Conglomerable coherence

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Abstract

We contrast Williams' and Walley's theories of coherent lower previsions in the light of conglomerability. These are two of the most credited approaches to a behavioural theory of imprecise probability. Conglomerability is the notion that distinguishes them the most: Williams' theory does not consider it, while Walley aims at embedding it in his theory. This question is important, as conglomerability is a major point of disagreement at the foundations of probability, since it was first defined by de Finetti in 1930. We show that Walley's notion of joint coherence (which is the single axiom of his theory) for conditional lower previsions does not take all the implications of conglomerability into account. Considered also some previous results in the literature, we deduce that Williams' theory should be the one to use when conglomerability is not required; for the opposite case, we define the new theory of conglomerably coherent lower previsions, which is arguably the one to use, and of which Walley's theory can be understood as an approximation. We show that this approximation is exact in two important cases: when all conditioning events have positive lower probability, and when conditioning partitions are nested.

Keywords: Conglomerability, Williams' and Walley's theories of coherent lower previsions, sets of desirable gambles, coherence, conglomerable natural extension, infinite conditioning partitions.

1. Introduction

Theories of coherent lower previsions

Recent years have witnessed a considerable amount of research devoted to model uncertainty using sets of probabilities, which is sometimes referred to as *imprecise probability*. A leading approach in this line of research is Peter Walley's behavioural theory of *coherent lower previsions* [16]. A coherent lower prevision $\underline{P}$ is a lower expectation functional: it is the lower envelope of the expectations obtained through a set of finitely additive probabilities. A conditional coherent lower prevision $\underline{P}(\cdot|\mathcal{B})$ is defined similarly: it can be understood as a set of conditional lower expectation functionals $\underline{P}(\cdot|B)$ relative to the events B in a partition $\mathcal{B}$ of the possibility space Ω. Walley's core modelling unit is, however, not a single conditional coherent lower prevision but a collection of them, such as $\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$, which is obtained by considering m partitions of Ω.

These conditional lower previsions are not assumed to be derived from a single 'joint' model, as it is often the case in traditional, precise, probability. The theory takes them as given, while providing tools to check whether they express rational assessments, and otherwise to extend them in a least-committal way to a rational model—provided that there is one. The former is done by checking whether the conditional lower previsions are *jointly coherent*. The latter is done by a procedure of *natural extension*, which delivers the extended conditional lower previsions.

Joint coherence and natural extension are in fact the two pillars of Walley's theory. From a purely logical point of view, we can think of them respectively as the (single) axiom of the theory and its (single) inferential method.

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Walley developed his theory based on the earlier theory by Peter Williams [19] (these two theories are introduced in some detail in Section 2.1) who, in turn, was influenced by de Finetti [3]. In fact, Williams was the first to lay down the foundations for many of the concepts used nowadays, such as coherent lower previsions (that he called imprecise previsions), joint coherence and natural extension, among others. In addition, and as a matter of fact, Williams' theory is still very lively, and some researchers argue that it should be Williams' theory to be taken as the foundation for a behavioural theory of imprecise probability rather than Walley's (e.g., see [14]).

What are then the differences between these two theories? One is that Williams' notion of coherence is based on the combination of a finite number of conditional gambles, and the conditioning events are not required to be elements of a partition. In this sense, Williams' theory can be regarded as structure-free, unlike Walley's. Most importantly, if we formulate Williams' theory under Walley's terminology (that is, using partitions of conditioning events) the difference that remains—a fundamental one—arises when some of the above partitions $\mathcal{B}_i$ ($i = 1, \dots, m$) are infinite. An important consequence is that the two theories coincide in case the space of possibilities Ω is finite. In a sense, we could say that something as important as formalising a self-consistent behavioural theory of imprecise probability for the finite case has been settled by Williams as long as 38 years ago, even though this has gone largely unnoticed outside the community of imprecise probability so far.

When Ω (and in particular some partition $\mathcal{B}_i$) is not finite, as it is very often the case in statistics, then we face a dilemma. To better understand the terms of the question, we need to dig a bit more in some detail, as the question has to do with the notion of conglomerability.

It was de Finetti that in 1930 [1] discovered a puzzling phenomenon related to finitely, but not countably, additive probabilities. Denote by P a linear prevision, that is, the expectation taken with respect to a finitely additive probability, and by $P(\cdot|\mathcal{B})$ a linear prevision conditional on an infinite partition $\mathcal{B}$; this is defined in an analogous way. Let also f denote a bounded random variable (we call it a *gamble*). De Finetti discovered that even if P, $P(\cdot|\mathcal{B})$ can be argued to be a rational model (that is, two jointly coherent previsions in the sense of de Finetti and Williams), they may still lead to the following relation:

$$P(f) \notin \left[\inf_{B \in \mathcal{B}} P(f|B), \sup_{B \in \mathcal{B}} P(f|B) \right],$$

which obviously prevents $P(f)$ from being understood as a mixture of the conditional expectations. De Finetti called *conglomerable* a linear prevision P that cannot incur the previous situation.

The failure of conglomerability creates questionable, and controversial, situations (e.g., see [15, Section 2.2], [16, Example 6.8.5]). The controversy has originated the two most important schools of behavioural imprecise probabilities in the literature: on one side we find researchers like de Finetti who reject that conglomerability should be imposed as a rationality axiom, and whose work is at the core of Williams' approach to the imprecise case; on the other, we can find Walley, for instance, who instead endorses such a practice (see [16, Sections 6.3.3 and 6.8.4] for a detailed view). The consequence is that we now have two behavioural theories of imprecise probability, Williams' and Walley's, that differ in the way they deal with conglomerability.[1] The difference is not without importance: the conclusions these theories may lead to can be very different on the same problem. Even more important, this has brought the quest for a *standard* behavioural theory of coherent lower previsions to a standstill.

This state of affairs has recently been revived by two new insights about the long-standing question of conglomerability. The first is an original justification of conglomerability through considerations of '(strong) temporal coherence'[2] [20]: loosely speaking, the idea is that conglomerability should indeed be a rationality axiom whenever one assumes, right from the start, that conditional beliefs will be used to determine his future behaviour. Here, given a conditioning event B, by conditional beliefs we refer to the uncertain transactions, such as the offer of a gamble, that are called off unless B occurs and that we would accept now (according to our current set of beliefs); it can be proven that if we use these conditional beliefs as models of our future behaviour when later some element B in a partition is observed, then if we do not require conglomerability the present and future behaviour can be clearly inconsistent with each other (in spite of depending on beliefs established at the same time). This gives support to Walley's theory—under the mentioned

[1]Note that requiring conglomerability makes also necessary to work with partitions of conditioning events, like in Walley's theory. Hence, the additional flexibility of Williams' theory, given by its being structure-free, can be exploited only if one is willing, or allowed, to neglect conglomerability.

[2]This notion of temporal coherence should not be confused with the one considered by Michael Goldstein in [5, 6].

assumption—while going to the detriment of Williams'.[3] On the other hand, the second insight tells us that Walley's procedure of natural extension does not fully take into account the implications of conglomerability [13]: this means that a basic procedure to construct rational models in Walley's theory is not always yielding models that comply with Walley's own rationality criteria.

Theories of coherent sets of desirable gambles

In order to better understand the last claim, we need to step back from coherent lower previsions to *sets of desirable gambles*. This is also a behavioural theory of imprecise probability; a theory that is more general and more fundamental than coherent lower previsions (and arguably also easier to deal with).

Unsurprisingly, there are two theories of coherent sets of desirable gambles (they are introduced in Section 2.2). In fact, it was again Williams who introduced sets of desirable gambles, and defined the axioms they should satisfy to express rational assessments: a set that satisfies these axioms is called coherent. In addition, and again not surprisingly at this point, there is also a procedure of natural extension for sets of desirable gambles that extends in a least-committal way an incoherent set to a coherent one, provided there is one.

Coherent sets of desirable gambles are the counterpart of Williams' theory of coherent lower previsions in terms of sets of gambles, in the sense that Williams' theory of coherent lower previsions can mathematically be derived from them.[4] This is not the case of Walley's theory, because Williams' axioms for sets of desirable gambles do not include any axiom of conglomerability. It was Walley in [16, Appendix F1], who showed that conglomerability can be expressed in a very natural and general way as a further axiom of desirability.[5] For this reason, we can say that Walley settled the problem of formalising a behavioural theory of imprecise probability for sets of desirable gambles based on conglomerability.

However, Walley did not investigate the rational extension of a set of desirable gambles under the further requirement of conglomerability. This has been done recently in [13]: it is called the *conglomerable natural extension* of a set.

At this point one might expect that, as in the case of Williams' theory, Walley's theory of coherent lower previsions can mathematically be derived from the theory of conglomerably coherent sets of desirable gambles. But this is not the case: for Walley's procedure of natural extension for coherent lower previsions can only be regarded as an approximation to the actual procedure that follows from the conglomerable natural extension [13], and hence, as we have mentioned already, it is not fully consistent with conglomerability.

A new theory of conglomerably coherent lower previsions

If we take all these insights seriously into account, what we should deduce is that whenever one takes for granted that conditional beliefs will be used to determine future behaviour, (i) Williams' theory of coherent lower previsions is not the one we are after because it does not require conglomerability; and (ii) Walley's theory of coherent lower previsions does not seem to be such a theory either, in this case because conglomerability is not fully accounted for.

Which is the actual theory we should consider then? The answer follows directly from the previous discussion: the theory of conglomerably coherent lower previsions that can be derived from conglomerably coherent sets of desirable gambles. We define the basic notions of this new theory in Section 2.3: in particular, we define $\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$ as *conglomerably coherent* when there is a set of desirable gambles that induces them and is conglomerably coherent; and we say that these conditional lower previsions *avoid conglomerable partial loss* when they can be extended to conglomerably coherent conditional lower previsions.

This is not the end of the story, however, and for two reasons:

(i) We have not investigated Walley's notion of joint coherence in the light of conglomerability yet: for even if Walley's natural extension has a problem with conglomerability, it could still be the case that Walley's definition of joint coherence is the appropriate notion to use under conglomerability; or, in other words, that Walley's joint coherence coincides with conglomerable coherence.

[3] Note that the justification of conglomerability in [20], although somewhat related, is different from the one used by Walley in [16, Section 6.3.3] (which is based on his *updating* and *conglomerative* principles), mainly because Walley's work is not focused on temporal considerations.

[4] This has been detailed for the case of finite partitions, where Williams' and Walley's agree, in [11].

[5] Actually Walley formalised 'full conglomerability', that is, conglomerability with respect to all the possible partitions of Ω. This is a more questionable concept than the weaker $\mathcal{B}$-conglomerability, that is, conglomerability with respect to a given partition. For the time being, we are neglecting these subtleties that will be better detailed later on.

(ii) Conglomerability is difficult to work with, as it has been made clear in [13]: the conglomerable natural extension needs limits to be computed, so it is hardly going to give rise to closed formulas in general. If Walley's coherence agreed with conglomerable coherence, then we would have a case where conglomerability is instead easier to deal with.

In order to analyse these questions, we need tools to convert $\underline{P}(\cdot|\mathcal{B}_1), \ldots, \underline{P}(\cdot|\mathcal{B}_m)$ into a set of desirable gambles $\mathcal{R}$. This is described in Section 2.4.

We start our actual investigation in Section 3. We show that $\underline{P}(\cdot|\mathcal{B}_1), \ldots, \underline{P}(\cdot|\mathcal{B}_m)$ avoid conglomerable partial loss if and only if the conglomerable natural extension $\mathcal{F}$ of $\mathcal{R}$ exists. Moreover, we show that $\underline{P}(\cdot|\mathcal{B}_1), \ldots, \underline{P}(\cdot|\mathcal{B}_m)$ are conglomerably coherent if and only if $\mathcal{F}$ not only exists but also induces them. We show also that the conglomerable coherence of $\underline{P}(\cdot|\mathcal{B}_1), \ldots, \underline{P}(\cdot|\mathcal{B}_m)$ implies their joint coherence in Walley's sense.

We prove analogous results, for completeness, in the case of Williams' theory. In this context the former role of the conglomerable natural extension $\mathcal{F}$ is taken by the (simpler) natural extension $\mathcal{E}$ of $\mathcal{R}$. In addition, we consider what happens when we start with a set of desirable gambles rather than conditional lower previsions: we prove that a coherent set of desirable gambles induces Williams-coherent conditional lower previsions $\underline{P}(\cdot|\mathcal{B}_1), \ldots, \underline{P}(\cdot|\mathcal{B}_m)$; if such a set is also conglomerable with respect to all the partitions $\mathcal{B}_1, \ldots, \mathcal{B}_m$ (we say that it is $\mathcal{B}_{1:m}$-conglomerable), then $\underline{P}(\cdot|\mathcal{B}_1), \ldots, \underline{P}(\cdot|\mathcal{B}_m)$ are jointly coherent in Walley's sense.

However, we do not know yet whether conglomerable coherence and Walley's joint coherence coincide or not. We settle this problem in Example 1, which is the most important outcome of this paper: we illustrate a case where the conditional lower previsions $\underline{P}(\cdot|\mathcal{B}_1), \ldots, \underline{P}(\cdot|\mathcal{B}_m)$ are jointly coherent in Walley's sense, but not conglomerably coherent. The significance of this result is much related to the impact it has on Walley's theory of coherent lower previsions: it shows that also the second pillar of this theory, besides the natural extension, does not take all the implications of conglomerability into account. It tells us that there are two behavioural theories of imprecise probability that one should consider at this point: Williams' theory for the case when conglomerability can be neglected, and the new theory for the case when it cannot.

What is the role of Walley's theory then? We show that it may be understood as an approximation to the actual theory based on conglomerability: if the conditional lower previsions $\underline{P}(\cdot|\mathcal{B}_1), \ldots, \underline{P}(\cdot|\mathcal{B}_m)$ are jointly coherent in Walley's sense, then they are induced by a set that satisfies a condition which is related to, but weaker than, conglomerability.

This approximation becomes exact in some important cases. In Section 4, we consider *weakly coherent* conditional lower previsions $\underline{P}(\cdot|\mathcal{B}_1), \ldots, \underline{P}(\cdot|\mathcal{B}_m)$. This notion is equivalent to the existence of an unconditional lower prevision $\underline{P}$ on $\mathcal{L}(\Omega)$ that is compatible with them (in a way that will be detailed later). Under the assumption that $\underline{P}$ assigns positive probability to all events in $\cup_{i=1}^{m}\mathcal{B}_i$, it turns out (see Theorem 13) that $\underline{P}(\cdot|\mathcal{B}_1), \ldots, \underline{P}(\cdot|\mathcal{B}_m)$ are not only jointly coherent in Walley's sense, but also conglomerably coherent. This tells us that Walley's coherence can break down only when some conditioning event is given zero lower probability. Apart from clarifying the scope of the technical issues originated by conglomerability, this result has practically useful consequences: whenever zero probabilities are not originated in a problem, in the sense that all the conditioning events have positive lower probability, we can live with Walley's simpler notion of coherence, as it will be equivalent to conglomerable coherence.

In Section 5, we focus on another important special case: when the partitions are nested, that is, finer and finer. This situation is common in some applications of probability. Under these conditions, and through quite an involved proof, we show again that Walley's joint coherence is equivalent to conglomerable coherence; in addition we provide the explicit expression for the conglomerable natural extension of set $\mathcal{R}$.

Our concluding comments are in Section 6, and all the proofs can be found in the Appendix.

2. Behavioural theories of imprecise probability: basic notions

In this section we introduce several concepts needed for later developments. In particular, we give a brief introduction to coherent lower previsions in Section 2.1 and to the more general model of sets of desirable gambles in Section 2.2. In these sections, we describe together both the theories of Walley and Williams (which can both be discussed either at the level of coherent lower previsions or at that of desirable gambles). We do this because these two theories are formally very similar, although they differ with respect to the notion of conglomerability. In Section 2.3 we define a new uncertainty theory based on conditional lower previsions and conglomerability. In Section 2.4 we give some notation and preliminary results.

We refer the interested reader to [16] for an in-depth study of Walley's theory (see [7] for a survey), and to [14, 19] for Williams' theory.

2.1. Coherent lower previsions

2.1.1. The behavioural interpretation

Given a possibility space Ω, a *gamble* f is a bounded real-valued function on Ω. This function represents a random reward $f(\omega)$, which depends on the a priori unknown value ω of Ω. We shall denote by $\mathcal{L}(\Omega)$ the set of all gambles on Ω, or by $\mathcal{L}$ when there is no confusion about the possibility space we are dealing with (or because we want to write a statement independently of a specific possibility space), and by $\mathcal{L}^+(\Omega)$ (or just $\mathcal{L}^+$) the set of so-called positive gambles: $\{f \in \mathcal{L} : f \gneq 0\}$.[6] A *lower prevision* $\underline{P}$ is a real functional defined on some set of gambles; throughout this paper, we shall only deal with the case of full domains, where the lower previsions are always defined on the entire set $\mathcal{L}$. A lower prevision is used to represent a subject's supremum acceptable buying prices for these gambles, in the sense that for all $\varepsilon > 0$ and all f in $\mathcal{L}$ the subject is disposed to accept the uncertain reward $f - \underline{P}(f) + \varepsilon$, or, in other words, he desires to buy the gamble f in exchange of $\underline{P}(f) - \varepsilon$.

From any lower prevision $\underline{P}$ we can define an upper prevision $\overline{P}$ using conjugacy: $\overline{P}(f) := -\underline{P}(-f)$. $\overline{P}(f)$ can be interpreted as the infimum acceptable selling price for the gamble f. Because of this relationship, it will suffice for the purposes of this paper to concentrate on lower previsions for the most part.

We can also consider the supremum buying prices for a gamble, *conditional* on an element of a partition $\mathcal{B}$ of Ω. Given such a set $B \in \mathcal{B}$ and a gamble f on Ω, the lower prevision $\underline{P}(f|B)$ represents the subject's supremum acceptable buying price for the gamble f, provided he later comes to know that the unknown value ω belongs to B, and nothing else. Equivalently, it can also be seen as the supremum value of ε for which our subject is disposed to accept the transaction given by the gamble $B(f - \varepsilon)$,[7] where to simplify the notation we use B to denote also the indicator function $\mathbb{I}_B$ of the set B. If we consider a partition $\mathcal{B}$ of Ω (for instance a set of categories), then we shall represent by $\underline{P}(f|\mathcal{B})$ the gamble on Ω that takes the value $\underline{P}(f|B)$ if and only if ω belongs to the element B of the partition $\mathcal{B}$. The functional $\underline{P}(\cdot|\mathcal{B})$ that maps any gamble f on its domain into the gamble $\underline{P}(f|\mathcal{B})$ is called a *conditional lower prevision*. To any conditional lower prevision $\underline{P}(\cdot|\mathcal{B})$ we can associate a *conditional upper prevision* $\overline{P}(\cdot|\mathcal{B})$ by $\overline{P}(f|\mathcal{B}) := -\underline{P}(-f|\mathcal{B})$. It will represent the infimum acceptable selling price for the gamble f contingent on the element of the partition $\mathcal{B}$ that we observe.

A gamble f on Ω is called *$\mathcal{B}$-measurable* when it is constant over the elements of $\mathcal{B}$. This is for instance the case of the conditional lower prevision $\underline{P}(f|\mathcal{B})$.

We shall also use the notations

$$G(f|B) := B(f - \underline{P}(f|B)),\ G(f|\mathcal{B}) := \sum_{B\in\mathcal{B}} G(f|B) = f - \underline{P}(f|\mathcal{B})$$

for all $f \in \mathcal{L}$ and all $B \in \mathcal{B}$. By $G(f|B)$ we represent the transaction where the gamble f is bought at price $\underline{P}(f|B)$ under the assumption that B happens, and which is called off otherwise. By definition of conditional lower prevision, $G(f|B) + \varepsilon B$ is desirable for our subject for all $\varepsilon > 0$: we say that $G(f|B)$ is an *almost-desirable* gamble, in the sense that shall be detailed in Section 2.2. Moreover, $G(f|\mathcal{B}) + \varepsilon = \sum_{B\in\mathcal{B}}(G(f|B) + \varepsilon B)$ is desirable according to Walley, given that it is defined piece-wise on a partition of Ω by means of desirable gambles. Since this holds for all $\varepsilon > 0$, then also $G(f|\mathcal{B})$ is almost-desirable for Walley. The situation is different in the case of Williams, because he assumes that sums of desirable gambles are desirable only when these sums are finite. In other words, $G(f|\mathcal{B})$ is almost-desirable for Williams if $\mathcal{B}$ is finite. This is in fact the crucial difference in the theories by Williams and Walley: Williams' theory is entirely finitary in this respect, while Walley's is not. This difference is tightly related to Walley's acceptance (and Williams' neglect) of the notion of conglomerability introduced later on.

