_n)=(1+\alpha)/2^{n+2}+(1-\alpha)/2^{n+2}=1/2^{n+1}>0$ for every α. Given $f\in\mathcal{L}$, we have that

$$Q_\alpha(f|B_n)=\frac{1+\alpha}{2}f(n)+\frac{1-\alpha}{2}f(-n).$$

Since $\underline{P}(B_n) > 0$, it follows from Eq. (4) and the expression for (GBR) in [17, Theorem 6.4.2] that for every gamble f the natural extension of $\underline{P}$ conditional on B_n is

$$\begin{aligned}\underline{P}(f|B_n) &= \min\{Q_\alpha(f|B_n) : \alpha \in [a,b]\} = \min\{Q_a(f|B_n), Q_b(f|B_n)\} \\ &= \begin{cases} \frac{1+a}{2} f(n) + \frac{1-a}{2} f(-n) & \text{if } f(n) \geq f(-n) \\ \frac{1+b}{2} f(n) + \frac{1-b}{2} f(-n) & \text{if } f(n) \leq f(-n), \end{cases}\end{aligned}$$

where $Q_\alpha(f|B_n)$ is the natural extension of Q_α conditional on B_n. Hence, if for a gamble f we let $A := \{n \in \mathbb{N} : f(n) \leq f(-n)\}$, then given $n \in A$,

$$\begin{cases} G_{\underline{P}}(f|B_n)(n) = \frac{1-b}{2}[f(n) - f(-n)] \leq 0 \\ G_{\underline{P}}(f|B_n)(-n) = \frac{1+b}{2}[f(-n) - f(n)] \geq 0. \end{cases}$$

Similarly, given $n \notin A$,

$$\begin{cases} G_{\underline{P}}(f|B_n)(n) = \frac{1-a}{2}[f(n) - f(-n)] \geq 0 \\ G_{\underline{P}}(f|B_n)(-n) = \frac{1+a}{2}[f(-n) - f(n)] \leq 0. \end{cases}$$

Now we would like to check for which values of α it is the case that $Q_\alpha(G_{\underline{P}}(f|\mathcal{B})) \geq 0$ for all $f \in \mathcal{L}$, because by Proposition 5, $\mathcal{M}(\underline{E}_1) = \{Q_\alpha : Q_\alpha(G_{\underline{P}}(f|\mathcal{B})) \geq 0 \text{ for all } f \in \mathcal{L}\}$.

Given a gamble f, its associated set $A := \{n \in \mathbb{N} : f(n) \leq f(-n)\}$ and $C := \cup_{n \in A} B_n$, it holds by Remark 1 that $G_{\underline{P}}(f|\mathcal{B}) = G_{\underline{P}}(Cf|\mathcal{B}) + G_{\underline{P}}(C^c f|\mathcal{B})$. Let $g' := G_{\underline{P}}(Cf|\mathcal{B}), g'' := G_{\underline{P}}(C^c f|\mathcal{B})$. We proceed to determine when $Q_\alpha(g') \geq 0, Q_\alpha(g'') \geq 0$.

▷ Let us consider $Q_\alpha(g')$. If $n \notin A$, then $g'(-n) = g'(n) = 0$; if $n \in A$, then $g'(-n) = \frac{1+b}{2}[f(-n) - f(n)]$ and $g'(n) = \frac{1-b}{2}[f(n) - f(-n)]$. As a consequence, $g'(-n) = -\frac{1+b}{1-b} g'(n) \geq 0$. Then:

$$\begin{aligned} P_2(g') &= \sum_{n\in\mathbb{N}} g'(n) \frac{1}{2^{n+1}} + \frac{1}{2} P(g'^-) \\ P_4(g') &= \sum_{n\in\mathbb{N}} g'(n) \frac{1}{2^{n+2}} + \sum_{n\in\mathbb{N}} g'(-n) \frac{1}{2^{n+2}} + \frac{3}{8} P(g'^+) + \frac{1}{8} P(g'^-) \\ &= \sum_{n\in\mathbb{N}} g'(n) \frac{1}{2^{n+2}} \left(1 - \frac{1+b}{1-b}\right) + P(g'^-) \left(\frac{1}{8} - \frac{3}{8}\frac{1-b}{1+b}\right) \\ &= \sum_{n\in\mathbb{N}} g'(n) \frac{1}{2^{n+2}} \frac{-2b}{1-b} + P(g'^-) \frac{1}{8}\frac{4b-2}{1+b} \\ &= \sum_{n\in\mathbb{N}} g'(n) \frac{1}{2^{n+1}} \frac{-b}{1-b} + P(g'^-) \frac{1}{4}\frac{2b-1}{1+b}. \end{aligned}$$

