Learning Patterns and Pattern Sequences by Self-Organizing Nets of Threshold Elements

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Abstract—Various information-processing capabilities of self-organizing nets of threshold elements are studied. A self-organizing net, learning from patterns or pattern sequences given from outside as stimuli, "remembers" some of them as stable equilibrium states or state-transition sequences of the net. A condition where many patterns and pattern sequences are remembered in a net at the same time is shown. The stability degree of their remembrance and recalling under noise disturbances is investigated theoretically. For this purpose, the stability of state transition in an autonomous logical net of threshold elements is studied by the use of characteristics of threshold elements.

It is also shown that a self-organizing net forms a representative pattern from a given set of stimulus patterns and fixes it as a stable state. The representative pattern can be recalled from any member of the set. This kind of self-organizing net may be regarded as a model for associative memory, sequential recalling, and concept formation.

Index Terms—Associative memory, brain model, concept formation, logic nets of threshold elements, self-organization, sequential recalling, stability of state transition.

I. Introduction

SINCE the nerve cell is considered to be a kind of threshold element, it is expected that some aspects of information processing in the brain may be revealed by investigating the characteristics of nets of threshold elements. Nets of threshold elements have so far been investigated from various standpoints and many interesting results have been obtained (see e.g., [1]–[12]). However, by virtue of their nonlinear character, it is extremely difficult to analyze the behavior of nets of threshold elements in general, and we are yet far from a full understanding of information-processing capabilities of nets of threshold elements. This paper aims at elucidating theoretically those characteristics of self-organizing nets of threshold elements that may have some relation to learning and recalling of patterns and pattern sequences. The present model may be considered as a simplified version of the self-organizing random net system proposed in [1], and has close relation with the four-layer perceptron [5], the associatron [11], and the correlation matrix memories [16].

A self-organizing net has the property that its structure gradually varies, depending on the stimulus patterns applied to the net from the outside. When stimulus patterns or sequences of patterns are repeatedly applied to the net, it is expected that the net learns from these stimuli and fixes some of them by self-organization as stable equilibrium states or sequences of state transitions. It may be said that the net "remembers" these patterns or pattern sequences. Once a pattern is remembered as a stable equilibrium state, it will be recalled and reproduced correctly when a neighboring pattern is given as a stimulus. Similarly, once a pattern sequence is remembered, by giving a pattern in the sequence as a cue stimulus, the part of the sequence that follows the cue pattern is successively recalled. A self-organizing net also has the capability to form a representative pattern for each of the given sets of pattern stimuli. All of these capabilities are studied theoretically.

The present paper is divided into two parts. In the first part, the stability of state transition is investigated for a general autonomous net of threshold elements. Stability numbers of equilibrium states, state transition, and state-transition sequences are defined, and they can easily be calculated from the constants of a net. It will be shown, for example, that a net reaches an equilibrium state within k state-transition times if its initial state is located within a distance of the kth stability number from the equilibrium state, in the sense of the Hamming distance. Similar results are proved for the stability of state-transition sequences.

In the second part, three features of self-organizing nets are investigated by the use of stability numbers. One feature is concerned with learning many patterns under noise disturbances. A condition where many patterns are remembered in the net at the same time as equilibrium states is given, and by the use of stability numbers, the stability of remembrance and recalling is shown. Another feature is the learning of many pattern sequences at the same time. A condition for remembrance is given and the stability of sequential recall is shown. The third feature is concerned with the formation of a common pattern from a given set of stimulus patterns. The condition for two representative patterns to be formed, respectively, from two given sets of pat-

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terns is given. The representative pattern of a set is recalled from any member in the set.

This paper might give a new theoretical method of approach to self-organizing nets such as the perceptron [5], the associatron [11], and the nerve network model [13].