In the case of an unconditional lower prevision $\underline{P}$, we shall let $G(f) := f - \underline{P}(f)$ for any gamble f in its domain. This is equivalent to have a conditional lower prevision $\underline{P}(\cdot|\mathcal{B})$ with $\mathcal{B} = \{\Omega\}$. Moreover, in this case $G(f|\{\Omega\}) = G(f)$.

[6]In this paper we shall use $f < g$ to denote that $f(\omega) < g(\omega)$ for all $\omega \in \Omega$, and $f \leq g$ when $f(\omega) \leq g(\omega)$ for all $\omega \in \Omega$. The notation $f \lneq g$ (often adopted when either $f = 0$ or $g = 0$) is used in the case $f \leq g, f \neq g$, and similarly $f \gneq g$ means that $f \geq g, f \neq g$.

[7]These are called the *updated* and *contingent* interpretations of the conditional lower prevision, and represent our subject's beliefs *at the present time*, even if they take into account future scenarios.

These assessments modelled by a conditional lower prevision $\underline{P}(\cdot|\mathcal{B})$ can be made for many different partitions of Ω, and therefore it is not uncommon to model a subject's beliefs using a finite number of different conditional lower previsions. We should verify then that all the assessments modelled by these conditional lower previsions are coherent with one another. In this section we review the different consistency criteria.

2.1.2. Separate coherence

The first requirement we make is that for any partition $\mathcal{B}$, the conditional lower prevision $\underline{P}(\cdot|\mathcal{B})$ defined on $\mathcal{L}$ should be separately coherent.

Definition 1 (**Separate coherence**). A conditional lower prevision $\underline{P}(\cdot|\mathcal{B})$ with domain $\mathcal{L}$ is *separately coherent* if for all $B \in \mathcal{B}$, $f, g \in \mathcal{L}$, and $\lambda > 0$:

$$\underline{P}(f|B) \geq \inf_B f, \tag{SC1}$$

$$\underline{P}(\lambda f|B) = \lambda \underline{P}(f|B), \tag{SC2}$$

$$\underline{P}(f+g|B) \geq \underline{P}(f|B) + \underline{P}(g|B). \tag{SC3}$$

Separate coherence implies that, contingent on B, a subject's supremum acceptable buying price for a gamble f cannot be raised by taking into account his other acceptable transactions, and also that he should be prepared to bet on B at all odds after having observed it.

It is also useful for this paper to explicitly consider the particular case where $\mathcal{B} = \{\Omega\}$, that is, when we have unconditional information. We have then a(n *unconditional*) *lower prevision* $\underline{P}$ on $\mathcal{L}$. Separate coherence is simply called then *coherence*:

Definition 2 (**Coherence for a lower prevision**). An unconditional lower prevision $\underline{P}$ with domain $\mathcal{L}$ is *coherent* if for all $f, g \in \mathcal{L}$, and $\lambda > 0$:

$$\underline{P}(f) \geq \inf f, \tag{C1}$$

$$\underline{P}(\lambda f) = \lambda \underline{P}(f), \tag{C2}$$

$$\underline{P}(f+g) \geq \underline{P}(f) + \underline{P}(g). \tag{C3}$$

Its interpretation is similar to that of separate coherence, now with $B = \Omega$. Equivalently, $\underline{P}$ is coherent if and only if for every $f_0, f_1, \ldots, f_n$ in $\mathcal{L}$, it holds that

$$\sup_{\Omega} \left[\sum_{i=1}^{n} G(f_i) - G(f_0) \right] \geq 0.$$

2.1.3. (Williams-)Avoiding partial loss

Let $\mathcal{B}_1, \ldots, \mathcal{B}_m$ be partitions of Ω and let $\underline{P}(\cdot|\mathcal{B}_1), \ldots, \underline{P}(\cdot|\mathcal{B}_m)$ be separately coherent conditional lower previsions on $\mathcal{L}$. With each partition $\mathcal{B}_i$, $i \in \{1, \ldots, m\}$, and gamble $f \in \mathcal{L}$, we associate the notion of support:

Definition 3 (**Support**). Define the *$\mathcal{B}_i$-support* $S_i(f)$ of a gamble $f \in \mathcal{L}$ as

$$S_i(f) := \{B_i \in \mathcal{B}_i : B_i f \neq 0\},$$

i.e., it is the set of elements of the partition where f is not identically equal to the zero gamble.

Definition 4 (**Williams-avoiding partial loss for lower previsions**). We say that $\underline{P}(\cdot|\mathcal{B}_1), \ldots, \underline{P}(\cdot|\mathcal{B}_m)$ *Williams-avoid partial loss*[8] if for all $f_1, \ldots, f_m \in \mathcal{L}$ with finite supports, and not all of them equal to the zero gamble, there is some $B \in \cup_{j=1}^m S_j(f_j)$ such that

$$\sup_B \left[\sum_{j=1}^{m} G(f_j|\mathcal{B}_j) \right] \geq 0.$$

[8] Although Williams' conditions are not usually expressed in terms of partitions [18], we have opted for uniformity to use a similar notation as in Walley's work.

Observe that the requirement that $f_1, \dots, f_m \in \mathcal{L}$ have finite supports implies that the generic term $G(f_j|\mathcal{B}_j)$ is equivalent to a finite sum:

$$G(f_j|\mathcal{B}_j) = \sum_{B_j \in \mathcal{B}_j} G(f_j|B_j) = \sum_{B_j \in S_j(f_j)} G(f_j|B_j),$$

considered that $G(0|\mathcal{B}_j) = 0$ because of separate coherence. As we have discussed already, this implies that $G(f_j|\mathcal{B}_j)$ is almost-desirable under Williams' theory. In this light, we can see that the intuition behind the above definition is that for every combination of transactions that are almost-desirable to our subject, there should be an event B where the transactions are not trivial (i.e., not all equal to zero), conditional on which the subject cannot lose utiles for all the possible outcomes of the experiment.

The counterpart of this notion in the case of Walley's theory is the following:

Definition 5 (**Avoiding partial loss for lower previsions**). Let $\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$ be conditional lower previsions. They are said to *avoid partial loss* if for all $f_1, \dots, f_m \in \mathcal{L}$, not all of them equal to the zero gamble, there is some $B \in \cup_{j=1}^{m} S_j(f_j)$ such that

$$\sup_B \left[\sum_{j=1}^{m} G(f_j|\mathcal{B}_j) \right] \geq 0.$$

The intuition here is the same as before. Yet, by dropping the requirement that the supports be finite, Walley implicitly assumes that the generic term $G(f_j|\mathcal{B}_j)$ must be almost-desirable even when it is equal to an infinite sum of almost-desirable gambles.

2.1.4. Weak and strong coherence

We next give two notions that generalise the concept of separate coherence to multiple conditional lower previsions:

Definition 6 (**Weak coherence**). Let $\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$ be conditional lower previsions. We say that they are *weakly coherent* if for all $f_0, f_1, \dots, f_m \in \mathcal{L}$ and $B_0 \in \mathcal{B}_j$ for some $j \in \{1, \dots, m\}$, it holds that

$$\sup_\Omega \left[\sum_{i=1}^{m} G(f_i|\mathcal{B}_i) - G(f_0|B_0) \right] \geq 0. \tag{1}$$

With this condition we require that our subject should not be able to raise his supremum acceptable buying price $\underline{P}(f_0|B_0)$ for a gamble f_0 contingent on B_0 by taking into account the implications of other conditional assessments: if Eq. (1) does not hold and the supremum is strictly negative then we can deduce that there is some $\varepsilon > 0$ such that $G(f_0|B_0) - \varepsilon$ is also a desirable gamble, which means that $\underline{P}(f_0|B_0) + \varepsilon$ is an acceptable buying price for f_0 contingent on B_0.

The following lemma rephrases, for the case of partitions, a characterisation of weak coherence given in [9, Theorem 1] in terms of variables.

Lemma 1. *$\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$ are weakly coherent if and only if there is some coherent lower prevision $\underline{P}$ on $\mathcal{L}$ such that for all $j = 1, \dots, m$, it holds that*

$$\underline{P}(G(f|B_j)) = 0 \text{ and } \underline{P}(G(f|\mathcal{B}_j)) \geq 0 \tag{2}$$

for all $f \in \mathcal{L}$ and $B_j \in \mathcal{B}_j$.

However, under the behavioural interpretation, weakly coherent conditional lower previsions can still present some forms of inconsistency with one another. See [16, Chapter 7], [10] and [17] for some discussion. On the other hand, weak coherence neither implies nor is implied by the notion of avoiding partial loss. Because of these two facts, we consider another notion which is stronger than both, and which is called (*joint* or *strong*) *coherence*:

Definition 7 (**Coherence for lower previsions**). Let $\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$ be separately coherent conditional lower previsions on $\mathcal{L}$. We say that they are *coherent*[9] if for all $f_0, f_1, \dots, f_m \in \mathcal{L}$, and for every $B_0 \in \mathcal{B}_j$ for some

[9]The distinction between (joint) coherence and the unconditional notion of coherence from Definition 2 will always be clear from the context. It is also useful to note that the latter is a special case of Eq. (3) obtained when $m = 1$ and $\mathcal{B}_1 = \{\Omega\}$. More generally, when $m = 1$, Eq. (3) can be used as a characterisation of separate coherence.

$j \in \{1, \dots, m\}$, there is some $B \in \{B_0\} \bigcup \cup_{j=1}^m S_j(f_j)$ such that

$$\sup_B \left[\sum_{j=1}^m G(f_j|\mathcal{B}_j) - G(f_0|B_0) \right] \geq 0. \tag{3}$$

The coherence of a collection of conditional lower previsions implies their weak coherence; although the converse does not hold in general, it does in the particular case when we only have a conditional and an unconditional lower prevision $\underline{P}, \underline{P}(\cdot|\mathcal{B})$. To see this, it is enough to reconsider Lemma 1 in the light of the following result by Walley:

Theorem 1 ([16, Theorem 6.5.3]). *Consider an unconditional coherent lower prevision $\underline{P}$ on $\mathcal{L}$ and a separately coherent conditional lower prevision $\underline{P}(\cdot|\mathcal{B})$ on $\mathcal{L}$. $\underline{P}, \underline{P}(\cdot|\mathcal{B})$ are coherent if and only if for all $f \in \mathcal{L}$ and all $B \in \mathcal{B}$ it holds that*

$$\underline{P}(G(f|B)) = 0 \tag{GBR}$$

$$\underline{P}(G(f|\mathcal{B})) \geq 0. \tag{CNG}$$

Condition (GBR) is called the *Generalised Bayes Rule*. When $\underline{P}(B) > 0$, (GBR) can be used to determine the value $\underline{P}(f|B)$: it is then the *unique* value $\mu \in \mathbb{R}$ for which $\underline{P}(B(f - \mu)) = 0$ holds.

On the other hand, (CNG) represents a condition of conglomerability of $\underline{P}$ with respect to the conditional lower prevision $\underline{P}(\cdot|\mathcal{B})$. More generally speaking, we have the following definition:

Definition 8 (**Conglomerability for lower previsions**). Let $\underline{P}$ be a coherent lower prevision on $\mathcal{L}$, and $\mathcal{B}$ a partition of Ω. We say that $\underline{P}$ is $\mathcal{B}$-*conglomerable* if whenever $f \in \mathcal{L}$ and $B_1, B_2, \dots,$ are distinct sets in $\mathcal{B}$ such that $\underline{P}(B_n) > 0$ and $\underline{P}(B_n f) \geq 0$ for all $n \geq 1$, it holds that $\underline{P}(\sum_{n=1}^{\infty} B_n f) \geq 0$.

Note that $\underline{P}$ is trivially $\mathcal{B}$-conglomerable when $\mathcal{B}$ is finite, because of the super-additivity (C3) of coherent lower previsions. On the other hand, the assumption of $\underline{P}(B_n) > 0$, which effectively allows us to deal with countable partitions only, is related to the relationship of conglomerability with sets of *strictly desirable gambles*; see [16, Section 6.8] and Section 2.2 for more information.

Conglomerability and coherence are connected through the following:

Theorem 2 ([16, Theorem 6.8.2]). *Let $\underline{P}$ be a coherent lower prevision on $\mathcal{L}$. Then $\underline{P}$ is $\mathcal{B}$-conglomerable if and only if there is a separately coherent conditional lower prevision $\underline{P}(\cdot|\mathcal{B})$ coherent with $\underline{P}$, that is, such that* (GBR) *and* (CNG) *hold.*

This result helps to see more clearly the connection between this definition of conglomerability for coherent lower previsions and that of the precise case we have recalled in the first part of the Introduction (see also [4]). The latter is equivalent to the equality

$$P(f) = P(P(f|\mathcal{B})) \tag{4}$$

for every gamble f, or, in other words, to $P(f - P(f|\mathcal{B})) = 0$. When we extend this to the imprecise case, we require that $\underline{P}(f - \underline{P}(f|\mathcal{B})) \geq 0$ (i.e., condition (CNG)) and moreover that $\underline{P}(B(f - \underline{P}(f|B))) = 0$ for every B (GBR). Conglomerability in the precise case means that P satisfies (4) with respect to *some* conditional linear prevision $P(\cdot|\mathcal{B})$; it can be checked that if we want a similar property in the imprecise case we must require the inequality in Eq. (CNG). On the other hand, since $\underline{P}$ only determines $\underline{P}(\cdot|B)$ uniquely by means of (GBR) when $\underline{P}(B) > 0$, if we want to give a condition in terms of the unconditional model only we must focus on the conditioning events with positive lower probability. This is another reason for the condition in Definition 8.

The situation is more complicated when we consider two conditional lower previsions. Yet, we can still provide a relatively simple link between coherence and weak coherence, provided that we make some further assumption of positivity (the following result is related to [8, Theorem 11]):

Proposition 1. *Let $\mathcal{B}_1, \mathcal{B}_2$ be two partitions of Ω, and let $\underline{P}(\cdot|\mathcal{B}_1), \underline{P}(\cdot|\mathcal{B}_2)$ be two separately coherent conditional lower previsions. Assume they are weakly coherent with some coherent lower prevision $\underline{P}$ satisfying $\underline{P}(B) > 0$ for all $B \in \mathcal{B}_1 \cup \mathcal{B}_2$ except for one B. Then $\underline{P}(\cdot|\mathcal{B}_1), \underline{P}(\cdot|\mathcal{B}_2)$ are coherent.*

Let us finally give the notion of coherence corresponding to Williams' theory, which can be obtained from Walley's notion by restricting the attention to finite supports (and hence sums):

Definition 9 (**Williams-coherence for lower previsions**). Consider separately coherent conditional lower previsions $\underline{P}(\cdot|\mathcal{B}_1),\dots,\underline{P}(\cdot|\mathcal{B}_m)$ on $\mathcal{L}$. We say that they are *Williams-coherent* if for all $f_0,f_1,\dots,f_m$ with finite supports, and for every $B_0\in\mathcal{B}_j$ for some $j\in\{1,\dots,m\}$, there is some $B\in\{B_0\}\bigcup\cup_{j=1}^m S_j(f_j)$ such that

$$\sup_B\left[\sum_{j=1}^m G(f_j|\mathcal{B}_j)-G(f_0|B_0)\right]\geq 0. \tag{5}$$

An equivalent formulation of the previous definition is the following:

Theorem 3. *Consider separately coherent $\underline{P}(\cdot|\mathcal{B}_1),\dots,\underline{P}(\cdot|\mathcal{B}_m)$ on $\mathcal{L}$. They are Williams-coherent if and only if for all $g_0,\dots,g_n\in\mathcal{L}$, $B_i\in\cup_{j=1}^m\mathcal{B}_j$, $i=0,\dots,n$, it holds that*

$$\sup_B\left[\sum_{i=1}^n G(B_ig_i|\mathcal{B}_{j(i)})-G(B_0g_0|\mathcal{B}_{j(0)})\right]\geq 0, \tag{6}$$

where $B:=\cup_{i=0}^n B_i$ and $j(i)$ denotes an element of $\{1,\dots,m\}$ for which $B_i\in\mathcal{B}_{j(i)}$.

2.1.5. *Linear previsions*

Given a conditional lower prevision $\underline{P}(\cdot|\mathcal{B})$ with domain $\mathcal{L}$, we define its conjugate *conditional upper prevision* by $\overline{P}(f|B):=-\underline{P}(-f|B)$ for every $f\in\mathcal{L}$. As we said at the beginning of the section, the value $\overline{P}(f|B)$ can be interpreted as the infimum acceptable selling price for the gamble f contingent on B. When the supremum acceptable buying price for a gamble coincides with the infimum acceptable selling price, we obtain the so-called *conditional linear previsions*.

Definition 10 (**Linear conditional previsions**). We say that a conditional lower prevision $\underline{P}(\cdot|\mathcal{B})$ with domain $\mathcal{L}$ is *linear* if and only if it is separately coherent and moreover $\underline{P}(f+g|B)=\underline{P}(f|B)+\underline{P}(g|B)$ for all $B\in\mathcal{B}$ and $f,g\in\mathcal{L}$.

When a separately coherent conditional lower prevision $\underline{P}(\cdot|\mathcal{B})$ is linear we shall denote it by $P(\cdot|\mathcal{B})$; in the unconditional case, we shall use the notation P. It can be checked that for conditional linear previsions Definitions 5 and 7 are equivalent: they are coherent if and only if they avoid partial loss.

Conditional linear previsions correspond to conditional expectations with respect to a probability. In particular, an unconditional linear prevision P is the expectation with respect to the finitely additive probability which is the restriction of P to events.

2.1.6. *Extension of conditional lower previsions*

We next show how to determine the behavioural consequences of the assessments modelled by some conditional lower previsions.

Definition 11 (**Natural extension for lower previsions**). Consider separately coherent conditional lower previsions $\underline{P}(\cdot|\mathcal{B}_1),\dots,\underline{P}(\cdot|\mathcal{B}_m)$ on $\mathcal{L}$ that avoid partial loss. Their *natural extensions* are defined, for every gamble f and every $B_0\in\mathcal{B}_k$, with $k\in\{1,\dots,m\}$ by

$$\underline{E}(f|B_0):=\sup\left\{\alpha:\exists f_1,\dots,f_m\in\mathcal{L},B\in\{B_0\}\bigcup\cup_{j=1}^m S_j(f_j)\text{ s.t. }\sup_B\left[\sum_{j=1}^m G(f_j|\mathcal{B}_j)-B_0(f-\alpha)\right]<0\right\}. \tag{7}$$

The natural extensions are conditional lower previsions that 'correct' the initial assessments by taking into account the implications of Walley's notion of coherence (and as a consequence—but only partly—conglomerability). For this reason, they dominate $\underline{P}(\cdot|\mathcal{B}_1),\dots,\underline{P}(\cdot|\mathcal{B}_m)$, and moreover coincide with them if and only if $\underline{P}(\cdot|\mathcal{B}_1),\dots,\underline{P}(\cdot|\mathcal{B}_m)$ are coherent. More in general, they constitute a lower bound of any coherent lower previsions $\underline{Q}(\cdot|\mathcal{B}_1),\dots,\underline{Q}(\cdot|\mathcal{B}_m)$

that dominate $\underline{P}(\cdot|\mathcal{B}_1), \ldots, \underline{P}(\cdot|\mathcal{B}_m)$, even if they may not provide the smallest dominating coherent conditional lower previsions [16, Example 8.1.3]. This contrasts with Williams' notion of coherence, for which we can determine the smallest dominating Williams' coherent conditional lower previsions; see [18, Section 4] and Theorem 7 later on.

We can also use the notion of natural extension to define an unconditional coherent lower prevision $\underline{E}$ out of $\underline{P}(\cdot|\mathcal{B}_1), \ldots, \underline{P}(\cdot|\mathcal{B}_m)$. To this end, it is enough to take B_0 equal to Ω and to apply (7). This will create the so-called *unconditional natural extension* $\underline{E}$ of $\underline{P}(\cdot|\mathcal{B}_1), \ldots, \underline{P}(\cdot|\mathcal{B}_m)$, i.e., a lower prevision on $\mathcal{L}$ that embodies the coherent implications of $\underline{P}(\cdot|\mathcal{B}_1), \ldots, \underline{P}(\cdot|\mathcal{B}_m)$ on unconditional assessments. The form of the unconditional natural extension can be seen in the proof of Lemma 1 in the Appendix (Eq. (A.1)). When the conditional lower previsions $\underline{P}(\cdot|\mathcal{B}_1), \ldots, \underline{P}(\cdot|\mathcal{B}_m)$ are coherent, $\underline{E}$ is the smallest coherent lower prevision that is coherent with them.