This implies that

$$\begin{aligned} Q_\alpha(g') &= \sum_{n\in\mathbb{N}} g'(n) \frac{1}{2^{n+1}} \left[\alpha - \frac{(1-\alpha)b}{1-b}\right] + P(g'^-) \frac{1}{4} \left[2\alpha + \frac{1-\alpha}{1+b}(2b-1)\right] \\ &= \underbrace{\underbrace{\sum_{n\in\mathbb{N}} g'(n) \frac{1}{2^{n+1}}}_{\leq 0} \underbrace{\frac{\alpha - b}{1-b}}_{\leq 0}}_{\geq 0} + \underbrace{P(g'^-) \frac{1}{4}}_{\geq 0} \frac{3\alpha + 2b - 1}{1+b}. \end{aligned} \tag{A.8}$$

▷ Let us focus now on $Q_\alpha(g'')$. It holds that $g''(n) = -\frac{1-a}{1+a} g''(-n) \geq 0$ for every $n \notin A$, and trivially also for $n \in A$, given that in that case $g''(n) = g''(-n) = 0$. Then:

$$\begin{aligned}
P_2(g'') &= \sum_{n\in\mathbb{N}} g''(n)\frac{1}{2^{n+1}} + \frac{1}{2}P(g''^{-}) \\
P_4(g'') &= \sum_{n\in\mathbb{N}} (g''(n) + g''(-n))\frac{1}{2^{n+2}} + \frac{3}{8}P(g''^{+}) + \frac{1}{8}P(g''^{-}) \\
&= \sum_{n\in\mathbb{N}} g''(n)\frac{1}{2^{n+2}}\left(1 - \frac{1+a}{1-a}\right) + P(g''^{-})\frac{1}{8}\left(1 - 3\frac{1-a}{1+a}\right) \\
&= \sum_{n\in\mathbb{N}} g''(n)\frac{1}{2^{n+1}}\frac{-a}{1-a} + P(g''^{-})\frac{1}{4}\frac{2a-1}{1+a}.
\end{aligned}$$

This implies that

$$Q_\alpha(g'') = \underbrace{\sum_{n\in\mathbb{N}} g''(n)\frac{1}{2^{n+1}}\frac{\alpha-a}{1-a}}_{\geq 0} + \underbrace{P(g''^{-})\frac{1}{4}}_{\leq 0}\frac{3\alpha+2a-1}{1+a}. \tag{A.9}$$

This allows us to depict a number of possibilities:

▷ If $3\alpha + 2b - 1 \geq 0$ and $3\alpha + 2a - 1 \leq 0$ (note that we can attain this case given that $3b + 2b - 1 \geq 0$ and $3a + 2a - 1 \leq 0$ if and only if $a \leq \frac{1}{5} \leq b$), then it follows from Eqs. (A.8), (A.9) that $Q_\alpha(g') \geq 0, Q_\alpha(g'') \geq 0$ and therefore $Q_\alpha(G_{\underline{P}}(f|\mathcal{B})) \geq 0$; using Proposition 5 we obtain that $Q_\alpha \in \mathcal{M}(\underline{E}_1)$.

▷ If $3\alpha + 2b - 1 < 0$, fix $n^\star \in \mathbb{N}$ and let $f := \mathbb{I}_{\{m\in-\mathbb{N}: m\leq -n^\star\}}$. Then $C = \cup_{\{n\in\mathbb{N}: f(n)\leq f(-n)\}} B_n = \Omega$, so given $g' = G_{\underline{P}}(f|\mathcal{B})$, it holds that

$$\sum_{n\in\mathbb{N}} g'(n)\frac{1}{2^{n+1}} = -\frac{1-b}{2}\sum_{n\geq n^\star}\frac{1}{2^{n+1}} \text{ and } P(g'^{-}) = \frac{1+b}{2}.$$

If in particular we choose $n^\star$ such that

$$\sum_{n\geq n^\star}\frac{1}{2^{n+1}} < \frac{3\alpha+2b-1}{4(\alpha-b)},$$

we deduce from Eq. (A.8) that $Q_\alpha(g') < 0$. We conclude that we can always find some gamble f such that $\underline{Q}_\alpha(G_{\underline{P}}(f|\mathcal{B})) = Q_\alpha(g') < 0$ when $3\alpha + 2b - 1 < 0$. Applying Proposition 5, we deduce that $Q_\alpha \notin \mathcal{M}(\underline{E}_1)$.