II. Stability of State Transition in a Net of Threshold Elements

A. Nets of Threshold Elements

A threshold element having n inputs is specified by n quantities $w_1, w_2, \cdots, w_n$, called weights, and a quantity h, called a threshold. The element is denoted by $E(w_1, \cdots, w_n; h)$ or briefly $E(w_i; h)$. Let the n input variables be $y_1, \cdots, y_n$ and assume that the variables take on two values 1 and -1. The output is 1 when the weighted sum $\sum_i w_i y_i$ exceeds h. Otherwise, the output is -1. Therefore, the input–output relation of the element is described as

$$x = \operatorname{sgn}\left(\sum_i w_i y_i - h\right) \tag{1}$$[1]

where

$$\operatorname{sgn}(u) = \begin{cases} 1, & u > 0 \\ -1, & u < 0. \end{cases} \tag{2}$$

From this it can easily be shown that two elements $E(w_i; h)$ and $E(cw_i; ch)$, where c is a positive constant, have the same characteristics. Therefore, for the sake of definiteness

$$|w_i| \leqq 1$$

is assumed. It is desirable that the weights be so normalized as to satisfy

$$\max_i |w_i| = 1.$$

Consider an autonomous net composed of n threshold elements $E_1, \cdots, E_n$, E_j being $E(w_{j1}, w_{j2}, \cdots, w_{jn}; h)$. These elements are interconnected in such a manner that the output of E_i, denoted by x_i, is connected with the ith inputs of all the elements with unit time delay. Therefore, output signal x_i of E_i is multiplied by w_{ji} when it enters into E_j. The net is assumed to work synchronously at fixed time intervals. The net is specified by the w_{ji} and h_j. We call the $n \times n$ matrix $W = (w_{ji})$ a weight matrix and n-vector $h = (h_j)$ a threshold vector where $|w_{ji}| \leqq 1$ is also assumed.

An n-dimensional column vector $x = (x_i)$, where x_i is the present output of E_i, is called the present state of the net. The next state x' is uniquely determined by the present state, because output x_i' of E_i at the next time is determined by

$$x_i' = \operatorname{sgn}\left(\sum_j w_{ij} x_j - h_i\right). \tag{3}$$[2]

By denoting the state-transition operator by T, the next state is written as

$$x' = Tx.$$

Only a few properties are yet known about the state-transition operators of nets of threshold elements.[3]

A state x which is invariant under T,

$$x = Tx$$

is called an equilibrium state. An ordered set of states $B = \{x_1, x_2, \cdots, x_m\}$, for which

$$x_{i+1} = Tx_i, \qquad i = 1, 2, \cdots, m-1$$

holds, is called a state-transition sequence of length m. A state-transition sequence $C = \{x_1, \cdots, x_m\}$, for which $Tx_m = x_1$ holds, is called a length m cycle.

Since the number of states in a net is finite, the sequence of states beginning at an arbitrary x_0

$$x_0, Tx_0, T^2x_0, \cdots$$

either converges to an equilibrium state or falls into a cycle within a finite number of state transitions. Equilibrium states and cycles can be considered as patterns that the net can retain persistently without any input from the outside.

B. Stability Numbers of State Transition

Several preliminary definitions are given first. Let dis (x, y) be the distance between two states $x = (x_i)$ and $y = (y_i)$ defined by

$$\operatorname{dis}(x, y) = \frac{1}{2} \sum_i |x_i - y_i|.$$

This is the well-known Hamming distance, representing the number of different components. The set of those states that are located within a distance of k from x, is called the k-neighborhood of x and is denoted by $N(x, k)$. Hence,

$$N(x, k) = \{y \mid \operatorname{dis}(y, x) \leqq k\}.$$

We define n functions $u_i(x, y)$ $(i = 1, 2, \cdots, n)$ of two states x and y by

[1] In order to avoid unnecessary complicated discussion, it is assumed throughout the paper that the case $\sum_i w_i y_i - h = 0$ never occurs.