2.2. *Sets of desirable gambles*

The above theories can be generalised using *sets of desirable gambles*. In the theory of sets of desirable gambles, it is assumed that we evaluate the gambles in a set $\mathcal{Q} \subseteq \mathcal{L}(\Omega)$, coming up with a subset $\mathcal{R} \subseteq \mathcal{Q}$ of those that represent acceptable transactions for us, i.e., the gambles that we desire. In case $\mathcal{Q} = \mathcal{L}(\Omega)$, the rationality of our desirability assessments is characterised by four axioms.

Definition 12 (**Coherence for gambles**). Let $\mathcal{R}$ be a set of gambles. We consider the following rationality axioms for desirability:

D1. $\mathcal{L}^+ \subseteq \mathcal{R}$.

D2. $0 \notin \mathcal{R}$.

D3. $f \in \mathcal{R}, \lambda > 0 \Rightarrow \lambda f \in \mathcal{R}$.

D4. $f, g \in \mathcal{R} \Rightarrow f + g \in \mathcal{R}$.

A set of desirable gambles satisfying these four axioms is called *coherent* relative to $\mathcal{L}(\Omega)$, or simply coherent.

These axioms are actually derived from a more primitive definition of coherence for $\mathcal{R}$ [11, Definition 12] that holds for any subset $\mathcal{Q}$ of $\mathcal{L}(\Omega)$. A set that satisfies such a definition is called coherent relative to $\mathcal{Q}$. In the following we shall mostly focus on sets of gambles coherent relative to $\mathcal{L}(\Omega)$, although on some occasions we shall consider the remaining situation. We shall point out when this is the case.

Given a set of desirable gambles $\mathcal{R}$, we define

$$\operatorname{posi}(\mathcal{R}) \coloneqq \left\{ \sum_{k=1}^{n} \lambda_k f_k \colon f_k \in \mathcal{R}, \lambda_k > 0, n \geq 1 \right\}.$$

We call $\mathcal{R}$ a *convex cone* if it is closed under positive linear combinations, meaning that $\operatorname{posi}(\mathcal{R}) = \mathcal{R}$. This is equivalent to $\mathcal{R}$ satisfying conditions D3 and D4. Therefore we see that coherent sets of gambles are convex cones that include all the positive gambles, thanks to D1, and exclude all $f \leq 0$; the latter is a consequence of D1, D2 and D4 (see [11, Corollary 2]).

It may happen that $\mathcal{R}$ is not coherent; in this case, it is important that it can at least be extended into a coherent set. We formalise this property by the following definition:

Definition 13 (**Avoiding partial loss for gambles**). Let $\mathcal{R}$ be a set of gambles. We say that $\mathcal{R}$ *avoids partial loss* if it is included in a coherent set.

In fact, in this case (and only then) it is possible to extend $\mathcal{R}$ into a coherent set. The way to do it exploits the natural extension:

Definition 14 (**Natural extension for gambles**). Let $\mathcal{R}$ be a set of gambles. The set $\operatorname{posi}(\mathcal{R} \cup \mathcal{L}^+)$ is called the *natural extension* of $\mathcal{R}$, and corresponds to its smallest coherent superset—provided $\mathcal{R}$ avoids partial loss.

A set of desirable gambles $\mathcal{R}$ avoids partial loss if and only if its natural extension does not include the zero gamble (and hence all $f \leq 0$) [11, Proposition 3(d)]. Equivalently, $\mathcal{R}$ avoids partial loss if and only if there is no gamble $f \leq 0$ in $\operatorname{posi}(\mathcal{R})$.

The description done so far about sets of desirable gambles can be understood as a synthetic overview of the foundations of Williams' theory. This theory is finitary, as we have mentioned already, in particular because axiom D4, once we use it recursively, says that a *finite* sum of desirable gambles is desirable.

This limitation to finite sums may however be perceived as too restrictive in some cases, for example when we consider special gambles like Bf, where B is an element of a partition $\mathcal{B}$ of Ω. Bf should better be understood as a conditional gamble: in fact it rewards our subject with f in case B happens, and the transaction is instead zero (i.e., it is called-off) when B does not.[10]

Now, say that our subject desires gamble Bf for all $B \in \mathcal{B}$. Should we deduce from this that he desires f? Of course we should do it when $\mathcal{B}$ is finite, because this is entailed by D4, but what about the case of an infinite $\mathcal{B}$? Here the situation is controversial: Williams does not assume that such an f should be desirable in the infinite case, while Walley does (and this is the key difference between Definitions 7 and 9). In order to do so, Walley introduces a further axiom in his theory, besides D1– D4, which is an axiom of conglomerability:

Definition 15 (**Conglomerability for gambles**). Let $\mathcal{R}$ be a coherent set of desirable gambles and $\mathcal{B}$ a partition of Ω. $\mathcal{R}$ is called *$\mathcal{B}$-conglomerable* when it satisfies the following axiom:

D5. $f \in \mathcal{L}, Bf \in \mathcal{R} \cup \{0\}$ for all $B \in \mathcal{B} \Rightarrow f \in \mathcal{R} \cup \{0\}$.

Observe that D5 is a consequence of D4 when $\mathcal{B}$ is finite. This notion can be used to define a special type of natural extension [13]:

Definition 16 (**Conglomerable natural extension for gambles**). Given a set of desirable gambles $\mathcal{R}$ and a partition $\mathcal{B}$ of Ω, the *$\mathcal{B}$-conglomerable natural extension* of $\mathcal{R}$, if it exists, is the smallest set $\mathcal{F}$ that contains $\mathcal{R}$ and satisfies D1–D5.

Walley's idea for a general theory of uncertainty is then summarised by axioms D1–D5 that we can find for example in [16, Appendix F1], with an important difference however: Walley requires D5 with respect to all the possible partitions of Ω, which is sometimes referred to as 'full conglomerability'. Throughout this paper instead, we stick to the weaker notion of 'partial conglomerability' where conglomerability is required only with respect to the collection of partitions under consideration. This difference is important and yet it is a relatively minor point in the context of this paper. For this reason, in the following we shall sometimes neglect the difference and just say that Walley's theory is based on axioms D1–D5.

Now we proceed to establish the relations between the different concepts of conglomerability for lower previsions and for sets of desirable gambles. In order to do this, we introduce two additional concepts for sets of desirable gambles.

Definition 17 (**Strict and almost desirability**). A set $\mathcal{R}$ is called a *coherent set of strictly desirable gambles* when it is coherent and moreover

$$\forall f \in \mathcal{R} \setminus \mathcal{L}^+ \, \exists \, \varepsilon > 0 : f - \varepsilon \in \mathcal{R},$$

and it is called a *set of almost-desirable gambles* when it satisfies axiom

D0′. $f + \varepsilon \in \mathcal{R} \, \forall \varepsilon > 0 \Rightarrow f \in \mathcal{R}$,

the following modified versions of axioms D1 and D2:

D1′. $\inf f > 0 \Rightarrow f \in \mathcal{R}$,

D2′. $\sup f < 0 \Rightarrow f \notin \mathcal{R}$,

as well as axioms D3 and D4.

Note that a set of almost-desirable gambles is not a coherent set of desirable gambles: axioms D0′–D1′ imply that any set of almost-desirable gambles includes the zero gamble, and as a consequence it violates D2.

Given a coherent lower prevision $\underline{P}$, we define its associated coherent set of *strictly desirable gambles* by

$$\underline{\mathcal{R}} \coloneqq \mathcal{L}^+ \cup \{f \in \mathcal{L} \colon \underline{P}(f) > 0\}\,, \tag{8}$$

[10] It is useful to observe that the 'conditional' gamble $Bf \in \mathcal{L}(\Omega)$ can be represented also through the gamble $f_B \in \mathcal{L}(B)$ defined by $f_B(\omega) \coloneqq f(\omega)$ for all $\omega \in B$. At the same time, one should keep in mind that Bf and f_B are two logically different objects, because the gamble Bf is defined for all $\omega \in \Omega$, while f_B is not defined outside B.

and its associated set of *almost-desirable gambles* by

$$\overline{\mathcal{R}} \coloneqq \{f \in \mathcal{L} \colon \underline{P}(f) \geq 0\}\,. \tag{9}$$

Moreover, $\underline{\mathcal{R}} \subseteq \overline{\mathcal{R}}$, and $\overline{\mathcal{R}}$ contains all non-negative gambles and is closed under dominance.

Conversely, given a coherent set of desirable gambles $\mathcal{R}$, we can define a lower prevision by

$$\underline{P}(f) \coloneqq \sup\{\mu \colon f - \mu \in \mathcal{R}\} \text{ for all } f \in \mathcal{L}. \tag{10}$$

It follows from [11, Theorem 6] that $\underline{P}$ is a coherent lower prevision. Moreover, if we consider the sets $\underline{\mathcal{R}}$ and $\overline{\mathcal{R}}$ given by Eqs. (8) and (9), it follows from [16, Theorem 3.8.1] that

$$\sup\{\mu \colon f - \mu \in \underline{\mathcal{R}}\} = \underline{P}(f) = \sup\left\{\mu \colon f - \mu \in \overline{\mathcal{R}}\right\}.$$

As a consequence, any set $\mathcal{R}$ such that $\underline{\mathcal{R}} \subseteq \mathcal{R} \subseteq \overline{\mathcal{R}}$ induces the same lower prevision $\underline{P}$ by means of (10) [16, Theorem 3.8.1].

The set $\overline{\mathcal{R}}$ is the closure of $\underline{\mathcal{R}}$ (and as a consequence also of any $\underline{\mathcal{R}} \subseteq \mathcal{R} \subseteq \overline{\mathcal{R}}$) in the topology of uniform convergence [11, Proposition 4]:

$$\overline{\mathcal{R}} = \{f \in \mathcal{L} \colon f + \varepsilon \in \mathcal{R} \text{ for all } \varepsilon > 0\}\,,$$

and on the other hand:

$$\underline{\mathcal{R}} = \mathcal{L}^{+} \cup \{f \in \mathcal{R} \colon f - \varepsilon \in \mathcal{R} \text{ for some } \varepsilon > 0\}\,,$$

for any $\underline{\mathcal{R}} \subseteq \mathcal{R} \subseteq \overline{\mathcal{R}}$.

Hence, any coherent lower prevision is in correspondence with an infinite class of coherent sets of desirable gambles: all the coherent $\mathcal{R}$ such that $\underline{\mathcal{R}} \subseteq \mathcal{R} \subseteq \overline{\mathcal{R}}$.

We can finally use the notion of strict desirability in order to define a variant of the notion of conglomerability for sets of desirable gambles:

Definition 18 (**Weak conglomerability**). A set of desirable gambles $\mathcal{R}$ is called *weakly $\mathcal{B}$-conglomerable* if and only if $\underline{\mathcal{R}}$ is $\mathcal{B}$-conglomerable.

It follows that for a set of strictly desirable gambles, conglomerability and weak conglomerability coincide. We have the following characterisation of weak conglomerability:

Theorem 4 (**[13, Theorem 2]**). *Let $\mathcal{R}$ be a coherent set of desirable gambles. Then $\mathcal{R}$ is weakly $\mathcal{B}$-conglomerable if and only if*

$$f \in \mathcal{L}, Bf \in \underline{\mathcal{R}} \cup \{0\} \text{ for all } B \in \mathcal{B} \Rightarrow f \in \overline{\mathcal{R}}.$$

Moreover, weak conglomerability turns out to be a bridge between the conglomerability properties of sets of desirable gambles and lower previsions:

Theorem 5 (**[13, Theorem 3]**). *Let $\mathcal{R}$ be a coherent set of desirable gambles, and let $\underline{P}$ be the coherent lower prevision it induces by means of Eq.* (10). *Then $\underline{P}$ is $\mathcal{B}$-conglomerable if and only if $\mathcal{R}$ is weakly $\mathcal{B}$-conglomerable.*

This result, together with Theorem 4, implies that a coherent lower prevision $\underline{P}$ is $\mathcal{B}$-conglomerable if and only if its associated set of strictly desirable gambles is $\mathcal{B}$-conglomerable. In addition, this shows that the notion of conglomerability for sets of desirable gambles, given by D5, is stronger (more restrictive) than the one for coherent lower previsions, given in Definition 8.

2.3. New concepts for a theory of conglomerably coherent lower previsions

One aim of this paper is to study how the requirement of conglomerability shapes a theory of coherent lower previsions. In fact, if the focus is on desirable gambles we know that axioms D1–D5 already make up the foundations of such a theory. With coherent lower previsions the situation is less clear, because the link with conglomerability has been only partly explored so far. In particular, we know that Williams' lower previsions do not take conglomerability into account by definition. We know also that Walley's lower previsions are instead related to conglomerability, but we do not know precisely to which extent. The crucial point here is that we do not know how the conglomerability of Walley's lower previsions is related to the conglomerability of sets of desirable gambles: since Walley's theory (as well as Williams') is implicitly derived from the theory of desirable gambles, which is in fact more primitive and fundamental, we expect that the conglomerability properties of coherent lower previsions should be obtained, and especially justified, from sets of desirable gambles.

For the time being, we start by detailing how we understand that the conglomerable properties of coherent lower previsions should be in fact related to those of desirable gambles. We start with a property that concerns only sets of desirable gambles and is a variant of avoiding partial loss. Remember that the original definition of avoiding partial loss for a set of desirable gambles is equivalent to having the possibility to extend it to a coherent set. When we focus on conglomerability, one desirable property would be to be able to extend our assessments to a greater set while preserving conglomerability:

Definition 19 (**Avoiding $\mathcal{B}$-conglomerable partial loss for sets**). Let $\mathcal{R}$ be a set of desirable gambles and $\mathcal{B}$ a partition of Ω. We say that it *avoids $\mathcal{B}$-conglomerable partial loss* if it has a $\mathcal{B}$-conglomerably coherent superset.

Now we focus on a collection of conditional lower previsions $\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$, where $\mathcal{B}_1, \dots, \mathcal{B}_m$ are partitions of Ω. A set of desirable gambles $\mathcal{R}$ induces a conditional lower prevision $\underline{P}(\cdot|\mathcal{B}_i)$ on $\mathcal{L}(\Omega)$ by means of the formula

$$\underline{P}(f|B_i) \coloneqq \sup\{\mu : B_i(f - \mu) \in \mathcal{R}\}, \tag{11}$$

whenever $f \in \mathcal{L}(\Omega)$ and $B_i \in \mathcal{B}_i$.[11] In particular, in the unconditional case (that is, when $\mathcal{B} = \{\Omega\}$) we recover Eq. (10).

We introduce the following notation, for short: when a set of gambles is conglomerable with respect to all the partitions $\mathcal{B}_1, \dots, \mathcal{B}_m$, we write that it is $\mathcal{B}_{1:m}$-conglomerable; we shall use this notation also more generally when we want to refer simultaneously to the conglomerability with respect to all the partitions. Now we are ready to define the new notion of joint coherence for conditional lower previsions, obtained taking conglomerability explicitly into account:

Definition 20 (**Conglomerable coherence for lower previsions**). Let $\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$ be conditional lower previsions. They are called *conglomerably coherent* if there is a $\mathcal{B}_{1:m}$-conglomerable coherent set of desirable gambles that induces them by means of (11).

Now that conglomerable coherence is defined, it is a small step to define also a notion of avoiding partial loss for conditional lower previsions so as to take conglomerability into account:

Definition 21 (**Avoiding conglomerable partial loss for lower previsions**). Let $\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$ be conditional lower previsions. We say that they *avoid conglomerable partial loss* if they have dominating conglomerably coherent extensions.

Remark 1. One could wonder whether or not conglomerably coherent conditional lower previsions can be characterised as lower envelopes of conglomerably coherent conditional linear previsions. Envelope characterisations (or 'envelope theorems') are useful in that they often help both for theoretical developments and applications. It is well known that conditional lower previsions coherent in Williams' sense are in one-to-one relation with envelopes of conditional linear previsions coherent in Williams' (or, which is the same, de Finetti's) sense. It is also known that this holds only in one direction in the case of Walley's: an envelope of coherent conditional linear previsions originates coherent conditional lower previsions, but the converse is not true in general.

It turns out that the situation for conglomerably coherent lower previsions is similar to Walley's. If we focus on the special case made by one unconditional lower prevision $\underline{P}$ and one conditional lower prevision $\underline{P}(\cdot|\mathcal{B})$, we know from [13, Theorem 25] that they are coherent if and only if they are conglomerably coherent; and in [16, Sections 6.6.9–6.6.10] there are examples of a coherent pair $\underline{P}, \underline{P}(\cdot|\mathcal{B})$ that is not dominated by any coherent linear pair $P, P(\cdot|\mathcal{B})$. What we can show, however, is that taking the lower envelope of conglomerably coherent conditional lower previsions originates again conglomerably coherent conditional lower previsions: to see this, it suffices to use that the intersection of a family of conglomerably coherent sets of gambles $\mathcal{F}^\lambda$ (with λ in some index-set Λ) is again conglomerably coherent (see [13, Proposition 5]), and it induces the lower envelope of the conditional lower previsions originated by $\mathcal{F}^\lambda$, $\lambda \in \Lambda$. ♦

[11] Observe that when $\mathcal{R}$ is coherent, then $B_i(f - (\underline{P}(f|B_i) - \varepsilon))$ belongs to $\mathcal{R}$ for all $\varepsilon > 0$: from (11), there must be a positive $\varepsilon' < \varepsilon$ such that $B_i(f - (\underline{P}(f|B_i) - \varepsilon')) \in \mathcal{R}$, and since $B_i(f - (\underline{P}(f|B_i) - \varepsilon)) \geq B_i(f - (\underline{P}(f|B_i) - \varepsilon'))$, we have that $B_i(f - (\underline{P}(f|B_i) - \varepsilon)) \in \mathcal{R}$ because $\mathcal{R}$ is coherent.

2.4. Some final notation and preliminary results

Formula (11) has shown to us how to create a conditional lower prevision from a set of desirable gambles. In the following we shall often need to follow the converse path and induce a set of desirable gambles from one or more conditional lower previsions. We shall use the following formulas and notational conventions:

- A separately coherent conditional lower prevision $\underline{P}(\cdot|\mathcal{B}_i)$ on $\mathcal{L}$ induces a coherent set of strictly desirable gambles on B_i for each conditioning event $B_i \in \mathcal{B}_i$:

$$\mathcal{R}_i\rfloor B_i := \{g \in \mathcal{L}(B_i) : \underline{P}(g|B_i) > 0 \text{ or } g \gneq 0\};$$

  this follows from [16, Theorem 3.8.1]. We represent this set equivalently by gambles on the entire possibility space Ω as follows:

$$\mathcal{R}_i|B_i := \{G(f|B_i) + \varepsilon B_i : f \in \mathcal{L}(\Omega), \varepsilon > 0\} \cup \{f \in \mathcal{L}(\Omega) : f = B_i f \gneq 0\}, \tag{12}$$

  using the equivalence: $\underline{P}(B_i g|B_i) > 0$ if and only if $B_i g = G(f|B_i) + \varepsilon B_i$ for some $\varepsilon > 0$. It is not difficult to see that $\mathcal{R}_i|B_i$ satisfies axioms D2–D4, and that it is coherent relative to $\mathcal{Q} := \{f \in \mathcal{L}(\Omega) : f = B_i f\}$ (in the sense of [11, Definition 12]). Any gamble in $\mathcal{R}_i|B_i$ is a gamble on Ω that is zero outside B_i, and as a consequence there is a one-to-one correspondence between $\mathcal{R}_i|B_i$ and $\mathcal{R}_i\rfloor B_i$.