▷ Finally, if $3\alpha + 2a - 1 > 0$, fix $n^\star \in \mathbb{N}$ and let $f := \mathbb{I}_{\{m\in\mathbb{N}: m\geq n^\star\}}$. Then $C = \cup_{\{n\in\mathbb{N}: f(n)\leq f(-n)\}} B_n = \cup_{\{n<n^*\}} B_n$, and $g' = G_{\underline{P}}(Cf|\mathcal{B}) = 0$, so given $g'' = G_{\underline{P}}(f|\mathcal{B})$, it holds that

$$\sum_{n\in\mathbb{N}} g''(n)\frac{1}{2^{n+1}} = \frac{1-a}{2}\sum_{n\geq n^\star}\frac{1}{2^{n+1}} \text{ and } P(g''^{-}) = -\frac{1+a}{2}.$$

If in particular we choose $n^\star$ such that

$$\sum_{n\geq n^\star}\frac{1}{2^{n+1}} < \frac{3\alpha+2a-1}{4(\alpha-a)},$$

we deduce from Eq. (A.9) that $Q_\alpha(g'') < 0$. We conclude that we can always find some gamble f such that $\underline{Q}_\alpha(G_{\underline{P}}(f|\mathcal{B})) = Q_\alpha(g'') < 0$ when $3\alpha + 2a - 1 > 0$. Applying again Proposition 5, we deduce that $Q_\alpha \notin \mathcal{M}(\underline{E}_1)$.

Thus, recalling that $\mathcal{M}(\underline{P}) = \{Q_\alpha : \alpha \in [a,b]\}$, with $0 < a < \frac{1}{5} < b < 1$, it follows that

$$\mathcal{M}(\underline{E}_1) = \left\{ Q_\alpha : \alpha \in \left[\max\left\{ a, \frac{1-2b}{3} \right\}, \min\left\{ \frac{1-2a}{3}, b \right\} \right], 0 < a < \frac{1}{5} < b < 1 \right\}.$$

Note that since $a < \frac{1}{5} < b$, it must be the case that $[\max\{a, \frac{1-2b}{3}\}, \min\{\frac{1-2a}{3}, b\}] \subsetneq [a,b]$, because it is not possible that both $a \geq \frac{1-2b}{3}$ and $b \leq \frac{1-2a}{3}$ hold. This means that at least one of the two extreme points of $[a,b]$ must change. Moreover, note that the new interval will have still to contain the value $\frac{1}{5}$ properly, in the sense that $\frac{1}{5}$ will have to be an interior point of the new interval, because

$$a < \frac{1}{5} < b \Rightarrow \max\left\{ a, \frac{1-2b}{3} \right\} < \frac{1}{5} \text{ and}$$
$$a < \frac{1}{5} < b \Rightarrow \min\left\{ b, \frac{1-2a}{3} \right\} > \frac{1}{5}.$$

This means that the infinite sequence $\underline{P}, \underline{E}_1, \underline{E}_2, \ldots$ will be in correspondence with an infinite sequence of intervals of strictly decreasing length, each one containing $\frac{1}{5}$ properly.

Let us show now that $\frac{1}{5}$ is actually the limit of the sequence of these intervals. We must consider a number of cases:

- ▷ If in the passage from $\mathcal{M}(\underline{P})$ to $\mathcal{M}(\underline{E}_1)$ both extreme points of the interval change, then we go from $[a,b]$ to $[\frac{1-2b}{3}, \frac{1-2a}{3}]$, and the length of the new interval is two thirds of the length of the previous one.
- ▷ Assume otherwise that in the passage from $\mathcal{M}(\underline{P})$ to $\mathcal{M}(\underline{E}_1)$ only the left extreme of the interval $[a,b]$ changes (if it were the right extreme, we would eventually obtain analogous conclusions). We can then rewrite the interval as $[\max\{a, \frac{1-2b}{3}\}, \min\{\frac{1-2a}{3}, b\}] = [\frac{1-2b}{3}, \min\{\frac{1-2a}{3}, b\}]$. If we now do one more step, to get to $\mathcal{M}(\underline{E}_2)$, we see that the left extreme cannot change and hence the new interval will be