[2] This may be regarded as an approximate expression of the state equation of a random net system [1]. It has been shown in [1] that the activity level z_i' at the next time of the ith component random net is given by

$$z_i' = \Phi\left(\sum_j W_{ij} z_j - \Theta_i\right)$$

where z_j is the present activity level of the jth component random net, w_{ji} is the coupling coefficient from the jth to the ith random net, Θ_i is the average threshold of the ith random net, and Φ is defined by

$$\Phi(u) = 2 \int_0^u \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt.$$

If function Φ is approximated by the sgn function, then the above state equation reduces to (3). In this approximation, a component random net plays the same role as a single threshold element does in the present model.

[3] See, e.g., [6]–[9].

$$u_i(x,y) = y_i\Big(\sum_j w_{ij}x_j - h_i\Big). \tag{4}$$

Obviously, $u_i(x, y)$ is positive when the ith component of Tx coincides with that of y, and is negative otherwise.

The following operators min (k), $k=1, 2, \cdots$, play an important role in studying the stability of state transition. For n real numbers $u_1, u_2, \cdots, u_n$, min $(k)\{u_i\}$ denotes the $(k+1)$st smallest number of u_i's (the largest number when $k \geqq n$), i.e., when the u_i are rearranged in order of magnitude as

$$u_{i_1} \leqq u_{i_2} \leqq \cdots \leqq u_{i_n}$$

$$\min(k)\{u_i\} = \begin{cases} u_{i_{k+1}}, & \text{when } 0 \leqq k < n \\ u_{i_n}, & \text{when } k \geqq n. \end{cases}$$

The following lemma elucidates the role of functions $u_i(x, y)$ and operators min (k).

Lemma 1: A necessary and sufficient condition that state x falls in the k-neighborhood of y after a single step of state transition, i.e., $Tx \in N(y; k)$ holds, is

$$\min(k)\{u_i(x,y)\} > 0. \tag{5}$$

Proof: The inequality $\min(k)\{u_i\} > 0$ implies that the number of negative u_i is not more than k and vice versa. On the other hand, $u_i(x, y)$ is negative, when, and only when, y_i is not equal to $\mathrm{sgn}(\sum_j w_{ij}x_j - h_i)$ or the ith component of Tx. Therefore, the number of negative u_i is equal to dis (Tx, y). Hence, inequality (5) is equivalent to

$$\mathrm{dis}\,(Tx, y) \leqq k$$

which means $Tx \in N(y, k)$.

We can now define the degree of stability of state transition. The k-stability number $s(x, k)$ of state transition $x \to Tx$ is defined, for nonnegative k by

$$s(x, k) = [r(x, k)] \tag{6}$$

$$r(x, k) = \tfrac{1}{2} \min(k)\{u_i(x, Tx)\} \tag{7}$$[4]

where $[r]$ denotes the integer part of r. Obviously, for $k \geqq k'$,

$$s(x, k) \geqq s(x, k')$$

holds. The following lemma clarifies the meaning of the k-stability number.

Lemma 2: When z belongs to the $s(x, k)$-neighborhood of x (i.e., when $z \in N\{x, s(x, k)\}$), Tz belongs to the k-neighborhood of Tx (i.e., $Tz \in N(Tx, k)$). In particular, if z belongs to the $s(x, 0)$-neighborhood of x, then $Tz = Tx$.

Proof: Let z be a state belonging to $N\{x, s(x, k)\}$. Then,

$$\mathrm{dis}\,(z, x) \leqq s(x, k).$$

If we put

$$z = x + 2e$$

e is a vector whose components are 0 or ± 1, and

$$\sum_i |e_i| = \mathrm{dis}\,(z, x) \leqq s(x, k)$$

because $|e_i| = 1$ when $x_i \neq z_i$, and $e_i = 0$, otherwise. For the purpose of obtaining the distance between Tz and Tx, a lower bound of $u_i(z, Tx)$ is obtained as follows where we put $x' = Tx$:

$$\begin{aligned} u_i(z, Tx) &= x_i'\Big(\sum_j w_{ij}z_j - h_i\Big) \\ &= x_i'\Big(\sum w_{ij}x_j - h_i + 2\sum w_{ij}e_j\Big) \\ &\geqq x_i'\Big(\sum w_{ij}x_j - h_i\Big) - 2\sum |w_{ij}||e_j| \\ &\geqq u_i(x, Tx) - 2\sum |e_j| \geqq u_i(x, Tx) - 2s(x, k) \end{aligned}$$

where $|w_{ij}| \leqq 1$ is used. From this follows:

$$\begin{aligned} \min(k)\{u_i(z, Tx)\} &\geqq \min(k)\{u_i(x, Tx)\} - 2s(x, k) \\ &= 2r(x, k) - 2[r(x, k)] > 0 \end{aligned}$$[5]

which by virtue of Lemma 1 means

$$Tz \in N(Tx, k).$$

By putting $k=0$, the latter half of the lemma is obtained.

C. Stability Numbers of Equilibrium States

A necessary and sufficient condition for x to be an equilibrium state is easily obtained, by putting $y = x$ and $k = 0$ in Lemma 1.

Theorem 1: State x is equilibrium iff

$$\min_i \{u_i(x, x)\} > 0.$$

The stability numbers of an equilibrium state x are defined by the use of $s(x, k)$ as follows. The first stability number $s_1(x)$ of x is defined by

$$s_1(x) = s(x, 0) = \big[\tfrac{1}{2} \min(0)\{u_i(x, x)\}\big].$$

The jth stability number $s_j(x)$ is recursively defined by using the $(j-1)$th stability number as

$$s_j(x) = s\{x, s_{j-1}(x)\}. \tag{8}$$

In other words, $s_j(x)$ is the $s_{j-1}(x)$-stability number of state transition $x \to Tx = x$.

Lemma 3: The sequence $s_1(x), s_2(x), \cdots$ of the stability numbers is monotonically nondecreasing, converging to the limit $s(x)$ within a finite number of terms.

Proof: Since x is an equilibrium state, $s_1(x) \geqq 0$ holds. From the monotonicity that $s(x, k) \geqq s(x, k')$ for $k \geqq k'$

$$s_2(x) = s\{x, s_1(x)\} \geqq s(x, 0) = s_1(x)$$

holds. Similarly, $s_{j+1}(x) \geqq s_j(x)$ can be proved from $s_j(x) \geqq s_{j-1}(x)$. Hence, $s_j(x)$ is monotonically nondecreasing. Since $s_j(x)$ coincides with one of $u_i(x, x)$'s, $s_{j_0}(x) = s_{j_0+1}(x)$ must hold for some $j_0 \leqq n$. In this case,

[4] Note that $u_i(x, Tx) = |\sum_j w_{ij}x_j - h_i|$.

[5] It has been assumed that $u_i(x, Tx)$ is not equal to 0.

$s_{j_0+1}(x) = s_{j_0+2}(x) = \cdots$, and the sequence converges to $s(x) = s_{j_0}(x)$. The limit $s(x)$ will be called the stability number of equilibrium state x.

We call an equilibrium state x stable if $s(x) > 0$ and unstable if $s(x) = 0$. Obviously, when and only when $s_1(x) = 0$, $s(x) = 0$ holds and the state is unstable.

Let us call the $s_k(x)$-neighborhood $N\{x, s_k(x)\}$ of an equilibrium state x the kth stability domain, and denote it by $D_k(x)$

$$D_k(x) = N\{x, s_k(x)\}. \tag{9}$$

Let $D_0(x)$ and $D(x)$ be, respectively,

$$D_0(x) = N(x, 0) = \{x\} \tag{10}$$

and

$$D(x) = N\{x, s(x)\}. \tag{11}$$

We call $D(x)$ simply the stability domain of x. From the monotonicity of $\{s_k(x)\}$

$$D_0(x) \subset D_1(x) \subset D_2(x) \subset \cdots \subset D(x)$$

holds.