  The natural extension of the sets $\mathcal{R}_i|B_i$ ($B_i \in \mathcal{B}_i$) is given by

$$\mathcal{E}_i := \text{posi}(\mathcal{L}^+(\Omega) \cup (\cup_{B_i \in \mathcal{B}_i} \mathcal{R}_i|B_i)). \tag{13}$$

- From [20, Proposition 1], their $\mathcal{B}_i$-conglomerable natural extension is instead

$$\mathcal{F}_i := \left\{ f \in \mathcal{L} : 0 \neq f = \sum_{B_i \in \mathcal{B}_i} B_i f_i, B_i f_i \in \mathcal{R}_i|B_i \cup \{0\} \right\}. \tag{14}$$

  This is the smallest $\mathcal{B}_i$-conglomerably coherent set of desirable gambles that extends the originating sets. Obviously, it need not be $\mathcal{B}_j$-conglomerable for another partition $\mathcal{B}_j$, and actually we can show the following:

  Proposition 2. *$\mathcal{F}_i$ is $\mathcal{B}_j$-conglomerable if and only if $\mathcal{R}_i|B_i$ is $\mathcal{B}_j$-conglomerable for all $B_i \in \mathcal{B}_i$.*

- Similarly, the collection of separately coherent conditional lower previsions $\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$ induces the overall set of desirable gambles $\cup_{i=1}^m \cup_{B_i \in \mathcal{B}_i} \mathcal{R}_i|B_i$, whose natural extension is given by

$$\mathcal{E} := \text{posi}(\mathcal{L}^+ \cup (\cup_{i=1}^m \cup_{B_i \in \mathcal{B}_i} \mathcal{R}_i|B_i)) = \text{posi}(\cup_{i=1}^m \mathcal{E}_i), \tag{15}$$

  where $\mathcal{E}_i$ is given by Eq. (13). We shall also consider the natural extension of the sets $\mathcal{F}_1, \dots, \mathcal{F}_m$, which is equal to

$$\mathcal{F}_1 \oplus \dots \oplus \mathcal{F}_m := \left\{ f \in \mathcal{L} : 0 \neq f = \sum_{i=1}^m f_i, f_i \in \mathcal{F}_i \cup \{0\} \right\},$$

  taking into account that these sets are coherent. Note, however, that this is not necessarily a coherent set, because it may be that $\cup_{i=1}^m \mathcal{F}_i$ incurs partial loss; it will only be the smallest coherent superset of $\cup_{i=1}^m \mathcal{F}_i$ when one such superset exists.

- Finally, we shall often denote by $\mathcal{F}$ the $\mathcal{B}_{1:m}$-conglomerable natural extension of $\cup_{i=1}^m \cup_{B_i \in \mathcal{B}_i} \mathcal{R}_i|B_i$, provided that it exists. Note that $\mathcal{F}$ is also the $\mathcal{B}_{1:m}$-conglomerable natural extension of $\cup_{i=1}^m \mathcal{F}_i$, because any $\mathcal{B}_{1:m}$-conglomerable superset of $\cup_{i=1}^m \cup_{B_i \in \mathcal{B}_i} \mathcal{R}_i|B_i$ must include in particular the $\mathcal{B}_i$-conglomerable natural extension of $\cup_{B_i \in \mathcal{B}_i} \mathcal{R}_i|B_i$, and this for $i = 1, \dots, m$.

Using some of these notations we can re-formulate one of Williams' basic results in our language, where lower previsions are conditional on partitions:

Proposition 3. *Let $\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$ be separately coherent conditional lower previsions. Let $\mathcal{E}$ be the natural extension of the desirable gambles they induce, given by Eq.* (15)*. It follows that:*

1. *If $\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$ are Williams-coherent, then $\mathcal{E}$ is coherent.*
2. *Moreover, $\mathcal{E}$ induces the conditional lower previsions $\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$ by means of* (11).

3. Fundamental relationships about consistency notions

3.1. Avoiding (conglomerable) partial loss

Let us start by illustrating the relationships that exist between the notions of avoiding (conglomerable) partial loss for desirable gambles and coherent conditional lower previsions.

Theorem 6. *Let $\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$ be separately coherent conditional lower previsions, and let*

$$\mathcal{R} := \cup_{i=1}^{m} \cup_{B_i \in \mathcal{B}_i} \mathcal{R}_i | B_i, \tag{16}$$

where the sets of gambles $\mathcal{R}_i|B_i$ are determined by Eq. (12). *Then*

1. *If $\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$ avoid partial loss, then $\mathcal{R}$ avoids partial loss.*
2. *$\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$ avoid conglomerable partial loss if and only if the $\mathcal{B}_{1:m}$-conglomerable natural extension $\mathcal{F}$ of $\mathcal{R}$ exists.*
3. *The smallest dominating conglomerably coherent extensions are induced by the $\mathcal{B}_{1:m}$-conglomerable natural extension $\mathcal{F}$ of $\mathcal{R}$.*

We can do something similar with respect to Williams-avoiding partial loss.

Theorem 7. *Let $\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$ be separately coherent conditional lower previsions, and let $\mathcal{R}$ be given by* (16). *Then*

1. *If $\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$ Williams-avoid partial loss, then $\mathcal{R}$ avoids partial loss.*
2. *If $\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$ Williams-avoid partial loss, then the smallest dominating Williams-coherent extensions are those induced by the natural extension $\mathcal{E}$ of $\mathcal{R}$.*

3.2. Coherence, Williams-coherence, and conglomerable coherence

Now we move on to characterise the different forms of coherence. We start by a preliminary result: we detail how the coherence properties of a set of desirable gambles affect those of the conditional lower previsions it induces.

Theorem 8. *Let $\mathcal{R}$ be a coherent set of desirable gambles, and for every $i = 1, \dots, m$, $B_i \in \mathcal{B}_i$, let $\underline{P}(\cdot|\mathcal{B}_i)$ denote the conditional lower prevision on $\mathcal{L}$ it induces by* (11). *Then:*

1. *$\underline{P}(\cdot|\mathcal{B}_i)$ is separately coherent for all $i = 1, \dots, m$.*
2. *$\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$ are Williams-coherent.*
3. *If $\mathcal{R}$ is in addition $\mathcal{B}_{1:m}$-conglomerable, then $\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$ are coherent.*[12]

From Theorem 8 and Proposition 3, a finite collection of conditional lower previsions $\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$ are Williams' coherent if and only if they are induced by a coherent set of desirable gambles. In the remainder of this section, we shall show that when we let conglomerability enter the picture, Walley's notion of coherence is not equivalent to conglomerable coherence, and therefore it is not true in general that a finite set of conditional lower previsions $\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$ are coherent if and only if they are induced by a conglomerably coherent set of desirable gambles: conglomerable coherence is sufficient, but not necessary, for (Walley's) coherence.

Theorem 9. *Let $\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$ be separately coherent conditional lower previsions. For all $i = 1, \dots, m$, let $\mathcal{F}_i$ be the $\mathcal{B}_i$-conglomerable natural extension of $\cup_{B_i \in \mathcal{B}_i} \mathcal{R}_i | B_i$, as given by* (14). *Then*

[12] This last statement has already been mentioned in [16, Appendix F3] and proved within [16, Sections 7.1.2 and 7.1.4].

1. $\underline{P}(\cdot|\mathcal{B}_1),\ldots,\underline{P}(\cdot|\mathcal{B}_m)$ *are conglomerably coherent if and only if the* $\mathcal{B}_{1:m}$*-conglomerable natural extension* $\mathcal{F}$ *of* $\cup_{i=1}^m \mathcal{F}_i$ *exists and it induces them by means of Eq.* (11)*. In the case of linear conditional previsions for the equivalence it suffices that the* $\mathcal{B}_{1:m}$*-conglomerable natural extension* $\mathcal{F}$ *exists.*

2. *If* $\underline{P}(\cdot|\mathcal{B}_1),\ldots,\underline{P}(\cdot|\mathcal{B}_m)$ *are conglomerably coherent, then* $\mathcal{F}_i$ *is* $\mathcal{B}_j$*-conglomerable for all* i,j *in* $\{1,\ldots,m\}$*.*

3. *If* $\underline{P}(\cdot|\mathcal{B}_1),\ldots,\underline{P}(\cdot|\mathcal{B}_m)$ *are conglomerably coherent, then they are coherent.*

Remark 2. Note that if $\underline{P}(\cdot|\mathcal{B}_1),\underline{P}(\cdot|\mathcal{B}_2)$ are coherent and we take their associated sets of desirable gambles $\mathcal{F}_1,\mathcal{F}_2$ (which are coherent and conglomerable), it can be that $\mathcal{F}_1\cup\mathcal{F}_2$ is not coherent: consider $\Omega:=\{1,2,3,4\}$, $\mathcal{B}_1:=\{\{1,2\},\{3,4\}\}$, $P(f|\{1,2\}):=\frac{f(1)+f(2)}{2}$, $P(f|\{3,4\}):=f(3)$, $\mathcal{B}_2:=\{\{1,3\},\{2,4\}\}$, $P(f|\{1,3\}):=\frac{f(1)+f(3)}{2}$, $P(f|\{2,4\}):=f(2)$. Then these are the conditional previsions induced by the mass function $(\frac{1}{3},\frac{1}{3},\frac{1}{3},0)$, which moreover satisfies $P(B)>0$ for all $B\in\mathcal{B}_1\cup\mathcal{B}_2$. Applying [8, Theorem 5], $P(\cdot|\mathcal{B}_1),P(\cdot|\mathcal{B}_2)$ are coherent. On the other hand, the gamble $f:=(3,-2,1,0)\in\mathcal{F}_1$ and $g:=(3,1,-2,0)\in\mathcal{F}_2$, but $f+g=(6,-1,-1,0)\notin\mathcal{F}_1\cup\mathcal{F}_2$, so this union is not coherent.

This is why in Theorem 9 we must consider a natural extension of the set $\cup_{i=1}^m\mathcal{F}_i$ (and in particular its conglomerable natural extension $\mathcal{F}$). ♦

At this point we have characterised some important relationships between coherence and conglomerable coherence. Yet, we have not addressed the most important issue: whether or not these two notions are equivalent. The next example settles the problem showing that they are not, and hence—using Theorem 9(3)—that conglomerable coherence is indeed stronger than coherence.

Example 1. Let Ω be the set of natural numbers (without zero), and a coherent lower prevision $\underline{P}$ on $\mathcal{L}(\Omega)$ which is not $\mathcal{B}$-conglomerable for some partition $\mathcal{B}$ of Ω but such that there exists a dominating $\mathcal{B}$-conglomerable linear prevision with $P(B)>0$ for all $B\in\mathcal{B}$ (one such $\underline{P}$ is given in [13, Example 5]).

Let us define $\Omega_1:=\Omega\cup-\Omega$, and the partitions of Ω_1

$$\mathcal{B}_1:=\{\Omega,-\Omega\},\text{ and }\mathcal{B}_2:=\{B\cup-B:B\in\mathcal{B}\}.$$

Define $\underline{P}(\cdot|\mathcal{B}_1)$ on $\mathcal{L}(\Omega_1)$ by $\underline{P}(f|\Omega):=\underline{P}(f^+)$ and $\underline{P}(f|-\Omega):=\underline{P}(f^-)$, where

$$\begin{aligned}f^+:\Omega&\to\mathbb{R}\\ \omega&\hookrightarrow f(\omega)\end{aligned}\quad\text{and}\quad\begin{aligned}f^-:\Omega&\to\mathbb{R}\\ \omega&\hookrightarrow f(-\omega).\end{aligned}\tag{17}$$

It follows from the coherence of $\underline{P}$ that $\underline{P}(\cdot|\mathcal{B}_1)$ is separately coherent.

From the linear prevision P on $\mathcal{L}$ considered above we can derive a linear prevision P_1 on $\mathcal{L}(\Omega_1)$ by $P_1(f):=P(f^+)$, where f^+ is given by Eq. (17). Then P_1 is a linear prevision satisfying $P_1(B\cup-B)=P(B)>0$ for any $B\in\mathcal{B}$, and moreover $P_1(\Omega)=1$. Define $P_1(\cdot|\mathcal{B}_2)$ by (GBR). Then for any gamble f on Ω_1 it holds that

$$P_1(G_1(f|\mathcal{B}_2))=P_1(\mathbb{I}_\Omega G_1(f|\mathcal{B}_2))=P(G(f|\mathcal{B}))\geq 0,$$

where: $G(f|\mathcal{B})=f-P(f|\mathcal{B})$ and $P(f|\mathcal{B})$ is obtained by P through Bayes' rule (remember that we can do so because $P(B)>0$ for all $B\in\mathcal{B}$); and using in the last passage that P is $\mathcal{B}$-conglomerable by assumption. This means that $P_1,P_1(\cdot|\mathcal{B}_2)$ are coherent.

On the other hand, if we consider the conditional lower prevision $\underline{P}_1(\cdot|-\Omega):=\underline{P}(\cdot|-\Omega)$, it holds that $P_1(G_1(f|-\Omega))=0$ because $P_1(-\Omega)=0$, where $G_1(f|-\Omega)=\mathbb{I}_{-\Omega}(f-\underline{P}(f|-\Omega))$. Define $P_1(\cdot|\Omega)=:\underline{P}_1(\cdot|\Omega)$ from P_1 by (GBR). Then we deduce that $P_1,\underline{P}_1(\cdot|\mathcal{B}_1)$ satisfy (GBR), and since $\mathcal{B}_1$ is finite, Theorem 1 implies that $P_1,\underline{P}_1(\cdot|\mathcal{B}_1)$ are coherent. Hence, $P_1,\underline{P}_1(\cdot|\mathcal{B}_1),P_1(\cdot|\mathcal{B}_2)$ are weakly coherent, and applying Proposition 1 we deduce that $\underline{P}_1(\cdot|\mathcal{B}_1),P_1(\cdot|\mathcal{B}_2)$ are coherent.

Similarly, if we consider the linear prevision P_2 on $\mathcal{L}(\Omega_1)$ given by $P_2(f):=P(f^-)$, we can repeat the above reasoning and define $P_2(\cdot|\mathcal{B}_2)$ and $P_2(\cdot|-\Omega)=:\underline{P}_2(\cdot|-\Omega)$ by (GBR), and let $\underline{P}_2(\cdot|\Omega)$ be equal to $\underline{P}(\cdot|\Omega)$ and we conclude that $\underline{P}_2(\cdot|\mathcal{B}_1),P_2(\cdot|\mathcal{B}_2)$ are coherent. By taking lower envelopes, we obtain coherent $\underline{Q}(\cdot|\mathcal{B}_1),\underline{Q}(\cdot|\mathcal{B}_2)$ (see [16, Theorem 7.1.6]), and the above construction implies that $\underline{Q}(\cdot|\mathcal{B}_1)=\underline{P}(\cdot|\mathcal{B}_1)$, taking into account that

$$\underline{P}_1(f|\Omega)=P_1(f|\Omega)=P(f^+)\geq\underline{P}(f^+)=\underline{P}(f|\Omega)=\underline{P}_2(f|\Omega)$$

and

$$\underline{P}_2(f|-\Omega) = P_2(f|-\Omega) = P(f^-) \geq \underline{P}(f^-) = \underline{P}(f|-\Omega) = \underline{P}_1(f|-\Omega)$$

for any gamble f.

Now, assume ex-absurdo that $\underline{P}(\cdot|\mathcal{B}_1), \underline{Q}(\cdot|\mathcal{B}_2)$ are conglomerably coherent. Then Theorem 9(2) implies that the set $\mathcal{F}_1$ induced by $\underline{P}(\cdot|\mathcal{B}_1)$ is $\mathcal{B}_2$-conglomerable, and Proposition 2 implies then that $\mathcal{R}_1|\Omega$ is $\mathcal{B}_2$-conglomerable. But

$$\mathcal{R}_1|\Omega = \{G(f|\Omega) + \varepsilon\Omega : f \in \mathcal{L}(\Omega_1), \varepsilon > 0\} \cup \{f \in \mathcal{L}(\Omega_1) : f = \mathbb{I}_\Omega f \gneq 0\}$$

is in a one-to-one correspondence with the set $\mathcal{R}_1\rfloor\Omega$, which coincides with the set of strictly desirable gambles induced by $\underline{P}$. From Theorem 5, since $\underline{P}$ is not $\mathcal{B}$-conglomerable, its associated set of strictly desirable gambles is not $\mathcal{B}$-conglomerable, whence there is some $0 \neq f \in \mathcal{L}(\Omega)$ such that

$$\underline{P}(Bf) > 0 \text{ or } Bf \geq 0 \text{ for all } B \in \mathcal{B}, \tag{18}$$

while f is not strictly desirable with respect to $\underline{P}$, meaning that $f \ngeq 0$ and $\underline{P}(f) \leq 0$. This means that if we consider the gamble $0 \neq \mathbb{I}_\Omega f \in \mathcal{L}(\Omega_1)$, it holds that $\mathbb{I}_\Omega f \ngeq 0, \underline{P}(\mathbb{I}_\Omega f|\Omega) = \underline{P}(f) \leq 0$, so $\mathbb{I}_\Omega f$ does not belong to $\mathcal{R}_1|\Omega$, while

$$(\underline{P}(B_2\mathbb{I}_\Omega f|\Omega) > 0 \text{ or } B_2\mathbb{I}_\Omega f \geq 0)\forall B_2 \in \mathcal{B}_2 \Rightarrow B_2\mathbb{I}_\Omega f \in \mathcal{R}_1|\Omega \cup \{0\}\ \forall B_2 \in \mathcal{B}_2,$$

using Eq. (18). Hence, $\mathcal{R}_1|\Omega$ is not $\mathcal{B}_2$-conglomerable, whence neither is $\mathcal{F}_1$ and as a consequence $\underline{P}(\cdot|\mathcal{B}_1), \underline{Q}(\cdot|\mathcal{B}_2)$ cannot be conglomerably coherent. ♦

This finding is important because it tells us that Walley's notion of coherence does not take into account all the implications of conglomerability, a requirement Walley himself had been arguing in favour of. In this light, the state of affairs with respect to the different coherence notions can be summarised as follows: Williams-coherence is the fundamental coherence notion for the case where conglomerability is not required; conglomerable coherence, on the other hand, should be the notion used in the opposite case.

It remains to understand what is the role of Walley's notion. We can be helped in this by the following theorem.

Theorem 10. *Let $\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$ be coherent conditional lower previsions on $\mathcal{L}$, and let us define the sets of gambles $\mathcal{F}_1, \dots, \mathcal{F}_m$ by Eq. (14). Let $\mathcal{F}' := \mathcal{F}_1 \oplus \dots \oplus \mathcal{F}_m$ be the natural extension of $\mathcal{F}_1, \dots, \mathcal{F}_m$. Then:*

1. *$\mathcal{F}'$ induces the conditional lower previsions $\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$.*
2. *$\mathcal{F}'$ is weakly $\mathcal{B}_{1:m}$-conglomerable.*

Hence, coherent conditional lower previsions $\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$ are always *weakly conglomerably coherent*, in the sense that they are always induced by a coherent set of desirable gambles which is weakly $\mathcal{B}_{1:m}$-conglomerable. However, they are not conglomerably coherent in general, as Example 1 points out. In addition, our next example shows also that the coherence of conditional lower previsions is not necessarily equivalent to being induced by a weakly conglomerable set:

Example 2. Let Ω be the set of natural numbers (without zero), $\mathcal{B} := \{\{2n-1, 2n\} : n \in \Omega\}$, and let $\mathcal{R}$ be the set of gambles

$$\left\{f : (\exists n \in \Omega) f\mathbb{I}_{\{n,n+1,\dots\}} \in \mathcal{L}^+ \text{ and } (\forall n \in \Omega)(\min\{f(2n) + f(2n-1), f(2n)\} \geq 0)\right\}.$$

Examples 2 and 3 in [13] show that this is a coherent set of desirable gambles that is weakly $\mathcal{B}$-conglomerable but not $\mathcal{B}$-conglomerable. Moreover, it is trivially $\mathcal{B}'$-conglomerable if we consider the partition $\mathcal{B}' := \{\Omega\}$. Let $\underline{P}, \underline{P}(\cdot|\mathcal{B})$ be the lower previsions it induces. To see that they are not coherent, consider the gamble $f := \mathbb{I}_{\{2n:n\in\Omega\}} - \mathbb{I}_{\{2n-1:n\in\Omega\}}$. Then

$$\underline{P}(f|\{2n-1, 2n\}) = 0\ \forall n \in \Omega \Rightarrow f = G(f|\mathcal{B}),$$

and

$$\underline{P}(G(f|\mathcal{B})) = \underline{P}(f) = \sup\{\mu : f - \mu \in \mathcal{R}\} = -1,$$

as showed in [13, Example 4]. Hence, $\underline{P}, \underline{P}(\cdot|\mathcal{B})$ are not coherent. ♦

Conglomerable coherence

$\downarrow$

Walley's coherence

$\downarrow$

Weak conglomerable coherence

$\downarrow$

Williams' coherence

Figure 1: Relationships between the coherence conditions.

We summarise the relationships among the different coherence conditions in Figure 1.

Overall, we obtain that from Walley's notion of coherence we can only deduce that the conditional lower previsions are weakly conglomerably coherent, and more broadly speaking, that such a notion can only be regarded as an approximation to conglomerable coherence. Such an approximation, however, matches conglomerable coherence in some important cases. This is detailed in the next sections.