  $$\left[\frac{1-2b}{3}, \min\left\{ \frac{1+4b}{9}, b \right\} \right] = \left[\frac{1-2b}{3}, \frac{1+4b}{9} \right],$$

  taking into account that $b > \frac{1}{5}$. Hence, in two steps we go from $[a,b]$ to $[\frac{1-2b}{3}, \frac{1+4b}{9}]$, and the length of the latter interval is $\frac{10b-2}{9}$. Now, since $a \leq \frac{1-2b}{3}$, we deduce that $3a + 2b \leq 1$, and as a consequence

  $$\frac{3}{2} \frac{10b-2}{9} = \frac{5b-1}{3} \leq b - a.$$

  This means that the length of $[\frac{1-2b}{3}, \frac{1+4b}{9}]$ is at most two thirds of the length of $[a,b]$.

By iterating the argument, we conclude that every two steps the length of the intervals decreases exponentially fast by $\frac{2}{3}$. As a consequence, given that $\frac{1}{5}$ is always included in the intervals, the sequence $\underline{P}, \underline{E}_1, \underline{E}_2, \ldots$ will converge towards $Q_{\frac{1}{5}}$, which, being conglomerable, is the conglomerable natural extension of $\underline{P}$. ♦

Example 5 (**The limit of the sequence of marginal extensions may differ from the conglomerable natural extension; $\underline{P}$ conglomerable $\not\Rightarrow \underline{P} = \underline{P}(\underline{P}(\cdot|\mathcal{B}))$**). Let $P_1, \ldots, P_4$ be the linear previsions on $\mathcal{L}(\Omega)$ given by Eqs. (A.3)–(A.6), and let $\underline{P} := \min\{P_1, P_2, P_4\}$. Given the gamble $h := \mathbb{I}_{-\mathbb{N}}$, it holds that:

$$\begin{aligned}
P_1(h) &= \frac{1}{2}, \\
P_2(h) &= \frac{1}{2}0 + \frac{1}{2}1 = \frac{1}{2}, \\
P_3(h) &= \frac{3}{4}0 + \frac{1}{4}1 = \frac{1}{4}, \\
P_4(h) &= \frac{1}{2}\frac{1}{2} + \frac{1}{2}\frac{1}{4} = \frac{3}{8},
\end{aligned}$$

so $\underline{P}(h) = \frac{3}{8}$.

Let $\underline{P}(\cdot|\mathcal{B})$ be the conditional natural extension of $\underline{P}$. In [13, Example 5] it is shown that the natural extension of $\underline{P}, \underline{P}(\cdot|\mathcal{B})$ is given by

$$\underline{E} = \min\left\{P_1, P_4, \frac{1}{3}P_2 + \frac{2}{3}P_4\right\},$$

that the conditional natural extension of $\underline{E}$ is given by

$$\underline{E}(f|B_n) = \min\left\{\frac{f(n)+f(-n)}{2}, \frac{2f(n)+f(-n)}{3}\right\},$$

for every $B_n \in \mathcal{B}$ and that $P_4(G_{\underline{E}}(f|\mathcal{B})) < 0$ for some f, so $\underline{E}$ is not conglomerable. Let us give an upper bound of the natural extension $\underline{E}_2$ of $\underline{E}, \underline{E}(\cdot|\mathcal{B})$.

- ▷ Since P_1 is σ-additive, it is conglomerable, so $P_1(G_{\underline{E}}(f|\mathcal{B})) \geq P_1(G_{P_1}(f|\mathcal{B})) \geq 0$ for all $f \in \mathcal{L}$, where $P_1(\cdot|\mathcal{B})$ is the conditional natural extension of P_1.
- ▷ Let us show that given $P_5 := \frac{1}{3}P_2 + \frac{2}{3}P_4$, also $P_5(G_{\underline{E}}(f|\mathcal{B})) \geq 0$.