Theorem 2: Let x be an equilibrium state. Then, the net arrives at state x after a finite number of state transitions if its initial state belongs to $D(x)$. More specifically, the net arrives at state x within k transition times if the initial state belongs to $D_k(x)$.

Proof: Let z be a state belonging to $D_k(x)$. Then,

$$z \in N\{x, s_k(x)\}.$$

Since $s_k(x) = s\{x, s_{k-1}(x)\}$, z belongs to the $s\{x, s_{k-1}(x)\}$-neighborhood of x. Hence, by virtue of Lemma 2, Tz belongs to the $s_{k-1}(x)$-neighborhood of $Tx = x$, i.e., $Tz \in D_{k-1}(x)$. Therefore, $T^k z \in D_0(x) = \{x\}$ which verifies the latter half of the theorem. The former half is trivial, since $D(x)$ coincides with some $D_k(x)$, $k \leqq n$.

Example: The stability numbers are calculated for a net consisting of three elements. Let the weight matrix and threshold vector be, respectively,

$$W = (w_{ij}) = \begin{bmatrix} 0.6 & 1.0 & 0.5 \\ 1.0 & 0.6 & 0.6 \\ 0.5 & 1.0 & 0.8 \end{bmatrix} \quad h = (h_i) = \begin{bmatrix} 0 \\ -1.8 \\ -4.0 \end{bmatrix}.$$

There are eight states. For state $x = (1, 1, 1)^t$, where "t" denotes the transposition of a vector, $u_1(x, x) = 2.1$, $u_2(x, x) = 4.0$, and $u_3(x, x) = 6.3$, so that min $\{u_i(x, x)\} = 2.1 > 0$. Hence, this is equilibrium. It is easy to show that there are no other equilibrium states. The stability numbers of $x = (1, 1, 1)^t$ are calculated recursively

$$s_1(x) = [\tfrac{1}{2} \times 2.1] = 1$$
$$s_2(x) = [\tfrac{1}{2} \times \min (1)\{u_i\}] = 2$$
$$s_3(x) = [\tfrac{1}{2} \times \min (2)\{u_i\}] = 3.$$

Hence, x is a stable equilibrium state and its stability number is $s(x) = 3$.

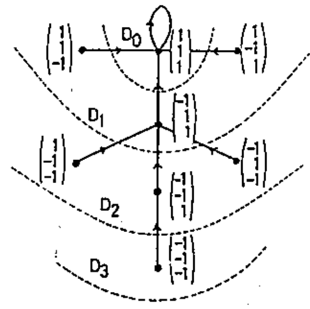

Fig. 1. State-transition diagram.

The stability domain of x is given by

$$D_0(x) = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$$D_1(x) = \left\{ D_0, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

$$D_2(x) = \left\{ D_1, \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

$$D(x) = D_3(x) = \{\text{all states}\}.$$

Therefore, in whatever initial state it is, the net arrives at state $(1, 1, 1)^t$ at most after three state transitions. This can indeed be shown by the state-transition diagram of the net in Fig. 1.

D. Stability Numbers of Sequences and Cycles

Let $B = \{y_1, y_2, \cdots, y_m\}$ be a state-transition sequence satisfying $y_{i+1} = Ty_i$. Starting at

$$s_B(y_m) = 0$$

the following quantities are calculated recursively

$$s_B(y_{i-1}) = s\{y_{i-1}, s_B(y_i)\}, \qquad i = m, m-1, \cdots, 2. \tag{12}$$

We call $s_B(y_i)$ the stability number of y_i in state-transition sequence B.

It is obvious from the definition and Lemma 2 that, if state z belongs to the $s_B(y_i)$-neighborhood of y_i, Tz belongs to the $s_B(y_{i+1})$-neighborhood of y_{i+1}. Hence, if the initial state is in the $s_B(y_i)$-neighborhood of y_i, the net, passing successively through the $s_B(y_j)$-neighborhood of y_j $(j = i+1, i+2, \cdots, m-1)$, arrives at y_m after $m-i$ state transitions.