4. Weak coherence and positivity

From Lemma 1, if the conditional lower previsions $\underline{P}(\cdot|\mathcal{B}_1), \ldots, \underline{P}(\cdot|\mathcal{B}_m)$ are weakly coherent, then there is an unconditional lower prevision $\underline{P}$ on $\mathcal{L}$ (their unconditional natural extension) that is pairwise coherent with them. We can use this property to show that there is a direct connection between weak coherence and conglomerability:

Theorem 11. *$\underline{P}(\cdot|\mathcal{B}_1), \ldots, \underline{P}(\cdot|\mathcal{B}_m)$ are weakly coherent if and only if there are coherent sets $\mathcal{R}, \mathcal{F}_1, \ldots, \mathcal{F}_m$ and a coherent lower prevision $\underline{P}$ such that for all $i = 1, \ldots, m$ $\mathcal{R} \cup \mathcal{F}_i$ is $\mathcal{B}_i$-conglomerably coherent and it induces $\underline{P}, \underline{P}(\cdot|\mathcal{B}_i)$.*

However, since the coherence of $\underline{P}(\cdot|\mathcal{B}_1), \ldots, \underline{P}(\cdot|\mathcal{B}_m)$ is a stronger notion that their weak coherence, we conclude that conglomerable coherence implies weak coherence but the converse is not true.

It is possible to create a stronger link between weak coherence and conglomerability by assuming that a further condition of positivity holds. In order to show this, we first need a lemma.

Lemma 2. *Let $\underline{P}(\cdot|\mathcal{B}_1), \underline{P}(\cdot|\mathcal{B}_2)$ be coherent conditional lower previsions. Consider $B_1 \in \mathcal{B}_1, B_2 \in \mathcal{B}_2, f \in \mathcal{L}$. Then $\underline{P}(B_1 f|B_2) > 0$ implies that $\underline{P}(B_2 f|B_1) \geq 0$.*

Example 3. We may have $\underline{P}(B_1 f|B_2) > 0$ and $\underline{P}(B_2 f|B_1) = 0$. To see this, let Ω be the set of natural numbers (without zero), $B_n := \{2n-1, 2n\}$, $\mathcal{B}_1 := \{B_n : n \in \Omega\}$, $\mathcal{B}_2 := \{\Omega\}$ and $\underline{P}(\cdot|\mathcal{B}_1), \underline{P}$ be the vacuous conditional and unconditional lower previsions. From [16, Section 6.6.1], $\underline{P}(\cdot|\mathcal{B}_1), \underline{P}$ are coherent. If we let f be the indicator function of B_n, then $\underline{P}(f|B_n) = 1 > 0$, while $\underline{P}(B_n f) = 0$. ♦

At this point, we can start proving a preliminary result that exploits the positivity we mentioned.

Theorem 12. *Let $\underline{P}, \underline{P}(\cdot|\mathcal{B}_1), \ldots, \underline{P}(\cdot|\mathcal{B}_m)$ be coherent lower previsions. For any $i = 1, \ldots, m$, $B_i \in \mathcal{B}_i$, if $\underline{P}(B_i) > 0$ then $\mathcal{R}_i|B_i$ is $\mathcal{B}_j$-conglomerable for all $j = 1, \ldots, m$.*

Now, if the unconditional lower prevision $\underline{P}$ satisfies $\underline{P}(B) > 0$ for all $B \in \cup_{i=1}^m \mathcal{B}_i$, we finally deduce that $\underline{P}(\cdot|\mathcal{B}_1), \ldots, \underline{P}(\cdot|\mathcal{B}_m)$ are conglomerably coherent:

Theorem 13. *Let $\underline{P}(\cdot|\mathcal{B}_1), \ldots, \underline{P}(\cdot|\mathcal{B}_m)$ be separately coherent conditional lower previsions on $\mathcal{L}$ which are weakly coherent with some coherent lower prevision $\underline{P}$ satisfying that $\underline{P}(B) > 0$ for all $B \in \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_m$. Then:*

1. *$\mathcal{F}_i$ is $\mathcal{B}_j$-conglomerable for $i, j = 1, \dots, m$.*

2. *$\mathcal{F}_1 \oplus \dots \oplus \mathcal{F}_m$ is $\mathcal{B}_{1:m}$-conglomerable.*

3. *$\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$ are conglomerably coherent.*

Note that the hypothesis that all the sets in a partition have positive lower probability implies that such a partition is necessarily countable.

5. Nested partitions

Next, we focus on another important setup where coherence implies conglomerable coherence. Such a setup is characterised by partitions that are nested, that is, finer and finer. This situation is common in some applications of probability. For example, developing a knowledge-based system often means to create a joint probabilistic model over some variables of interest; this joint model, which we can think of as a coherent lower prevision $\underline{P}$ on $\mathcal{L}$, is then queried by computing a lower prevision conditional on the observation of some of the variables, which we can represent by $\underline{P}(\cdot|\mathcal{B}_1)$ (the elements of $\mathcal{B}_1$ correspond in this case to the elements of the variables that are observed). When more and more variables become observed, by querying the system accordingly, we create a sequence of conditional lower previsions $\underline{P}(\cdot|\mathcal{B}_i)$, $i = 1, \dots, m$ related to the sequence of observations, with the property that their corresponding partitions are finer and finer: this happens because observations accumulate, that is, because we keep on adding variables after the conditioning bar without ever removing any of them.

In order to study the case of nested partitions we first focus on a technical result: we deduce some coherence conditions similar to those in Theorem 1, but this time we establish them in the case of two *conditional* lower previsions.

Lemma 3. *Let $\mathcal{B}_1, \mathcal{B}_2$ be two partitions of Ω, and assume that $\mathcal{B}_2$ is finer than $\mathcal{B}_1$. For every $B_1 \in \mathcal{B}_1$, denote by $\mathcal{B}_2(B_1)$ the partition of B_1 given by $\{B_2 \in \mathcal{B}_2 : B_2 \subseteq B_1\}$. Let $\underline{P}(\cdot|\mathcal{B}_1), \underline{P}(\cdot|\mathcal{B}_2)$ be two separately coherent conditional lower previsions on $\mathcal{L}$. If $\underline{P}(\cdot|\mathcal{B}_1), \underline{P}(\cdot|\mathcal{B}_2)$ are coherent, then*

$$\underline{P}(G(f|B_2)|B_1) = 0 \text{ and } \underline{P}(G(f|\mathcal{B}_2)|B_1) \geq 0$$

for all $f \in \mathcal{L}$, $B_1 \in \mathcal{B}_1$, $B_2 \in \mathcal{B}_2(B_1)$.

Now we move on to the general case made of partitions $\mathcal{B}_1, \dots, \mathcal{B}_m$ of Ω with the property that $\mathcal{B}_j$ is finer than $\mathcal{B}_{j-1}$ for all $j = 2, \dots, m$. Consider the corresponding separately coherent $\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$ on $\mathcal{L}$. For every $j = 1, \dots, m-1$, $B_j \in \mathcal{B}_j$ and $j' > j$, we define the following partition of B_j:

$$\mathcal{B}_{j'}(B_j) := \{B_{j'} \in \mathcal{B}_{j'} : B_{j'} \subseteq B_j\}.$$

Since this is a partition of B_j, it makes sense to study if a subset of $\mathcal{L}(B_j)$ is $\mathcal{B}_{j'}(B_j)$-conglomerable, i.e., if it satisfies D5. Let us also define the sets of gambles $\mathcal{F}_1, \dots, \mathcal{F}_m$ by Eq. (14).

The next theorem settles the case for nested partitions:

Theorem 14. 1. *$\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$ are coherent if and only if for all $f \in \mathcal{L}$, $B_i \in \mathcal{B}_i$, $i = 1, \dots, m-1$, $B_j \in \mathcal{B}_j(B_i)$, $j > i$, it holds that*

$$\underline{P}(G(f|B_j)|B_i) = 0 \tag{19}$$

$$\underline{P}(G(f|\mathcal{B}_j)|B_i) \geq 0. \tag{20}$$

2. *If $\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$ are coherent, then $\mathcal{R}_i \rfloor B_i$ is $\mathcal{B}_j(B_i)$-conglomerable and $\mathcal{R}_i|B_i$ is $\mathcal{B}_j$-conglomerable for all $B_i \in \mathcal{B}_i$, $i = 1, \dots, m$, $j > i$.*

3. *If $\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$ are coherent, then $\mathcal{F}_i$ is $\mathcal{B}_j$-conglomerable for all $i, j = 1, \dots, m$.*

4. *If $\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$ are coherent, then $\mathcal{F}_1 \oplus \dots \oplus \mathcal{F}_m$ is $\mathcal{B}_{1:m}$-conglomerable.*

5. *If $\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$ are coherent, then they are conglomerably coherent.*

This implies in particular that the notions of coherence and of conglomerable coherence are equivalent when we consider an unconditional lower prevision $\underline{P}$ and a conditional lower prevision $\underline{P}(\cdot|\mathcal{B})$ on $\mathcal{L}$. Hence, this result generalises [13, Theorem 25(1)].

On the other hand, the first point of Theorem 14 can actually be simplified as for it to hold it suffices to focus on consecutive partitions:

Corollary 1. *Let $\mathcal{B}_1, \ldots, \mathcal{B}_m$ be partitions of Ω such that $\mathcal{B}_j$ is finer than $\mathcal{B}_{j-1}$ for all $j = 2, \ldots, m$. Consider separately coherent conditional lower previsions $\underline{P}(\cdot|\mathcal{B}_1), \ldots, \underline{P}(\cdot|\mathcal{B}_m)$ on $\mathcal{L}$. Then $\underline{P}(\cdot|\mathcal{B}_1), \ldots, \underline{P}(\cdot|\mathcal{B}_m)$ are coherent if and only if for all gambles $f \in \mathcal{L}$, $B_{j-1} \in \mathcal{B}_{j-1}$, $B_j \in \mathcal{B}_j$, it holds that*

$$\underline{P}(G(f|B_j)|B_{j-1}) = 0 \tag{21}$$

$$\underline{P}(G(f|\mathcal{B}_j)|B_{j-1}) \geq 0. \tag{22}$$

6. Conclusions

When we restrict the attention to finite possibility spaces, the quest for a well-founded behavioural theory of coherent lower previsions was solved by Williams in 1975 [19]. Not long ago, the situation was not so clear for the more general case of infinite spaces (and in particular of infinite partitions), because of the controversy originated by conglomerability: there was uncertainty as to whether Williams' or Walley's theory should be opted for in that case.

This paper, together with some other recent work, seems to allow us to say a few conclusive words on this question: now we know in which cases conglomerability is justified from the behavioural point of view, based on considerations of consistency between present and future attitudes to gambles [20]; and we know that Walley's theory of coherent lower previsions does not consider all the implications of conglomerability. This has been initially shown in a recent paper [13] with respect to Walley's procedure of natural extension, and in this paper with regard to Walley's axiom of joint coherence, which is at the heart of this theory. These are important findings, which tell us that Walley's should be understood as an approximation to the theory of conglomerably coherent lower previsions described in the present paper.

We have showed that in two special cases we can use Walley's theory obtaining the same outcomes as with conglomerable coherence: when the coherent conditional lower previsions have a compatible lower prevision that assigns positive lower probability to all the conditioning events, and when the conditioning partitions are nested. Both cases are important in the applications of probability.

These results have implications also for the foundations of precise probability, because the question of conglomerability was pending in that case too: similarly to the imprecise case, before these recent contributions it was not clear to us which was the joint coherence condition to adopt in the case of linear conditional previsions. This problem is now solved by Theorem 9. The importance of this point should not be overlooked, since joint coherence can be regarded as the founding axiom for a behavioural theory of probability; therefore Theorem 9 contributes to give firmer foundations to probability theory in the case that conglomerability has to be accounted for.

In our view, now the most important next step to do is to try to make the new theory of practical use in general, not only in the cases already addressed in this paper. To this end, there is a main obstacle to overcome: the computation of the conglomerable natural extension of a set of desirable gambles. We know from [13] that we can approximate it by a sequence of sets and, from [12], that such a sequence may be made of infinitely many distinct elements. What we do not know yet is whether or not the conglomerable natural extension is always attained as the limit of the sequence. This is the main challenge that has to be faced in future work.

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Appendix A. Proofs of results

Proof of Lemma 1. Assume that $\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$ are weakly coherent. Let us consider the following lower prevision:

$$\underline{P}(f) := \sup \left\{ \alpha : \exists f_1, \dots, f_m \in \mathcal{L} \text{ s.t. } \sup_\Omega \left[\sum_{i=1}^m G(f_i|\mathcal{B}_i) - (f - \alpha) \right] < 0 \right\}. \tag{A.1}$$

To see that $\underline{P}$ is well defined, it suffices to note that $\sup f \geq \underline{P}(f) \geq \inf f$ for any gamble f: given $\alpha > \sup f$, there are no gambles $f_1, \dots, f_m$ satisfying the above equation or we contradict the weak coherence of the conditional lower previsions $\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$; and for any $\alpha < \inf f$ we can take $f_1 := \dots := f_m := 0$. It is also easy to see that $\underline{P}$ satisfies conditions (C1)–(C3), and as a consequence it is a coherent lower prevision.

To see that $\underline{P}$ satisfies Eq. (2) with respect to $\underline{P}(\cdot|\mathcal{B}_j)$ for $j = 1, \dots, m$, note that from Eq. (A.1) $\underline{P}(G(f|\mathcal{B}_j)) \geq 0$ for any gamble f and any $j = 1, \dots, m$, simply by considering $f_j := f$, $f_i := 0$ for all $i \neq j$ and $\alpha < 0$. As a consequence, we also have that $\underline{P}(G(f|B_j)) = \underline{P}(G(B_j f|\mathcal{B}_j)) \geq 0$ for all $B_j \in \mathcal{B}_j, j \in \{1, \dots, m\}$. Assume ex-absurdo that $\underline{P}(G(f|B_j)) > 0$; then there are gambles $f_1, \dots, f_m \in \mathcal{L}$ and $\alpha > 0$ such that

$$\sup_\Omega \left[\sum_{i=1}^m G(f_i|\mathcal{B}_i) - (G(f|B_j) - \alpha) \right] < 0,$$

whence in particular

$$\sup_\Omega \left[\sum_{i=1}^m G(f_i|\mathcal{B}_i) - G(f|B_j) \right] < 0,$$

a contradiction with the weak coherence of $\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$. We conclude that $\underline{P}(G(f|B_j)) = 0$ for every gamble f and every $B_j \in \mathcal{B}_j$, and as a consequence Eq. (2) holds.

Let us prove now the converse implication. Let us take $f_0, f_1, \dots, f_m \in \mathcal{L}$, $B_0 \in \mathcal{B}_j$ for some $j \in \{1, \dots, m\}$, and let us show that Eq. (1) holds. Let $g_i := G(f_i|\mathcal{B}_i), i = 1, \dots, m, g_0 := G(f_0|B_0)$. Then we deduce from the assumption that $\underline{P}(g_i) \geq 0$ for $i = 1, \dots, m$, and $\underline{P}(g_0) = 0$, whence $g_i = G(f_i|\mathcal{B}_i) \geq G(g_i)$ for $i = 1, \dots, m$ and $g_0 = G(f_0|B_0) = G(g_0)$. As a consequence,

$$\sup_\Omega \left[\sum_{i=1}^m G(f_i|\mathcal{B}_i) - G(f_0|B_0) \right] \geq \sup_\Omega \left[\sum_{i=1}^m G(g_i) - G(g_0) \right] \geq 0,$$

where the second inequality follows from the coherence of $\underline{P}$. We deduce that $\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$ are weakly coherent. $\square$

Proof of Proposition 1. Assume without loss of generality that the set $B \in \mathcal{B}_1 \cup \mathcal{B}_2$ for which $\underline{P}(B) = 0$ is $B = B_1' \in \mathcal{B}_1$. Consider gambles f, g, h on Ω, $B \in \mathcal{B}_1 \cup \mathcal{B}_2$, and let us show that

$$\sup_{B'} [G(f|\mathcal{B}_1) + G(g|\mathcal{B}_2) - G(h|B)] \geq 0 \tag{A.2}$$

for some $B' \in S_1(f) \cup S_2(g) \cup \{B\}$. Assume ex-absurdo that Eq. (A.2) does not hold. If there is some $B' \in S_1(f) \cup S_2(g) \cup \{B\}$ different from B_1', this means that there is some $\delta > 0$ such that

$$[G(f|\mathcal{B}_1) + G(g|\mathcal{B}_2) - G(h|B) + \delta B'](\omega) \leq 0$$

for all $\omega \in \Omega$ (remember that if Eq. (A.2) fails, then $[G(f|\mathcal{B}_1) + G(g|\mathcal{B}_2) - G(h|B)](\omega) \leq 0$ for all $\omega \in \Omega$), whence

$$G(f|\mathcal{B}_1) + G(g|\mathcal{B}_2) \leq G(h|B) - \delta B'.$$

As a consequence,

$$\underline{P}(G(f|\mathcal{B}_1) + G(g|\mathcal{B}_2)) \leq \underline{P}(G(h|B) - \delta B') \leq \underline{P}(G(h|B)) + \overline{P}(-\delta B') = \underline{P}(G(h|B)) - \underline{P}(\delta B') = -\delta \underline{P}(B') < 0,$$

where the second inequality follows from coherence [16, Theorem 2.6.1(e)] and the last equality from (C2) and Theorem 1. On the other hand, (C3) and the weak coherence of $\underline{P}$ with $\underline{P}(\cdot|\mathcal{B}_1), \underline{P}(\cdot|\mathcal{B}_2)$ imply via Lemma 1 that

$$\underline{P}(G(f|\mathcal{B}_1) + G(g|\mathcal{B}_2)) \geq \underline{P}(G(f|\mathcal{B}_1)) + \underline{P}(G(f|\mathcal{B}_2)) \geq 0,$$

a contradiction.

Finally, if there is no $B' \in S_1(f) \cup S_2(g) \cup \{B\}$ different from B'_1, then it must be $S_1(f) \subseteq B'_1, S_2(g) = \emptyset$ and $B = B'_1$, so Eq. (A.2) becomes

$$\sup_{B'_1}[G(f|B'_1) - G(h|B'_1)] \geq 0,$$

which follows from the separate coherence of $\underline{P}(\cdot|\mathcal{B}_1)$. We conclude that Eq. (A.2) holds and as a consequence the conditional lower previsions $\underline{P}(\cdot|\mathcal{B}_1), \underline{P}(\cdot|\mathcal{B}_2)$ are coherent. ☐

Proof of Theorem 3. Let us see how the sums in (5) and (6) can be mapped into each other. To avoid ambiguities, in the following we use subscripts in the G-gambles to make it clear which are the lower previsions they refer to.

Consider gambles $f_0, \ldots, f_m$ in $\mathcal{L}$ with finite supports, and let $B_0 \in \mathcal{B}_{j_0}$ for some $j_0 \in \{1, \ldots, m\}$. Then

$$\begin{aligned}\sum_{j=1}^{m} G_j(f_j|\mathcal{B}_j) - G_{j_0}(f_0|B_0) &= \sum_{j=1}^{m} \sum_{B_j \in S_j(f_j)} G_j(B_j f_j|B_j) - G_{j_0}(f_0|B_0) \\ &= \sum_{i=1}^{n} G_{j(i)}(B_i g_i|\mathcal{B}_{j(i)}) - G_{j(0)}(B_0 g_0|\mathcal{B}_{j(0)}),\end{aligned}$$

where n denotes the number of elements in the double sum, $g_i := f_j$, with f_j denoting the gamble in the i-th element in the same double sum, $B_i := B_j$, $j(i) := j$, $j(0) := j_0$, and $g_0 := f_0$. In the other direction, given gambles $g_0, \ldots, g_n \in \mathcal{L}, B_i \in \cup_{j=1}^{m}\mathcal{B}_j$, $i = 0, \ldots, n$, it holds that

$$\begin{aligned}\sum_{i=1}^{n} G_{j(i)}(B_i g_i|\mathcal{B}_{j(i)}) - G_{j(0)}(B_0 g_0|\mathcal{B}_{j(0)}) &= \sum_{i=1}^{n} G_{j(i)}(B_i g_i|B_i) - G_{j(0)}(B_0 g_0|B_0) \\ = \sum_{j=1}^{m} \sum_{i:j(i)=j} G_j(B_i g_i|B_i) - G_{j(0)}(B_0 g_0|B_0) &= \sum_{j=1}^{m} G_j(f_j|\mathcal{B}_j) - G_{j(0)}(f_0|B_0),\end{aligned}$$

where, for every $j = i, \ldots, m$, we have defined $f_j := \sum_{i:j(i)=j} B_i g_i$, which has obviously a finite support, and $f_0 := B_0 g_0$.