  Consider a gamble f and let $A := \{n \in \mathbb{N} : f(n) \leq f(-n)\}$. Then:

$$\begin{aligned}
\underline{E}(f|B_n) &= \begin{cases} \frac{2f(n)+f(-n)}{3} & \text{if } n \in A \\ \frac{f(n)+f(-n)}{2} & \text{if } n \notin A. \end{cases} \\
G_{\underline{E}}(f|B_n)(n) &= \begin{cases} \frac{f(n)-f(-n)}{3} \leq 0 & \text{if } n \in A \\ \frac{f(n)-f(-n)}{2} \geq 0 & \text{if } n \notin A. \end{cases} \\
G_{\underline{E}}(f|B_n)(-n) &= \begin{cases} \frac{2(f(-n)-f(n))}{3} \geq 0 & \text{if } n \in A \\ \frac{f(-n)-f(n)}{2} \leq 0 & \text{if } n \notin A. \end{cases}
\end{aligned}$$

  Letting $C := \cup\{B_n : n \in A\}$, we have by Remark 1 that $G_{\underline{E}}(f|\mathcal{B}) = G_{\underline{E}}(Cf|\mathcal{B}) + G_{\underline{E}}(C^c f|\mathcal{B})$. We deduce that $P_5(G_{\underline{E}}(f|\mathcal{B})) = P_5(G_{\underline{E}}(Cf|\mathcal{B})) + P_5(G_{\underline{E}}(C^c f|\mathcal{B}))$, and if we let $f' := Cf$ and $g' := G_{\underline{E}}(f'|\mathcal{B})$, then we get that $g'^- = -2g'^+ \geq 0$. Since

$$\begin{aligned}
P_2(g') &= \frac{1}{2}\sum_{n\in\mathbb{N}} g'(n)\frac{1}{2^n} + \frac{1}{2}P(g'^-) = \frac{1}{2}\sum_{n\in\mathbb{N}} g'(n)\frac{1}{2^n} - P(g'^+) \\
P_3(g') &= \frac{3}{4}P(g'^+) + \frac{1}{4}P(g'^-) = \frac{1}{4}P(g'^+) \\
P_4(g') &= \frac{1}{2}P_1(g') + \frac{1}{2}P_3(g') = \frac{1}{2}\sum_{n\in\mathbb{N}}[g'(n) + g'(-n)]\frac{1}{2^{n+1}} + \frac{1}{2}\frac{1}{4}P(g'^+) \\
&= -\frac{1}{2}\sum_{n\in\mathbb{N}} g'(n)\frac{1}{2^{n+1}} + \frac{1}{8}P(g'^+)
\end{aligned}$$

  and since $P_5(g') = \frac{1}{3}P_2(g') + \frac{2}{3}P_4(g')$, we obtain that

$$P_5(g') = \frac{1}{3}\left[\frac{1}{2}\sum_{n\in\mathbb{N}} g'(n)\frac{1}{2^n} - P(g'^+)\right] + \frac{2}{3}\left[-\frac{1}{2}\sum_{n\in\mathbb{N}} g'(n)\frac{1}{2^{n+1}} + \frac{1}{8}P(g'^+)\right] = -\frac{1}{4}P(g'^+) \geq 0.$$

  Now, let $f'' := C^c f$ and $g'' := G_{\underline{E}}(f''|\mathcal{B})$. We get that $g''^+ = -g''^- \geq 0$, whence

$$\begin{aligned}
P_2(g'') &= \sum_{n\in\mathbb{N}} g''^+(n)\frac{1}{2^{n+1}} + \frac{1}{2}P(g''^-) \\
P_3(g'') &= \frac{3}{4}P(g''^+) + \frac{1}{4}P(g''^-) = \frac{1}{2}P(g''^+) \\
P_4(g'') &= \frac{1}{2}P_1(g'') + \frac{1}{2}P_3(g'') = \frac{1}{2}\sum_{n\in\mathbb{N}}[g''(n) + g''(-n)]\frac{1}{2^{n+1}} + \frac{1}{2}\frac{1}{2}P(g''^+) = \frac{1}{4}P(g''^+),
\end{aligned}$$

so that

$$P_5(g'') = \frac{1}{3}P_2(g'') + \frac{2}{3}P_4(g'') = \frac{1}{3}\sum_{n\in\mathbb{N}} g''^{+}(n)\frac{1}{2^{n+1}} + \frac{1}{6}P(g''^{-}) + \frac{1}{6}P(g''^{+}) = \frac{1}{3}\sum_{n\in\mathbb{N}} g''^{+}(n)\frac{1}{2^{n+1}} \geq 0.$$