Similarly, we can proceed with the definition of stability of a cycle. Let $C = \{y_1, y_2, \cdots, y_m\}$ be a cycle of length m. Starting at $s^1(y_m) = 0$, the following $s^1(y_i)$ are recursively calculated

$$s^1(y_{i-1}) = s(y_{i-1}, s^1(y_i)).$$

Then we define $s^2(y_m)$ by

$$s^2(y_m) = s(y_m, s^1(y_1)).$$

The $s^2(y_i)$ are defined from $s^2(y_m)$ similarly $(i=m-1, \cdots, 2, 1)$, and we can then define $s^3(y_m)$, and so on. Since the sequence $s^1(y_i), s^2(y_i), \cdots$ is monotonically nondecreasing, it converges within a finite number of terms. The limit

$$s_C(y_i) = \lim_{k\to\infty} s^k(y_i) \tag{13}$$

is called the stability number of y_i in cycle C, and

$$s(C) = \min_i s_C(y_i)$$

the stability number of C. The following theorem is a direct consequence of the above definition and Lemma 2.

Theorem 3: Let $s_C(y_i)$ be the stability number of y_i in cycle C. Then, the state of the net falls into cycle C after a finite number of state transitions, provided the initial state belongs to the $s_C(y_i)$-neighborhood of y_i for some i.

III. Self-Organization of Threshold-Element Nets

It is believed that the structure of a nerve net changes adaptively according to the stimuli given from the outside. Such a net may be said to have self-organizing capability. The hypothesis that the brain is organized by changing the synaptic weights of neurons was proposed by Hebb [14]. Although the hypothesis is not yet proved physiologically, various engineering models of self-organization have been proposed based on this hypothesis (e.g., [5], [13]). The self-organization in the present paper is also of this kind.

Assume that patterns are applied to the net as stimuli from the outside, and that a pattern is represented by an n-dimensional vector whose components are ± 1, like a state vector where n is the number of elements in the net. It is also assumed that the net is forced to be in state $\mathbf{x}$ when stimulus pattern $\mathbf{x}$ is applied. Let $\mathbf{x}(t)$ be the stimulus pattern applied at time $t(t=1, 2, \cdots)$. The weights of the net change according to these stimulus patterns.

Two types of self-organizing nets are considered in the present paper. In a type I net, weight w_{ij} connecting the output of E_j to the input of E_i increases by a unit when the ith component of the stimulus pattern coincides with the jth component and decreases by the same unit, when they differ with each other. In a type II net, w_{ij} increases by a unit when the ith component of the present stimulus pattern coincides with the jth component of the previous stimulus pattern and decreases when they differ. In both nets, the weights always attenuate with a fixed time constant. The increments of the weights can be represented by

$$\Delta w_{ij}(t) = -\alpha w_{ij}(t) + \beta f_{ij}(t) \tag{14}$$

where α and β are small positive constants and

$$f_{ij}(t) = x_i(t)x_j(t)$$

for a type I net and

$$f_{ij}(t) = x_i(t)x_j(t-1)$$

for a type II net. Equation (14) can be solved to yield

$$w_{ij}(t) = (1-\alpha)^t w_{ij}^0 + \beta \sum_{\tau=0}^{t-1} (1-\alpha)^{t-\tau-1} f_{ij}(\tau)$$

where w_{ij}^0 is the initial value of $w_{ij}(t)$. Evidently, as t becomes large, the first term attenuates and the second dominates.