On the other hand, we may assume without loss of generality that all the f_j- and g_i-gambles in the previous sums are different from zero, because a G-gamble equals zero when it is evaluated in zero. Taking into account that the support of the generic gamble $B_i g_i$ is just B_i, we see that $\{B_0\} \bigcup \cup_{j=1}^{m} S_j(f_j) = \cup_{i=0}^{n}\{B_i\}$.

From the above discussion, it follows that after mapping one sum into the other, the only difference left between formulae (5) and (6) is concerned with the way the supremum is taken: in the first case, it is over a set from the collection $\cup_{i=0}^{n}\{B_i\}$, while in the second it is over the union $\cup_{i=0}^{n} B_i$. This makes it clear that if the supremum in (5) is non-negative then so must be that in (6). That the opposite is also true follows because set B in (6) is the union of the finitely many sets $B_0, \ldots, B_n$, and hence the supremum must be attained in one of them. ☐

Proof of Proposition 2. Let us first re-write the condition of $\mathcal{B}_j$-conglomerability for $\mathcal{R}_i|B_i$:

$$B_i f \in \mathcal{L}, B_j B_i f \in \mathcal{R}_i|B_i \cup \{0\} \text{ for all } B_j \in \mathcal{B}_j \Rightarrow B_i f \in \mathcal{R}_i|B_i \cup \{0\}. \tag{A.3}$$

To see that this condition is equivalent to the usual condition of conglomerability, namely

$$f \in \mathcal{L}, B_j f \in \mathcal{R}_i|B_i \cup \{0\} \text{ for all } B_j \in \mathcal{B}_j \Rightarrow f \in \mathcal{R}_i|B_i \cup \{0\},$$

it suffices to consider that any gamble g that belongs to $\mathcal{R}_i|B_i$ is such that $g = B_i g$.

Now, let us show that if (A.3) holds for all $B_i \in \mathcal{B}_i$, then $\mathcal{F}_i$ is $\mathcal{B}_j$-conglomerable. Take $f \in \mathcal{L}$ such that $B_j f \in \mathcal{F}_i \cup \{0\}$ for all $B_j \in \mathcal{B}_j$. This means that $B_j f = \sum_{B_i \in \mathcal{B}_i} B_i f_i$, with $B_i f_i \in \mathcal{R}_i|B_i \cup \{0\}$, whence $B_j B_i f = B_i f_i \in \mathcal{R}_i|B_i \cup \{0\}$. This holds for all $B_j \in \mathcal{B}_j$, and applying (A.3) we see that $B_i f \in \mathcal{R}_i|B_i \cup \{0\}$. By assumption, this holds for all $B_i \in \mathcal{B}_i$, so that $f \in \mathcal{F}_i \cup \{0\}$ by definition of $\mathcal{F}_i$.

Conversely, let us show that (A.3) holds for all $B_i \in \mathcal{B}_i$ if $\mathcal{F}_i$ is $\mathcal{B}_j$-conglomerable. Consider a gamble f such that $B_j B_i f \in \mathcal{R}_i|B_i \cup \{0\}$ for all $B_j \in \mathcal{B}_j$. Since $\mathcal{R}_i|B_i \subseteq \mathcal{F}_i$, we have that $B_j B_i f \in \mathcal{F}_i \cup \{0\}$ for all $B_j \in \mathcal{B}_j$, whence $B_i f \in \mathcal{F}_i \cup \{0\}$ given that this set is $\mathcal{B}_j$-conglomerable by assumption; this implies, by definition of $\mathcal{F}_i$, that $B_i f \in \mathcal{R}_i|B_i \cup \{0\}$. □

Proof of Proposition 3. Let us start by showing that $\cup_{i=1}^m \cup_{B_i \in \mathcal{B}_i} \mathcal{R}_i|B_i$ avoids partial loss. Assume this is not the case. Then $\mathcal{E}$ contains the zero gamble. Since each set $\mathcal{R}_i|B_i$ satisfies D3–D4, any positive linear combination of elements of $\mathcal{R}_i|B_i$ can be replaced by just one element of $\mathcal{R}_i|B_i$. We deduce that if $\cup_{i=1}^m \cup_{B_i \in \mathcal{B}_i} \mathcal{R}_i|B_i$ incurs partial loss, then there is a non-empty set of indexes $I \subseteq \{1, \dots, m\}$ and non-empty finite sets $\mathcal{B}'_i \subseteq \mathcal{B}_i$, $i \in I$, and gambles $g \in \mathcal{L}^+(\Omega) \cup \{0\}$, $f_{i,B_i} \in \mathcal{L}(\Omega)$, as well as positive constants ε_{i,B_i}, for all $B_i \in \mathcal{B}'_i, i \in I$, such that

$$g + \sum_{i \in I} \sum_{B_i \in \mathcal{B}'_i} G(f_{i,B_i}|B_i) + \varepsilon_{i,B_i} B_i = 0.$$

We can assume without loss of generality that for all $B_i \in \mathcal{B}'_i, i \in I$, it is $B_i f_{i,B_i} \neq 0$, because otherwise $G(f_{i,B_i}|B_i) + \varepsilon_{i,B_i} B_i = \varepsilon_{i,B_i} B_i \gneq 0$, and we could represent this positive contribution through g. We obtain that

$$\sum_{i \in I} \sum_{B_i \in \mathcal{B}'_i} G(f_{i,B_i}|B_i) \leq -\sum_{i \in I} \sum_{B_i \in \mathcal{B}'_i} \varepsilon_{i,B_i} B_i.$$

Let $f_i := \sum_{B_i \in \mathcal{B}'_i} B_i f_{i,B_i}$ and $\varepsilon_i := \min_{B_i \in S_i(f_i)} \varepsilon_{i,B_i}$ for all $i \in I$, and also $\varepsilon := \sum_{i \in I} \varepsilon_i$. Then

$$\sum_{i \in I} G(f_i|\mathcal{B}_i) \leq -\varepsilon \mathbb{I}_{\cup_{i \in I} S_i(f_i)}$$

or, in other words, that $\sup_{\cup_{i \in I} S_i(f_i)} \sum_{i \in I} G(f_i|\mathcal{B}_i) < 0$, which contradicts that $\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$ are Williams-coherent. Therefore $\cup_{i=1}^m \cup_{B_i \in \mathcal{B}_i} \mathcal{R}_i|B_i$ avoids partial loss, and hence $\mathcal{E}$ is coherent.

Now, let $\underline{Q}(\cdot|\mathcal{B}_1), \dots, \underline{Q}(\cdot|\mathcal{B}_m)$ be the Williams-coherent conditional lower previsions induced by $\mathcal{E}$. It follows that $\underline{Q}(\cdot|\mathcal{B}_1), \dots, \underline{Q}(\cdot|\mathcal{B}_m)$ dominate $\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$: in fact, for every $i = 1, \dots, m$, $B_i \in \mathcal{B}_i$ and $f \in \mathcal{L}$, $\underline{P}(f|B_i) = \sup\{\mu : B_i(f - \mu) \in \mathcal{R}_i|B_i\} \leq \sup\{\mu : B_i(f - \mu) \in \mathcal{E}\} = \underline{Q}(f|B_i)$.

Assume that there is B_0 in one of the partitions $\mathcal{B}_1, \dots, \mathcal{B}_m$ and a gamble f_0 such that $\underline{P}(f_0|B_0) < \underline{Q}(f_0|B_0)$. Then there is some $\delta > 0$ such that

$$G(f_0|B_0) - \delta B_0 \in \mathcal{E}.$$

Reasoning as at the beginning of the proof, there is a non-empty set of indexes $I \subseteq \{1, \dots, m\}$ and non-empty finite sets $\mathcal{B}'_i \subseteq \mathcal{B}_i$, $i \in I$, and gambles $g \in \mathcal{L}^+(\Omega) \cup \{0\}$, $f_{i,B_i} \in \mathcal{L}(\Omega)$, s.t. $B_i f_{i,B_i} \neq 0$, for all $B_i \in \mathcal{B}'_i, i \in I$, which lead to

$$G(f_0|B_0) - \delta B_0 = g + \sum_{i \in I} \sum_{B_i \in \mathcal{B}'_i} G(f_{i,B_i}|B_i) + \varepsilon_{i,B_i} B_i.$$

Using again the same line of reasoning, we deduce that there are gambles f_i, $i \in I$, with finite supports and $\varepsilon > 0$ such that

$$\sum_{i \in I} G(f_i|\mathcal{B}_i) - G(f_0|B_0) \leq -\delta B_0 - \varepsilon \mathbb{I}_{\cup_{i \in I} S_i(f_i)}$$

or, in other words, that for all $B \in B_0 \bigcup \cup_{i \in I} S_i(f_i)$,

$$\sup_B \left[\sum_{i \in I} G(f_i|\mathcal{B}_i) - G(f_0|B_0) \right] < 0,$$

which contradicts that $\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$ are Williams-coherent. It follows that $\underline{P}(\cdot|\mathcal{B}_i) = \underline{Q}(\cdot|\mathcal{B}_i)$ for all $i = 1, \dots, m$. □

Proof of Theorem 6. 1. Assume that $\mathcal{R}$ incurs partial loss. This means that there is a positive linear combination of gambles in $\mathcal{R}$ that is less than or equal to zero.

Since for every $i = 1, \dots, m$ and every $B_i \in \mathcal{B}$ the set $\mathcal{R}_i|B_i$ satisfies D3–D4, any positive linear combination of elements of $\mathcal{R}_i|B_i$ can be replaced by just one element of $\mathcal{R}_i|B_i$. We deduce that if $\mathcal{R}$ incurs partial loss, then there is a non-empty set of indexes $I \subseteq \{1, \dots, m\}$, and gambles $f_i \in \mathcal{L}$ with finite support (so that we make a finite combination of gambles from $\mathcal{R}$), $\varepsilon_{B_i} > 0$, $i \in I$, $B_i \in S_i(f_i)$ such that

$$\sum_{i \in I} \sum_{B_i \in S_i(f_i)} G(f_i|B_i) + \varepsilon_{B_i} B_i \leq 0,$$

which we can equivalently re-write as

$$\sum_{i \in I} G(f_i|\mathcal{B}_i) + \sum_{i \in I} \sum_{B_i \in S_i(f_i)} \varepsilon_{B_i} B_i \leq 0.$$

Whence, by choosing any $B_j \in S_j(f_j)$, for some $j \in I$, we obtain that

$$\sup_{B_j} \left[\sum_{i \in I} G(f_i|\mathcal{B}_i) \right] \leq -\varepsilon_{B_j},$$

and this implies that $\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$ incur partial loss.

2. Consider first the direct implication. Let $\underline{F}(\cdot|\mathcal{B}_1), \dots, \underline{F}(\cdot|\mathcal{B}_m)$ be the dominating conglomerably coherent extensions. Then they are induced by a conglomerably coherent set $\mathcal{F}$. This set $\mathcal{F}$ includes in particular the conglomerable natural extension of $\tilde{\mathcal{R}} := \cup_{i=1}^m \cup_{B_i \in \mathcal{B}_i} \tilde{\mathcal{R}}_i|B_i$, where $\tilde{\mathcal{R}}_i|B_i$ is induced by $\underline{F}(\cdot|B_i)$ by Eq. (12), because it induces $\underline{F}(\cdot|\mathcal{B}_1), \dots, \underline{F}(\cdot|\mathcal{B}_m)$. Since $\underline{F}(\cdot|\mathcal{B}_i) \geq \underline{P}(\cdot|\mathcal{B}_i)$ for all $i = 1, \dots, m$, it follows that $\mathcal{R} \subseteq \tilde{\mathcal{R}}$, and therefore $\mathcal{F}$ is a conglomerably coherent set that includes $\mathcal{R}$. Hence, the conglomerable natural extension of $\mathcal{R}$ exists.

Consider now the converse implication, that is, assume that the conglomerable natural extension $\mathcal{F}$ of $\mathcal{R}$ exists. Let $\underline{F}(\cdot|\mathcal{B}_1), \dots, \underline{F}(\cdot|\mathcal{B}_m)$ be the conditional lower previsions it induces. It follows by definition that these conditional lower previsions are conglomerably coherent. Since $\mathcal{R} \subseteq \mathcal{F}$ and $\mathcal{R}$ induces conditional lower previsions that dominate $\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$, we deduce that $\underline{F}(\cdot|\mathcal{B}_i) \geq \underline{P}(\cdot|\mathcal{B}_i)$ for all $i = 1, \dots, m$.

3. Let $\underline{Q}(\cdot|\mathcal{B}_1), \dots, \underline{Q}(\cdot|\mathcal{B}_m)$ be conglomerably coherent extensions of $\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$ that are dominated by the conditional lower previsions induced by $\mathcal{F}$, that we denote $\underline{F}(\cdot|\mathcal{B}_1), \dots, \underline{F}(\cdot|\mathcal{B}_m)$. Since the conditional lower previsions $\underline{Q}(\cdot|\mathcal{B}_1), \dots, \underline{Q}(\cdot|\mathcal{B}_m)$ are conglomerably coherent, they are induced by a conglomerably coherent set $\mathcal{F}'$; since they dominate $\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$, it must be the case that $\mathcal{F}' \supseteq \mathcal{R}$ and hence $\mathcal{F}' \supseteq \mathcal{F}$; but then the conditional lower previsions induced by $\mathcal{F}'$ must dominate those induced by $\mathcal{F}$, and as a consequence we must have $\underline{Q}(\cdot|\mathcal{B}_i) = \underline{F}(\cdot|\mathcal{B}_i)$ for all $i = 1, \dots, m$. □

Proof of Theorem 7. 1. From Theorem 6(1), if $\mathcal{R}$ incurs partial loss there is a non-empty set of indexes $I \subseteq \{1, \dots, m\}$, and gambles $f_i \in \mathcal{L}$ with finite support, $\varepsilon_{B_i} > 0$, $i \in I$, $B_i \in S_i(f_i)$ such that

$$\sum_{i \in I} G(f_i|\mathcal{B}_i) + \sum_{i \in I} \sum_{B_i \in S_i(f_i)} \varepsilon_{B_i} B_i \leq 0;$$

we deduce that

$$\sup_{\cup_{j \in I} \cup_{B_j \in S_j(f_j)} B_j} \left[\sum_{i \in I} G(f_i|\mathcal{B}_i) \right] \leq - \min_{B_j \in S_j(f) : j \in I} \varepsilon_{B_j},$$

and this contradicts that $\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$ Williams-avoid partial loss.

2. Assume that $\underline{P}(\cdot|\mathcal{B}_1),\dots,\underline{P}(\cdot|\mathcal{B}_m)$ Williams-avoid partial loss, and let $\mathcal{E}$ be the natural extension of $\mathcal{R}$. $\mathcal{E}$ is coherent because $\mathcal{R}$ avoids partial loss thanks to point 1, and hence the conditional lower previsions it induces, $\underline{E}(\cdot|\mathcal{B}_1),\dots,\underline{E}(\cdot|\mathcal{B}_m)$, are Williams-coherent.[13] Since $\mathcal{R}\subseteq\mathcal{E}$, and $\mathcal{R}$ induces conditional lower previsions that dominate $\underline{P}(\cdot|\mathcal{B}_1),\dots,\underline{P}(\cdot|\mathcal{B}_m)$, we deduce that $\mathcal{E}$ induces dominating Williams-coherent conditional lower previsions.

   To see that they are the smallest ones, denote by $\underline{Q}(\cdot|\mathcal{B}_1),\dots,\underline{Q}(\cdot|\mathcal{B}_m)$ the smallest dominating Williams-coherent conditional lower previsions. We have shown in Proposition 3 that $\underline{Q}(\cdot|\mathcal{B}_1),\dots,\underline{Q}(\cdot|\mathcal{B}_m)$ are induced by the natural extension $\tilde{\mathcal{E}}$ of $\tilde{\mathcal{R}}\coloneqq\cup_{i=1}^m\cup_{B_i\in\mathcal{B}_i}\tilde{\mathcal{R}}_i|B_i$, where the sets $\tilde{\mathcal{R}}_i|B_i$ are derived from $\underline{Q}(\cdot|\mathcal{B}_i)$ by means of Eq. (12). Since $\underline{Q}(\cdot|\mathcal{B}_i)\geq\underline{P}(\cdot|\mathcal{B}_i)$, we deduce that $\tilde{\mathcal{R}}_i|B_i\supseteq\mathcal{R}_i|B_i$, whence $\tilde{\mathcal{R}}\supseteq\mathcal{R}$. This implies $\mathcal{E}\subseteq\tilde{\mathcal{E}}$ and therefore $\underline{E}(\cdot|\mathcal{B}_i)\leq\underline{Q}(\cdot|\mathcal{B}_i)$ for all $i=1,\dots,m$. Hence, $\underline{E}(\cdot|\mathcal{B}_i)=\underline{Q}(\cdot|\mathcal{B}_i)$ for all $i=1,\dots,m$, and we conclude that these are the smallest Williams-coherent extensions. □

Proof of Theorem 8. 1. Let us show that (SC1)–(SC3) hold.

(SC1) Given $\mu<\inf_{B_i}f$, it holds that $B_i(f-\mu)\gneq 0$, whence $B_i(f-\mu)\in\mathcal{R}$ and therefore $\underline{P}(f|B_i)\geq\inf_{B_i}f$.

(SC2) Given $f\in\mathcal{L}$ and $\lambda>0$, $\underline{P}(\lambda f|B_i)=\sup\{\mu: B_i(\lambda f-\mu)\in\mathcal{R}\}=\sup\{\mu: B_i(f-\frac{\mu}{\lambda})\in\mathcal{R}\}=\lambda\sup\{\mu': B_i(f-\mu')\in\mathcal{R}\}=\lambda\underline{P}(f|B_i)$, where the second passage is due to the coherence of $\mathcal{R}$.

(SC3) Given $f,g\in\mathcal{L}$ and $\varepsilon>0$, it holds that $B_i(f-\underline{P}(f|B_i)+\frac{\varepsilon}{2}),B_i(g-\underline{P}(g|B_i)+\frac{\varepsilon}{2})\in\mathcal{R}$, whence $B_i(f+g-\underline{P}(f|B_i)-\underline{P}(g|B_i)+\varepsilon)\in\mathcal{R}$ and therefore $\underline{P}(f+g|B_i)\geq\underline{P}(f|B_i)+\underline{P}(g|B_i)-\varepsilon$ for all $\varepsilon>0$. This implies that $\underline{P}(f+g|B_i)\geq\underline{P}(f|B_i)+\underline{P}(g|B_i)$.

2. We use the equivalent formulation of Williams-coherence given in Theorem 3. Consider $f_0,\dots,f_n\in\mathcal{L}$, $B_i\in\cup_{j=1}^m\mathcal{B}_j$ for $i=0,\dots,n$, and let $j(i)$ denote an element of $\{1,\dots,m\}$ for which $B_i\in\mathcal{B}_{j(i)}$. Let us show that

$$\sup_B\left[\sum_{i=1}^n G(B_if_i|\mathcal{B}_{j(i)})-G(B_0f_0|\mathcal{B}_{j(0)})\right]\geq 0,$$

   with $B\coloneqq\cup_{i=0}^n B_i$. If this is not the case, then there is some $\delta>0$ such that

$$G(B_0f_0|\mathcal{B}_{j(0)})-\frac{\delta}{2}B\geq\sum_{i=1}^n G(B_if_i|\mathcal{B}_{j(i)})+\frac{\delta}{2}B.$$

   By definition, $G(B_if_i|\mathcal{B}_{j(i)})+\frac{\delta}{2n}B_i$ belongs to $\mathcal{R}$ for all $i=1,\dots,n$, whence $\sum_{i=1}^n G(B_if_i|\mathcal{B}_{j(i)})+\frac{\delta}{2}B$ belongs to $\mathcal{R}$ and therefore $G(B_0f_0|\mathcal{B}_{j(0)})-\frac{\delta}{2}B\in\mathcal{R}$. As a consequence, also $G(B_0f_0|\mathcal{B}_{j(0)})-\frac{\delta}{2}B_0\in\mathcal{R}$, but this means that we can increase the value $\underline{P}(B_0f_0|\mathcal{B}_{j(0)})$ by $\frac{\delta}{2}$. This contradicts the definition of $\underline{P}(B_0f_0|\mathcal{B}_{j(0)})$.