The analysis so far has allowed us to deduce that both $P_1(G_{\underline{E}}(\cdot|\mathcal{B})) \geq 0$ and $P_5(G_{\underline{E}}(\cdot|\mathcal{B})) \geq 0$. It follows from Proposition 5 that the natural extension $\underline{E}_2$ of $\underline{E}, \underline{E}(\cdot|\mathcal{B})$ is dominated by the lower envelope of $\{P_1, P_5\}$, from which we obtain that $\underline{E}_2(\cdot|B_n) \leq \min\{P_1(\cdot|B_n), P_5(\cdot|B_n)\}$ and in particular that

$$\begin{aligned}
\underline{E}_2(f|B_n) &\leq \min\left\{\frac{P_1(B_nf)}{P_1(B_n)}, \frac{P_5(B_nf)}{P_5(B_n)}\right\} = \min\left\{\frac{[f(n)+f(-n)]\frac{1}{2^{n+1}}}{\frac{1}{2^n}}, \frac{\frac{1}{3}P_2(B_nf)+\frac{2}{3}P_4(B_nf)}{\frac{1}{2^{n+1}}}\right\} \\
&= \min\left\{\frac{1}{2}[f(n)+f(-n)], \frac{\frac{1}{3}\left[\frac{1}{2}f(n)\frac{1}{2^n}+\frac{1}{2}0\right]+\frac{2}{3}\left[\frac{1}{2}P_1(B_nf)+\frac{1}{2}P_3(B_nf)\right]}{\frac{1}{2^{n+1}}}\right\} \\
&= \min\left\{\frac{1}{2}[f(n)+f(-n)], 2^{n+1}\left[\frac{1}{3}\frac{1}{2^{n+1}}f(n)+\frac{2}{3}\frac{1}{2}\frac{1}{2^{n+1}}[f(n)+f(-n)]+\frac{1}{2}0\right]\right\} \\
&= \min\left\{\frac{f(n)+f(-n)}{2}, \frac{2f(n)+f(-n)}{3}\right\} = \underline{E}(f|B_n),
\end{aligned}$$

which implies that $\underline{E}_2(f|B_n) = \underline{E}(f|B_n)$. This implies, by means of the sufficient condition for conglomerability in [13, Proposition 16], that $\underline{E}_2$ is conglomerable and therefore it is the conglomerable natural extension of $\underline{P}$.

Now, if we reconsider $h = \mathbb{I}_{-\mathbb{N}}$, then $\underline{E}_2(h|B_n) = \frac{1}{3}$ for all n, so if $\underline{E}_2$ were a marginal extension model, we would have $\underline{E}_2(h) = \underline{E}_2(\underline{E}_2(h|\mathcal{B})) = \underline{E}_2(\frac{1}{3}) = \frac{1}{3}$. But we know that $\underline{E}_2(h) \geq \underline{P}(f) = \frac{3}{8} > \frac{1}{3}$. This shows that the sequence of marginal extensions may not become constant, after a finite number of steps, on the conglomerable natural extension. ♦

Example 6 (**The sufficient conditions in Theorem 20 are not necessary**). Let P_1, P_2 be two σ-additive probabilities on $\mathcal{P}(\Omega)$ satisfying

$$\begin{aligned}
P_1(B_n) &= \frac{1}{2^n} \quad \text{for all } n \in \mathbb{N} \\
P_2(B_n) &= \begin{cases} \frac{1}{4^n} & \text{if } n \text{ is odd} \\ \frac{1}{2^{n/2}} + \frac{1}{2^{n/2-1}} - \frac{1}{4^{n-1}} & \text{if } n \text{ is even.} \end{cases}
\end{aligned}$$

Then $\sum_{n\in\mathbb{N}} P_1(B_n) = 1 = \sum_{n\in\mathbb{N}} P_2(B_n)$ because $P_2(B_{2n}) + P_2(B_{2n-1}) = P_1(B_{2n}) + P_1(B_{2n-1}) = \frac{1}{2^n} + \frac{1}{2^{n-1}}$ for all n. Inside each B_n we can assume that P_1 and P_2 are uniform (although this is not really necessary). Then $\underline{P} := \min\{P_1, P_2\}$ is conglomerable because it is a lower envelope of conglomerable models. On the other hand, given n odd, we have that

$$\frac{\overline{P}(B_n)}{\underline{P}(B_n)} = \frac{\frac{1}{2^n}}{\frac{1}{4^n}} = 2^n,$$

so $\left\{\frac{\overline{P}(B)}{\underline{P}(B)} : B \in \mathcal{B}\right\}$ is unbounded. Note also that in this case we have $\Delta' = \Delta = \{(\alpha, 1-\alpha) : \alpha \in [0,1]\}$, so we are not in the interior of the simplex, either. ♦

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