When sequence $\mathbf{x}(t)$ can be regarded as an ergodic stochastic process, $f_{ij}(t)$ can also be regarded as such, and the average

$$k_{ij} = \bar{f}_{ij}$$

where the overbar denotes the expectation, coincides with the time average of $f_{ij}(\tau)$ almost everywhere. Matrix K, defined by $K=(k_{ij})$ may be called a correlation matrix. In the case of a type I net

$$K = \overline{\mathbf{x}(t)\mathbf{x}(t)^t}$$

and in the case of a type II net

$$K = \overline{\mathbf{x}(t)\mathbf{x}(t-1)^t}.$$

When α is sufficiently small, for large t, the weighted average

$$\alpha \sum_{\tau=0}^{t-1} (1-\alpha)^{t-\tau-1} f_{ij}(\tau)$$

of $f_{ij}(t)$ can be approximated by the ordinary time average

$$k_{ij} = \frac{1}{t} \sum_{\tau=0}^{t-1} f_{ij}(\tau).$$

Therefore,

$$w_{ij}(t) = \frac{\beta}{\alpha} k_{ij}$$

holds approximately for large t. When $\alpha=\beta$, $w_{ij}(t)$ converges to k_{ij}.[6] In this case, $w_{ij}(t)$ satisfies the requirement

$$|w_{ij}| \leqq 1.$$

Therefore, it is assumed in the following that the weight matrix of the net approaches the correlation matrix of applied stimulus patterns by self-organization.

The self-organization manner of a type II net seems to be more natural and reasonable than that of a type I net. However, self-organization of a type I net may be

[6] More precisely speaking, $w_{ij}(t)$ converges to k_{ij} only on the average, fluctuating around it. The fluctuation can be made as small as desired by keeping α small. As α becomes large, the convergence speed of $w_{ij}(t)$ to k_{ij} becomes high, whereas the deviation of $w_{ij}(t)$ from k_{ij} becomes large. Such a tradeoff between convergence speed and accuracy exists in various learning problems (see e.g., [15]).

regarded as a version of type II. Consider the case where each stimulus continues for a considerably long period compared with a state-transition time. In this case, $x(t)=x(t-1)$ holds except at the time when a stimulus is replaced by another. Hence, $\overline{x(t)x(t-1)^t}$ can be approximated by $\overline{x(t)x(t)^t}$, and the type I self-organization can be realized by a special training program in a type II net.

Self-organization of a type I net is treated in the next section and that of a type II net in Section V.

IV. Pattern Learning by Type I Net

A. Learning a Single Pattern under Noise Disturbance

If the initial weights of a type I self-organizing net are given by

$$w_{ij}^0 = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

and if $h_i=0$, then all the states are initially equilibrium because

$$u_i(\mathbf{x}, \mathbf{x}) = x_i \sum_j w_{ij}^0 x_j = 1$$

and hence $\min \{u_i(\mathbf{x}, \mathbf{x})\} > 0$. Therefore, the net can persistently retain any initial state. In this sense, it can be said that the net is not organized at all. However, their stability numbers are 0, and all the states are unstable.

We assume, hereafter, $h_i=0$ for all i for simplicity's sake. It means that every component threshold element is self-dual. This implies that when the relation $T\mathbf{x}=\mathbf{y}$ holds, $T(-\mathbf{x})=-\mathbf{y}$ also holds. The behavior of $-\mathbf{x}$ is completely determined by that of $\mathbf{x}$. Therefore, if $\mathbf{x}$ is stable, then $-\mathbf{x}$ is stable, and the stability numbers of $\mathbf{x}$ and $-\mathbf{x}$ are the same. Hence, $-\mathbf{x}$ can be regarded as the reverse expression of the same pattern $\mathbf{x}$, and these two patterns are treated as one pattern in the following.

First of all, the net organization is shown, when a single stimulus pattern is applied repeatedly under noise disturbances. When only pattern $\mathbf{x}$ is applied repeatedly to the net, obviously the weight matrix converges to $\mathbf{x}\mathbf{x}^t$. If $\mathbf{x}$ is disturbed by noises, it changes to another similar pattern. Let $\tilde{\mathbf{x}}$ be a random pattern whose components are obtained by reversing the sign of each component of $\mathbf{x}$ independently with probability p. In other words

$$\tilde{x}_i = \begin{cases} x_i, & \text{with probability } 1-p \\ -x_i, & \text{with probability } p. \end{cases}$$