3. Consider $f_0,\dots,f_m\in\mathcal{L}$, and let us show that there is $B\in\{B_0\}\bigcup\cup_{i=1}^m S_i(f_i)$ such that

$$\sup_B\left[\sum_{i=1}^m G(f_i|\mathcal{B}_i)-G(f_0|B_0)\right]\geq 0. \tag{A.4}$$

   We can assume without loss of generality that $f_1,\dots,f_m$, $m\geq 1$, are all different from the zero gamble: in fact, if they are all equal to the zero gamble, then (A.4) holds by the separate coherence of $\underline{P}(\cdot|B_0)$, which follows from point 1; and if not all of them are equal to zero, it is sufficient to drop all the gambles in $f_1,\dots,f_m$ that are identically equal to zero.

   Assume ex-absurdo that (A.4) fails. Let $g\coloneqq\sum_{i=1}^m G(f_i|\mathcal{B}_i)-G(f_0|B_0)$; observe that $g\leq 0$. Define $\delta(B)\coloneqq-\frac{\sup_B g}{m+1}$ for all $B\in\{B_0\}\bigcup\cup_{i=1}^m S_i(f_i)$, so that $\delta(B)>0$. It follows that

$$B\delta(B)\leq-\frac{g}{m+1}B. \tag{A.5}$$

[13] This is essentially a result by Williams; we reproduce it in our language in Theorem 8(2).

By definition, $G(f_i|B_i) + B_i\delta(B_i)$ belongs to $\mathcal{R}$ for all $B_i \in S_i(f_i)$. Using the $\mathcal{B}_i$-conglomerability of $\mathcal{R}$, we get that $G(f_i|\mathcal{B}_i) + \sum_{B_i\in S_i(f_i)} B_i\delta(B_i) \in \mathcal{R}$, and eventually that

$$f + \sum_{i=1}^{m} \sum_{B_i\in S_i(f_i)} B_i\delta(B_i) \in \mathcal{R}, \tag{A.6}$$

where $f \coloneqq \sum_{i=1}^{m} G(f_i|\mathcal{B}_i)$.

Now, if $\omega \in B$ for some $B \in S_i(f_i)$, then $[\sum_{B_i\in S_i(f_i)} B_i\delta(B_i)](\omega) = \delta(B) \leq -\frac{g(\omega)}{m+1}$ by (A.5). Conversely, if $\omega \in B$ for some $B \notin S_i(f_i)$, then $[\sum_{B_i\in S_i(f_i)} B_i\delta(B_i)](\omega) = 0 \leq -\frac{g(\omega)}{m+1}$, recalling that $g \leq 0$. We deduce that

$$\sum_{B_i\in S_i(f_i)} B_i\delta(B_i) \leq -\frac{g}{m+1}. \tag{A.7}$$

As a consequence,

$$f + \left[\sum_{i=1}^{m} \sum_{B_i\in S_i(f_i)} B_i\delta(B_i)\right] - G(f_0|B_0) + B_0\delta(B_0) = g + \left[\sum_{i=1}^{m} \sum_{B_i\in S_i(f_i)} B_i\delta(B_i)\right] + B_0\delta(B_0)$$
$$\leq \frac{g}{m+1} + B_0\delta(B_0) \leq 0;$$

here, the one-but-last inequality follows from (A.7) and the last one holds trivially outside B_0 (recall that $g \leq 0$), and in B_0 it follows by (A.5).

Overall, we obtain that $f + \sum_{i=1}^{m} \sum_{B_i\in S_i(f_i)} B_i\delta(B_i) \leq G(f_0|B_0) - B_0\delta(B_0)$ and, through (A.6), that $G(f_0|B_0) - B_0\delta(B_0)$ belongs to $\mathcal{R}$. This means that we can increase the value $\underline{P}(f_0|B_0)$ by $\delta(B_0)$, which contradicts the definition of $\underline{P}(f_0|B_0)$. □

Proof of Theorem 9. 1. We consider the direct implication, as the converse is trivial. Let us first show that any $\mathcal{B}_i$-conglomerable set $\mathcal{F}'$ inducing $\underline{P}(\cdot|\mathcal{B}_i)$ must include $\mathcal{F}_i$. Consider $f \in \mathcal{F}_i$. Then $f = \sum_{B_i\in S_i(f)} B_i f_i$, with $B_i f_i \in \mathcal{R}_i|B_i$ and $S_i(f) \neq \emptyset$. For the generic term in the sum, it can either hold that $B_i f_i \gneq 0$, and hence, trivially, $B_i f_i \in \mathcal{F}'$; or that $\underline{P}(f_i|B_i) > 0$, and also in this case $B_i f_i = B_i(f_i - \underline{P}(f_i|B_i)) + B_i\underline{P}(f_i|B_i) \in \mathcal{F}'$, recalling that $\mathcal{F}'$ induces $\underline{P}(\cdot|B_i)$. Since $\mathcal{F}'$ is $\mathcal{B}_i$-conglomerable, we obtain that $f \in \mathcal{F}'$. This shows that $\mathcal{F}_i \subseteq \mathcal{F}'$.

Now, if $\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$ are conglomerably coherent, then there is a $\mathcal{B}_{1:m}$-conglomerably coherent set $\mathcal{F}'$ that induces them. We know that $\mathcal{F}'$ must include $\cup_{i=1}^{m}\mathcal{F}_i$, and this makes sure that the $\mathcal{B}_{1:m}$-conglomerable natural extension $\mathcal{F}$ of $\cup_{i=1}^{m}\mathcal{F}_i$ exists, given that taking the intersection of conglomerable sets preserves conglomerability. The conditional lower previsions induced by $\mathcal{F}$ must dominate $\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$ because $\mathcal{F}_i \subseteq \mathcal{F}$, for all $i = 1, \dots, m$, and at the same time be dominated by $\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$, because $\mathcal{F} \subseteq \mathcal{F}'$. This means that $\mathcal{F}$ induces $\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$.

Let us consider the second part of the proof, where we have linear conditional previsions $P(\cdot|\mathcal{B}_1), \dots, P(\cdot|\mathcal{B}_m)$. From the first part, we know that $P(\cdot|\mathcal{B}_1), \dots, P(\cdot|\mathcal{B}_m)$ are conglomerably coherent if and only if the conglomerable natural extension of $\cup_{i=1}^{m}\mathcal{F}_i$ exists and induces them. But if this conglomerable natural extension exists it induces conditional lower previsions that must dominate, and therefore coincide with (because they are linear) $P(\cdot|\mathcal{B}_1), \dots, P(\cdot|\mathcal{B}_m)$. Hence, this second condition is redundant in the linear case.

2. Consider $i \neq j$ in $\{1, \dots, m\}$, and assume ex-absurdo that $\mathcal{F}_i$ is not $\mathcal{B}_j$-conglomerable. If it does not have a $\mathcal{B}_j$-conglomerable natural extension, then we deduce that $\cup_{i=1}^{m}\mathcal{F}_i$ does not have a $\mathcal{B}_j$-conglomerable natural extension and applying point 1 we deduce that $\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$ cannot be conglomerably coherent, a contradiction.

   On the other hand, if $\mathcal{F}_i$ is not $\mathcal{B}_j$-conglomerable but it has a $\mathcal{B}_j$-conglomerable natural extension, we deduce from Proposition 2 that there is some $B_i \in \mathcal{B}_i$ such that $\mathcal{R}_i|B_i$ is not $\mathcal{B}_j$-conglomerable but has a $\mathcal{B}_j$-conglomerable natural extension $\tilde{\mathcal{F}}$. We also deduce that $\mathcal{R}_i\rfloor B_i$ is not $B_i\mathcal{B}_j$-conglomerable, where $B_i\mathcal{B}_j$ is the partition of B_i given by

$$B_i\mathcal{B}_j := \{B_jB_i : B_j \in \mathcal{B}_j \text{ and } B_jB_i \neq 0\}.$$

Given its associated conditional lower prevision $\underline{P}(\cdot|B_i)$, it follows from Theorems 2 and 5 that there is no conditional lower prevision $\underline{P}(\cdot|B_i\mathcal{B}_j)$ on $\mathcal{L}(B_i)$ which is coherent with it.

From the $\mathcal{B}_j$-conglomerable natural extension $\tilde{\mathcal{F}}$ of $\mathcal{R}_i|B_i$ we can induce a conditional lower prevision $\underline{P}'(\cdot|B_i)$ on $\mathcal{L}(B_i)$ and another conditional lower prevision $\underline{P}'(\cdot|B_i\mathcal{B}_j)$; since $\tilde{\mathcal{F}}$ is $\mathcal{B}_j$-conglomerable, and this trivially implies that it is $B_i\mathcal{B}_j$-conglomerable, we deduce from Theorem 8 that $\underline{P}'(\cdot|B_i), \underline{P}'(\cdot|B_i\mathcal{B}_j)$ are coherent. As a consequence, $\underline{P}'(\cdot|B_i)$ does not coincide with $\underline{P}(\cdot|B_i)$, and therefore the $\mathcal{B}_j$-conglomerable superset of $\cup_{i=1}^m\mathcal{F}_i$ does not induce $\underline{P}(\cdot|B_i)$ either. Applying point 1 we deduce that $\underline{P}(\cdot|\mathcal{B}_1), \ldots, \underline{P}(\cdot|\mathcal{B}_m)$ are not conglomerably coherent, a contradiction.

3. This follows at once from point 1 and Theorem 8(3). □

Proof of Theorem 10. Note first of all that the set $\mathcal{F}' := \mathcal{F}_1 \oplus \cdots \oplus \mathcal{F}_m$ is indeed a coherent of gambles: if there were $f_i \in \mathcal{F}_i \cup \{0\}, i = 1, \ldots, m$, not all of them equal to zero, such that $\sum_{i=1}^m f_i \leq 0$, then given $g_i := f_i\mathbb{I}_{\cup B_i : B_i f_i \not\geq 0}$ we would deduce that $\sup_{\omega\in B}\sum_{i=1}^m G(g_i|\mathcal{B}_i) < 0$ for every $B \in \cup_{i=1}^m S_i(g_i)$, a contradiction with the coherence of the conditional lower previsions $\underline{P}(\cdot|\mathcal{B}_1), \ldots, \underline{P}(\cdot|\mathcal{B}_m)$.

1. Consider $i \in \{1, \ldots, m\}$, and let $\underline{Q}(\cdot|\mathcal{B}_i)$ be the conditional lower prevision induced by $\mathcal{F}'$. If it does not coincide with $\underline{P}(\cdot|\mathcal{B}_i)$, this means that there is a gamble f on Ω and $B_i \in \mathcal{B}_i$ such that $\underline{Q}(f|B_i) > \underline{P}(f|B_i)$, whence there is some $\delta > 0$ such that $B_i(f - (\underline{P}(f|B_i) + \delta)) = G(f|B_i) - \delta B_i$ belongs to $\mathcal{F}'$. Note, in addition, that $G(f|B_i) - \delta B_i \not\geq 0$ because otherwise we would obtain $\underline{P}(f|B_i) < \inf_{B_i} f$, contradicting the separate coherence of $\underline{P}(\cdot|\mathcal{B}_i)$.

   Hence, there are gambles $f_j \in \mathcal{F}_j, f_j \not\geq 0, j \in J \subseteq \{1, \ldots, m\}, J \neq \emptyset$, such that

$$G(f|B_i) - \delta B_i \geq \sum_{j\in J} f_j,$$

   where the inequality follows by considering the possible exclusion of positive gambles. For every $f_j \in \mathcal{F}_j, f_j \not\geq 0$, there are gambles g_j on Ω and positive $\mathcal{B}_j$-measurable h_j such that $f_j = G(g_j|\mathcal{B}_j) + h_jS_j(g_j)$, whence

$$G(f|B_i) - \delta B_i \geq \sum_{j\in J} G(g_j|\mathcal{B}_j) + h_jS_j(g_j),$$

   or, equivalently,

$$\sum_{j\in J} G(g_j|\mathcal{B}_j) - G(f|B_i) \leq -\delta B_i - \sum_{j\in J} h_jS_j(g_j),$$

   which means that

$$\sup_{\omega\in B}\left[\sum_{j\in J} G(g_j|\mathcal{B}_j) - G(f|B_i)\right](\omega) < 0$$

   for any $B \in \cup_{j\in J}S_j(g_j) \cup \{B_i\}$. This is a contradiction with the coherence of $\underline{P}(\cdot|\mathcal{B}_1), \ldots, \underline{P}(\cdot|\mathcal{B}_m)$. As a consequence, these conditional lower previsions are induced by $\mathcal{F}'$.

2. Let $\underline{P}$ be the lower prevision induced by $\mathcal{F}'$, and let us show that it coincides with the unconditional natural extension $\underline{E}$ of $\underline{P}(\cdot|\mathcal{B}_1), \ldots, \underline{P}(\cdot|\mathcal{B}_m)$, given by

$$\underline{E}(f) := \sup\left\{\mu : f - \mu \geq \sum_{i=1}^m G(f_i|\mathcal{B}_i) \text{ for some } f_1, \ldots, f_m \in \mathcal{L}\right\}.$$

Consider a gamble f, and take $\mu < \underline{P}(f)$. Then there are gambles $f_i \in \mathcal{F}_i \cup \{0\}$, $i = 1, \dots, m$, such that $f - \mu \geq f_1 + \cdots + f_m$. Since $f_i \in \mathcal{F}_i \cup \{0\}$ implies that $\underline{P}(f_i|\mathcal{B}_i) \geq 0$, it follows that $f_i \geq G(f_i|\mathcal{B}_i)$, and thus $f - \mu \geq \sum_{i=1}^m G(f_i|\mathcal{B}_i)$. Hence, $\underline{E}(f) \geq \mu$ and from this we deduce that $\underline{E}(f) \geq \underline{P}(f)$.

Conversely, given $\mu < \underline{E}(f)$, there are gambles $f_1, \dots, f_m$ such that $f - \mu \geq \sum_{i=1}^m G(f_i|\mathcal{B}_i)$, whence for any $\varepsilon > 0$,

$$f - \mu + \varepsilon \geq \sum_{i=1}^{m} \Big[G(f_i|\mathcal{B}_i) + \frac{\varepsilon}{m}\Big] \geq \sum_{i=1}^{m} \Big[G(f_i|\mathcal{B}_i) + \frac{\varepsilon}{m} S_i(f_i)\Big];$$

since $G(f_i|\mathcal{B}_i) + \frac{\varepsilon}{m} S_i(f_i)$ belongs to $\mathcal{F}_i$ for all $i = 1, \dots, m$, we conclude that $\underline{P}(f) \geq \mu - \varepsilon$ for all $\varepsilon > 0, \mu < \underline{E}(f)$, whence $\underline{P}(f) \geq \underline{E}(f)$ and as a consequence $\underline{P}(f) = \underline{E}(f)$ for every f.

From [16, Theorem 8.1.8], $\underline{E}, \underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$ are coherent, which implies that $\underline{E}$ is $\mathcal{B}_{1:m}$-conglomerable. Since $\underline{E} = \underline{P}$ is induced by $\mathcal{F}'$, we deduce from Theorem 5 that $\mathcal{F}'$ is weakly $\mathcal{B}_{1:m}$-conglomerable. □

Proof of Theorem 11. From Lemma 1, $\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$ are weakly coherent if and only if there is a coherent lower prevision $\underline{P}$ that is pairwise coherent with them. The equivalence follows now from [13, Theorem 25(1)]. □

Proof of Lemma 2. It follows from the coherence of $\underline{P}(\cdot|\mathcal{B}_1), \underline{P}(\cdot|\mathcal{B}_2)$ that

$$\sup_{B_1 \cup B_2} [G(B_1 B_2 f|\mathcal{B}_2) - G(B_2 f|\mathcal{B}_1)] \geq 0.$$

Now,

$$[G(B_1 B_2 f|\mathcal{B}_2) - G(B_2 f|\mathcal{B}_1)](\omega) = \begin{cases} -\underline{P}(B_1 f|B_2) + \underline{P}(B_2 f|B_1) & \text{if } \omega \in B_1 \cap B_2, \\ \underline{P}(B_2 f|B_1) & \text{if } \omega \in B_1 \setminus B_2, \\ -\underline{P}(B_1 f|B_2) < 0 & \text{if } \omega \in B_2 \setminus B_1, \end{cases}$$

and for this supremum to be non-negative it must be $\underline{P}(B_2 f|B_1) \geq 0$. □

Proof of Theorem 12. Consider $j \in \{1, \dots, m\}$, and let f be a gamble such that $B_j f \in \mathcal{R}_i|B_i \cup \{0\}$ for every $B_j \in \mathcal{B}_j$; equivalently, the restriction of $B_j f$ to B_i belongs to $\mathcal{R}_i\rfloor B_i \cup \{0\}$ for every $B_j \in \mathcal{B}_j$.

We may assume without loss of generality that $j \neq i$, since trivially $\mathcal{R}_i|B_i$ is $\mathcal{B}_i$-conglomerable. For any $B_j \in \mathcal{B}_j$ such that $\underline{P}(B_j f|B_i) > 0$, it follows from Lemma 2 that $\underline{P}(B_i f|B_j) \geq 0$. On the other hand, if $B_j B_i f \geq 0$ then we also have $\underline{P}(B_i f|B_j) \geq 0$. Hence, $B_i f \geq G(B_i f|\mathcal{B}_j)$, whence $\underline{P}(B_i f) \geq \underline{P}(G(B_i f|\mathcal{B}_j)) \geq 0$. Since $\underline{P}(B_i) > 0$, this means that $\underline{P}(f|B_i) \geq 0$.

As a consequence, the restriction of f to B_i belongs to the closure of the set of strictly desirable gambles $\mathcal{R}_i\rfloor B_i$. This means that $\mathcal{R}_i\rfloor B_i$ is weakly $\mathcal{B}_j$-conglomerable, and since it is a coherent set of strictly desirable gambles, it follows from Theorem 4 that it is also $\mathcal{B}_j$-conglomerable. Hence, the restriction of f to B_i belongs to $\mathcal{R}_i\rfloor B_i$ and as a consequence $B_i f$ belongs to $\mathcal{R}_i|B_i$, from which we conclude that this set is $\mathcal{B}_j$-conglomerable. □

Proof of Theorem 13. From [8, Theorem 11], under these conditions $\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$ are coherent, and applying [16, Theorem 7.1.5], $\underline{P}, \underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$ are coherent. This implies that $\mathcal{F}_1 \oplus \cdots \oplus \mathcal{F}_m$ is coherent (see Theorem 10).

1. Consider $i, j \in \{1, \dots, m\}$, and let us show for instance that $\mathcal{F}_i$ is $\mathcal{B}_j$-conglomerable. Note that it suffices to show this for $j \neq i$. From Proposition 2, this is equivalent to $\mathcal{R}_i|B_i$ being $\mathcal{B}_j$-conglomerable for all $B_i \in \mathcal{B}_i$; and that $\mathcal{R}_i|B_i$ is $\mathcal{B}_j$-conglomerable follows from Theorem 12.

2. Consider $i \in \{1, \dots, m\}$, and let us show that $\mathcal{F}_1 \oplus \cdots \oplus \mathcal{F}_m$ is $\mathcal{B}_i$-conglomerable. Consider a gamble f such that $B_i f \in \mathcal{F}_1 \oplus \cdots \oplus \mathcal{F}_m \cup \{0\}$ for all $B_i \in \mathcal{B}_i$. Then if $B_i f \neq 0$ there are $g_j \in \mathcal{F}_j \cup \{0\}$, $j = 1, \dots, m$, such that $B_i f = \sum_{j=1}^m g_j$.

   If $g_j \geq 0$ for all $j = 1, \dots, m$, then $B_i f \geq 0$ and since it is non-zero we deduce that $B_i f \in \mathcal{F}_i$; on the other hand, if there is some $j \in \{1, \dots, m\}$ such that $g_j \ngeq 0$, then there must be some gamble h_j and a positive $\mathcal{B}_j$-measurable gamble h'_j such that $g_j = G(h_j|\mathcal{B}_j) + h'_j S_j(h_j)$, whence

$$\underline{P}(g_j) = \underline{P}(G(h_j|\mathcal{B}_j) + h'_j S_j(h_j)) \geq \underline{P}(G(h_j|\mathcal{B}_j)) + \underline{P}(h'_j S_j(h_j)) > 0,$$

where the last inequality follows because $\underline{P}(G(h_j|\mathcal{B}_j)) \geq 0$, thanks to the coherence of $\underline{P}, \underline{P}(\cdot|\mathcal{B}_j)$, and $\underline{P}(h'_j S_j(h_j)) > 0$. To see the latter, take $B_j \in S_j(h_j)$ so that h'_j equals a positive constant α over B_j; then $h'_j S_j(h_j) \geq B_j \alpha$ and hence $\underline{P}(h'_j S_j(h_j)) \geq \alpha \underline{P}(B_j) > 0$, using that $\underline{P}(B_j) > 0$ for all $B_j \in \mathcal{B}_j$.

Since $\underline{P}(g_k|\mathcal{B}_k) \geq 0$ for all $g_k \in \mathcal{F}_k \cup \{0\}$, $k = 1, \dots, m$, we have that $\underline{P}(g_k) \geq \underline{P}(G(g_k|\mathcal{B}_k)) \geq 0$, thanks to the coherence of $\underline{P}, \underline{P}(\cdot|\mathcal{B}_k)$; we deduce that

$$\underline{P}(B_i f) \geq \sum_{j=1}^{m} \underline{P}(g_j) > 0,$$

whence there is some positive δ such that $0 < \underline{P}(B_i f - \delta) \leq \underline{P}(B_i(f - \delta))$. If we now take into account that $\underline{P}(f|B_i)$ is uniquely determined by (GBR) because $\underline{P}(B_i) > 0$, we see that $\underline{P}(f|B_i) \geq \sup\{\mu : \underline{P}(B_i(f-\mu)) > 0\} \geq \delta > 0$. Therefore, $B_i f \in \mathcal{R}_i|B_i \subseteq \mathcal{F}_i$. We deduce that

$$f = \sum_{B_i f \neq 0} B_i f \in \mathcal{F}_i \subseteq \mathcal{F}_1 \oplus \cdots \oplus \mathcal{F}_m,$$

using that $\mathcal{F}_i$ is $\mathcal{B}_i$-conglomerable.

3. It suffices to use that $\mathcal{F}_1 \oplus \cdots \oplus \mathcal{F}_m$ induces $\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$ from Theorem 10, and that from the previous point it is $\mathcal{B}_{1:m}$-conglomerable. □

Proof of Lemma 3. Given $f \in \mathcal{L}$, it must be $\underline{P}(G(f|\mathcal{B}_2)|B_1) = \underline{P}(G(B_1 f|\mathcal{B}_2)|B_1)$, because of [16, Lemma 6.2.4] and the fact that $\mathcal{B}_2$ is finer than $\mathcal{B}_1$. Since $\underline{P}_1(\cdot|\mathcal{B}_1), \underline{P}_1(\cdot|\mathcal{B}_2)$ are coherent, we have then that

$$\sup_{B_1} [G(B_1 f|\mathcal{B}_2) - G(G(B_1 f|\mathcal{B}_2)|B_1)] \geq 0,$$

or, equivalently,

$$\sup_{B_1} [G(B_1 f|\mathcal{B}_2) - G(B_1 f|\mathcal{B}_2) + \underline{P}(G(B_1 f|\mathcal{B}_2)|B_1)] \geq 0,$$

which implies that $\underline{P}(G(B_1 f|\mathcal{B}_2)|B_1) \geq 0$ and in particular $\underline{P}(G(f|B_2)|B_1) \geq 0$ for any $B_2 \subseteq B_1$. On the other hand, given $B_2 \subseteq B_1$,

$$\sup_{B_1} [G(G(f|B_2)|\mathcal{B}_1) - G(f|B_2)] \geq 0,$$

and for any $\omega \in B_1$, this sum is equal to

$$\begin{cases} G(f|B_2) - \underline{P}(G(f|B_2)|B_1) - G(f|B_2) & \text{if } \omega \in B_2 \\ 0 - \underline{P}(G(f|B_2)|B_1) - 0 & \text{if } \omega \notin B_2, \end{cases}$$

so it must be that $\underline{P}(G(f|B_2)|B_1) \leq 0$. As a consequence, $\underline{P}(G(f|B_2)|B_1) = 0$. □

Proof of Theorem 14. 1. Assume that $\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$ are coherent, and consider $i, j \in \{1, \dots, m\}$, $i < j$. Then from Lemma 3 $\underline{P}(G(f|B_j)|B_i) = 0$ and $\underline{P}(G(f|\mathcal{B}_j)|B_i) \geq 0$ for all $f \in \mathcal{L}$, $B_i \in \mathcal{B}_i$, $B_j \in \mathcal{B}_j(B_i)$.

Conversely, consider gambles $f_1, \dots, f_m$ in $\mathcal{L}$, take $i \in \{1, \dots, m\}$, $B_i \in \mathcal{B}_i$, $f_0 \in \mathcal{L}$, and let us prove that there is $B \in \{B_i\} \bigcup \cup_{j=1}^{m} S_j(f_j)$ such that

$$\sup_{B} \left[\sum_{j=1}^{m} G(f_j|\mathcal{B}_j) - G(f_0|B_i) \right] \geq 0.$$

If this is not the case, we consider the smallest k, if it exists, such that there is some $B_k \in S_k(f_k)$ with $B_i \subseteq B_k$; in case it does not exist we set k equal to i. Then $k \leq i$ and

$$\sup_{B_k} \left[\sum_{j=1}^{m} G(f_j|\mathcal{B}_j) - G(f_0|B_i) \right] = \sup_{B_k} \left[\sum_{j=k}^{m} G(f_j|\mathcal{B}_j) - G(f_0|B_i) \right].$$

If this is negative, then there is some $\delta > 0$ such that on B_k it holds $\sum_{j=k}^{m} G(f_j|\mathcal{B}_j) - G(f_0|B_i) + \delta B_k \leq 0$, whence

$$B_k G(f_0|B_i) - \delta B_k \geq B_k \sum_{j=k}^{m} G(f_j|\mathcal{B}_j). \tag{A.8}$$

As a consequence,

$$-\delta \overset{(19)}{=} \underline{P}(G(f_0|B_i)|B_k) - \delta = \underline{P}(G(f_0|B_i) - \delta B_k|B_k)$$
$$\overset{(A.8)}{\geq} \underline{P}\left(\sum_{j=k}^{m} G(f_j|\mathcal{B}_j)|B_k\right) \geq \sum_{j=k}^{m} \underline{P}(G(f_j|\mathcal{B}_j)|B_k) \overset{(20)}{\geq} 0,$$

which is a contradiction (observe that when $i = k$ or $j = k$ the above passages are trivially based on separate coherence).

2. Consider a gamble f such that $B_i f \neq 0$ and $B_j f \in \mathcal{R}_i \rfloor B_i \cup \{0\}$ for every $B_j \in \mathcal{B}_j(B_i)$. Let us focus on the generic $B_j \in \mathcal{B}_j(B_i)$ for which $B_j f \ngeq 0$. Then $B_j f = G(h|B_i) + \varepsilon B_i$ for some gamble $h \neq 0$ and $\varepsilon > 0$. Hence,

   $$\underline{P}(B_j f|B_i) = \underline{P}(G(h|B_i)|B_i) + \varepsilon = \varepsilon > 0,$$

   where the last equality follows from separate coherence. From this we deduce that $\underline{P}(f|B_j) > 0$: if we had $\underline{P}(f|B_j) \leq 0$, then it would be

   $$0 \overset{(19)}{=} \underline{P}(G(f|B_j)|B_i) = \underline{P}(B_j(f - \underline{P}(f|B_j))|B_i) \geq \underline{P}(B_j f|B_i) > 0,$$

   a contradiction. Since this happens for every $B_j \in \mathcal{B}_j(B_i)$, $B_j f \ngeq 0$, we deduce that given the gamble $f' := \sum_{B_j \in \mathcal{B}_j(B_i): B_j f \ngeq 0} B_j f$, it holds that

   $$B_i f \geq \sum_{B_j \in \mathcal{B}_j(B_i): B_j f \ngeq 0} B_j f \geq \sum_{B_j \in \mathcal{B}_j(B_i): B_j f \ngeq 0} G(f|B_j) = G(f'|\mathcal{B}_j),$$

   so that

   $$\underline{P}(f|B_i) \geq \underline{P}(G(f'|\mathcal{B}_j)|B_i) \overset{(20)}{\geq} 0$$

   and consequently $B_i f$ belongs to the closure of $\mathcal{R}_i \rfloor B_i$. But since for strictly desirable gambles conglomerability and weak conglomerability are equivalent by Theorem 4, we conclude that $B_i f \in \mathcal{R}_i \rfloor B_i$. Hence, this set is $\mathcal{B}_j(B_i)$-conglomerable.

   The second part follows from the first, taking into account the one-to-one correspondence between $\mathcal{R}_i \rfloor B_i$ and $\mathcal{R}_i | B_i$.

3. Consider $i, j \in \{1, \ldots, m\}$, $j \neq i$, and a gamble $f \ngeq 0$ (as the opposite would lead to trivial cases) such that $B_j f \in \mathcal{F}_i \cup \{0\}$ for all $B_j \in \mathcal{B}_j$. There are two possibilities:

   - If $j > i$, it follows from Proposition 2 and the second statement that $f \in \mathcal{F}_i$.
   - If $j < i$, then $B_j f \in \mathcal{F}_i \cup \{0\}$ implies that $B_j f$ must be a sum of gambles from $\mathcal{R}_i | B_i \cup \{0\}$, $B_i \in \mathcal{B}_i(B_j)$. In other words, this means that

     $$B_j f \geq \sum_{B_i \in \mathcal{B}_i: B_j B_i f \ngeq 0} B_j B_i f = G(g|\mathcal{B}_i) + h S_i(g)$$

     for some $g \in \mathcal{L}$ and non-negative $\mathcal{B}_i(B_j)$-measurable gamble h, which is strictly positive on the $\mathcal{B}_i(B_j)$-support of g (note that if there is no $B_i \in \mathcal{B}_i$ s.t. $B_j B_i f \ngeq 0$, the inequality holds with $g = 0$). As a consequence, given $B_i \in \mathcal{B}_i(B_j)$, it holds that

     $$B_i f = B_i B_j f \geq G(g|B_i) + h B_i \mathbb{I}_{S_i(g)},$$

     whence Eq. (12) implies that $B_i f \in \mathcal{R}_i | B_i \cup \{0\}$ for all $B_i \in \mathcal{B}_i$, and therefore $f \in \mathcal{F}_i$, using Eq. (14).

It follows that $\mathcal{F}_i$ is $\mathcal{B}_{1:m}$-conglomerable for all $i = 1, \dots, m$.

4. Fix $j \in \{1, \dots, m\}$, and let us show that $\mathcal{F}_1 \oplus \cdots \oplus \mathcal{F}_m$ is $\mathcal{B}_j$-conglomerable. Consider a gamble $f \not\geq 0$ (otherwise the situation is trivial) such that $B_j f \in \mathcal{F}_1 \oplus \cdots \oplus \mathcal{F}_m \cup \{0\}$ for all $B_j \in \mathcal{B}_j$. Then there are gambles $g_1, \dots, g_m$ such that $g_i \in \mathcal{F}_i \cup \{0\}$ for all $i = 1, \dots, m$ and $B_j f = g_1 + \cdots + g_m$.

   We focus on the case $B_j f \not\geq 0$, and consider the smallest index i for which there is $B_i \in S_i(g_i)$, with $B_i \cap B_j \neq 0$, such that $B_i g_i \not\geq 0$. We may assume without loss of generality that $g_k = 0$ for all $k < i$ by renaming $g_i \coloneqq g_1 + \cdots + g_i$. There are two cases.

   (a) If $i \leq j$, we consider the mentioned $B_i \in \mathcal{B}_i$ (which in this case includes B_j), for which it must be $\underline{P}(g_i|B_i) > 0$ because $B_i g_i \not\geq 0$. Moreover, it follows from coherence that $\underline{P}(g_k|B_i) \geq 0$ for all $k > i$: by definition of $\mathcal{F}_k$ it holds that $\underline{P}(g_k|\mathcal{B}_k) \geq 0$ for all $g_k \in \mathcal{F}_k$, whence $g_k \geq G(g_k|\mathcal{B}_k)$, and applying Eq. (20) $\underline{P}(g_k|B_i) \geq \underline{P}(G(g_k|\mathcal{B}_k)|B_i) \geq 0$. As a consequence,

   $$\underline{P}(B_j f|B_i) \geq \underbrace{\underline{P}(g_i|B_i)}_{>0} + \underbrace{\underline{P}(g_{i+1}|B_i)}_{\geq 0} + \cdots + \underbrace{\underline{P}(g_m|B_i)}_{\geq 0} > 0,$$

   whence

   $$B_j B_i f = B_j f \in \mathcal{R}_i|B_i \subseteq \mathcal{F}_i, \tag{A.9}$$

   where the equality holds because $B_j \subseteq B_i$.

   (b) The other possibility is that $i > j$. In this case, consider any $k \geq i$ and $B_k \in \mathcal{B}_k$ such that $B_k \subseteq B_j$, $B_k g_k \not\geq 0$ and $\forall l = 1, \dots, k-1$, s.t. $B_k \subseteq B_l, B_l \notin S_l(g_l)$; we obtain that

   $$\underline{P}(B_j f|B_k) = \underline{P}(f|B_k) \geq \underbrace{\underline{P}(g_k|B_k)}_{>0} + \underbrace{\underline{P}(g_{k+1}|B_k)}_{\geq 0} + \cdots + \underbrace{\underline{P}(g_m|B_k)}_{\geq 0} > 0,$$

   whence

   $$B_j B_k f = B_k f \in \mathcal{R}_k|B_k, \tag{A.10}$$

   using for the equality that $B_k \subseteq B_j$. Now we would like to represent $B_j f$ as a sum of elements $B_k f \not\geq 0$, collected for some $k \geq i$, where B_k belongs to $S_k(g_k)$ and $B_k \subseteq B_j$. To this end, it is convenient in particular to exclude the elements $B_k f \geq 0$ from consideration: therefore, we assume without loss of generality that for all $k = i, \dots, m-1$, $B_k \in S_k(g_k)$ implies that $\underline{P}(g_k|B_k) > 0$; otherwise, if $B_k g_k \gneq 0$, we add $B_k g_k$ to g_m. Then we define

   $$\mathcal{H}_k(B_j) \coloneqq \{B_k \in S_k(g_k) : B_k \subseteq B_j \text{ and } \forall l < k \text{ s.t. } B_k \subseteq B_l, B_l \notin S_l(g_l)\}.$$

   The idea here is simpler to understand if we reason progressively from i to m. In the first case, $\mathcal{H}_i(B_j)$ contains all the events of $\mathcal{B}_i$ where g_i is not zero. For all events B_i on which g_i is instead zero, we consider the events B_{i+1} of $\mathcal{B}_{i+1}$ that are included in B_i and such that g_{i+1} is not zero. If these events B_{i+1} do not cover B_i completely, then this means that there are some events in $\mathcal{B}_{i+1}$, included in B_i, on which g_{i+1} is zero. In this case we move on to $\mathcal{B}_{i+2}$, in a recursive fashion, up to $\mathcal{B}_m$. We can also understand the above reasoning by noting that if $f(\omega) \neq 0$ for some $\omega \in B_j$, there must be some $k \geq j$ such that $g_k(\omega) \neq 0$; and we choose the smallest such k and the set $B_k \in \mathcal{B}_k$ that includes ω.

   All the events that are considered in this way are collected in the sets $\mathcal{H}_i(B_j), \dots, \mathcal{H}_m(B_j)$ (note also that these events are all disjoint). This allows us to write

   $$B_j f = \sum_{B_i \in \mathcal{H}_i(B_j)} \overbrace{B_i f}^{\in \mathcal{R}_i|B_i} + \sum_{B_i \in \mathcal{H}_{i+1}(B_j)} \overbrace{B_{i+1} f}^{\in \mathcal{R}_{i+1}|B_{i+1}} + \cdots + \sum_{B_m \in \mathcal{H}_m(B_j)} \overbrace{B_m f}^{\in \mathcal{R}_m|B_m}, \tag{A.11}$$

   where the generic $B_k f$ belongs to $\mathcal{R}_k|B_k$ thanks to (A.10).

At this point we split $\mathcal{B}_j$ in two subsets: $\mathcal{B}_j^{i\le j}$, which contains the events B_j that are in case (a), and $\mathcal{B}_j^{i>j}$, which contains the remaining events, those that are in case (b). We obtain that

$$\sum_{B_j\in\mathcal{B}_j^{i\le j}} B_j f \in \mathcal{F}_1 \oplus \cdots \oplus \mathcal{F}_m$$

because each of those $B_j f$ belongs to a certain $\mathcal{F}_i$, as it follows from (A.9), and considering that set $\mathcal{F}_i$ is $\mathcal{B}_j$-conglomerable for all i, thanks to the third statement. On the other hand,

$$\sum_{B_j\in\mathcal{B}_j^{i>j}} B_j f = \overbrace{\sum_{B_j\in\mathcal{B}_j^{i>j}} \sum_{B_i\in\mathcal{H}_i(B_j)} B_i f}^{\in\mathcal{F}_i} + \overbrace{\sum_{B_j\in\mathcal{B}_j^{i>j}} \sum_{B_{i+1}\in\mathcal{H}_{i+1}(B_j)} B_{i+1} f}^{\in\mathcal{F}_{i+1}} + \cdots + \overbrace{\sum_{B_j\in\mathcal{B}_j^{i>j}} \sum_{B_m\in\mathcal{H}_m(B_j)} B_m f}^{\in\mathcal{F}_m},$$

thanks to (A.11), and where the generic sum $\sum_{B_j\in\mathcal{B}_j^{i>j}} \sum_{B_k\in\mathcal{H}_k(B_j)} B_k f$ belongs to $\mathcal{F}_k$ because this set is $\mathcal{B}_k$-conglomerable. As a consequence,

$$f \ge \sum_{B_j\in\mathcal{B}_j^{i\le j}} B_j f + \sum_{B_j\in\mathcal{B}_j^{i>j}} B_j f \in \mathcal{F}_1 \oplus \cdots \oplus \mathcal{F}_m,$$

(where the inequality follows because at the beginning of the proof we have only focused on terms $B_j f \not\geq 0$) so that we finally deduce that $f \in \mathcal{F}_1 \oplus \cdots \oplus \mathcal{F}_m$.

5. This is a consequence of the previous point and of Theorem 10. □

Proof of Corollary 1. The direct implication follows from Theorem 14(1). For the converse one, consider $i, j \in \{1, \dots, m\}$, $j > i$, and $B_j \in \mathcal{B}_j$, $B_i \in \mathcal{B}_i$ and $f \in \mathcal{L}$.

Let us start by showing that $\underline{P}(G(f|B_j)|B_i) = 0$. In case $B_j \not\subseteq B_i$, this is trivial because $\underline{P}(G(f|B_j)|B_i) = \underline{P}(0|B_i) = 0$. If on the other hand $B_j \subseteq B_i$, then given $h_j := G(f|B_j) \in \mathcal{L}$, it follows from (21) that $\underline{P}(h_j|B_{j-1}) = 0$, whence $h_j = G(h_j|B_{j-1})$; then, given $B_{j-2} \supseteq B_{j-1}$, we have that $\underline{P}(G(h_j|B_{j-1})|B_{j-2}) = \underline{P}(h_j|B_{j-2}) = 0$, whence $h_j = G(h_j|B_{j-2})$; repeating this argument, we obtain $h_j = G(h_j|B_{i+1})$, whence $\underline{P}(G(f|B_j)|B_i) = \underline{P}(h_j|B_i) = \underline{P}(G(h_j|B_{i+1})|B_i) = 0$.

Let us now show that $\underline{P}(G(f|\mathcal{B}_j)|B_i) \ge 0$. Given $h_j := G(f|\mathcal{B}_j)$, it follows from (22) that $\underline{P}(h_j|B_{j-1}) \ge 0$ for all $B_{j-1} \in \mathcal{B}_{j-1}$, whence $h_j \ge G(h_j|\mathcal{B}_{j-1}) =: h_{j-1}$; then $\underline{P}(h_{j-1}|B_{j-2}) \ge 0$ for all $B_{j-2} \in \mathcal{B}_{j-2}$, whence $h_{j-1} \ge G(h_{j-1}|\mathcal{B}_{j-2}) =: h_{j-2}$; by repeating this argument, we obtain that $\underline{P}(G(f|\mathcal{B}_j)|B_i) = \underline{P}(h_j|B_i) \ge \underline{P}(h_{j-1}|B_i) \ge \cdots \ge \underline{P}(h_{i+1}|B_i) \ge 0$, because $h_{i+1} = G(h_{i+2}|\mathcal{B}_{i+1})$, with $h_{i+2} \in \mathcal{L}$.

Using Theorem 14(1), we conclude that $\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$ are coherent. □

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