When the applied stimulus pattern is disturbed by noises of probability p, it changes to $\tilde{\mathbf{x}}$. In this case the weight matrix of the net converges to

$$K = \overline{\tilde{\mathbf{x}}\tilde{\mathbf{x}}^t}. \tag{15}$$

From

$$\overline{\tilde{x}_i} = (1-2p)x_i$$

it follows that

$$k_{ij} = \overline{\tilde{x}_i\tilde{x}_j} = \begin{cases} 1, & i = j \\ (1-2p)^2 x_i x_j, & i \neq j. \end{cases} \tag{16}$$

Therefore, by putting $1-\sigma^2=(1-2p)^2$ or

$$\sigma^2 = 4p(1-p) \tag{17}$$

$$K = (1-\sigma^2)\mathbf{x}\mathbf{x}^t + \sigma^2 E \tag{18}$$

is obtained, where E is the unit matrix. The weight matrix eventually coincides with the above K by applying $\mathbf{x}$ under noise disturbances.

Parameter σ^2 represents the intensity of noise disturbances. It satisfies

$$0 \leqq \sigma^2 \leqq 1$$

and, when $p=0$, i.e., when no noises exist, $\sigma^2=0$. When $p=\frac{1}{2}$, $\sigma^2=1$. In the latter case $\tilde{\mathbf{x}}$ includes no information about original $\mathbf{x}$.

Let $\cos(\mathbf{x}, \mathbf{y})$ be the cosine of the angle between $\mathbf{x}$ and $\mathbf{y}$. Then

$$\cos(\mathbf{x}, \mathbf{y}) = \frac{1}{n}\mathbf{x}\cdot\mathbf{y}$$

holds, where the center dot denotes the inner product.

Theorem 4: When pattern $\mathbf{x}$ is applied repeatedly under noise disturbances of intensity σ^2, the net is self-organized in such a manner that we have the following.

a) Pattern $\mathbf{x}$ is stored in the net as an equilibrium state and its stability number is

$$s(\mathbf{x}) = \left[\frac{1-\sigma^2}{2}n + \frac{\sigma^2}{2}\right]. \tag{19}$$

b) Those $\mathbf{y}$ that satisfy

$$|\cos(\mathbf{x}, \mathbf{y})| < \frac{\sigma^2}{n(1-\sigma^2)} \tag{20}$$

remain unstable equilibrium states with stability number 0. All the other patterns change to $\mathbf{x}$ in a single state transition.

Proof: After the self-organization is completed

$$W = (1-\sigma^2)\mathbf{x}\mathbf{x}^t + \sigma^2 E$$

holds. Therefore, for an arbitrary $\mathbf{y}$

$$u_i(\mathbf{y}, \mathbf{y}) = y_i \sum_j w_{ij} y_j = \sigma^2 + (1-\sigma^2)x_i y_i \mathbf{x}\cdot\mathbf{y}.$$

For $\mathbf{y}=\mathbf{x}$

$$u_i(\mathbf{x}, \mathbf{x}) = (1-\sigma^2)n + \sigma^2 > 0.$$

Therefore, $\mathbf{x}$ is equilibrium and $s(\mathbf{x})$ is given by (19). For $\mathbf{y}\neq\mathbf{x}$

$$\begin{aligned} \min\{u_i(\mathbf{x}, \mathbf{x})\} &= \sigma^2 - (1-\sigma^2)|\mathbf{x}\cdot\mathbf{y}| \\ &= \sigma^2 - (1-\sigma^2)n|\cos(\mathbf{x}, \mathbf{y})| < \sigma^2 < 1. \end{aligned}$$

Hence, $\mathbf{y}$ is unstable equilibrium when (20) is satisfied. Lastly, when $\mathbf{y}$ is not equilibrium and $\mathbf{x}\cdot\mathbf{y}>0$

$$u_i(y, x) = x_i\{x_i(1-\sigma^2)x\cdot y + \sigma^2 y_i\}$$
$$\geqq (1-\sigma^2)n\,|\cos(x, y)| - \sigma^2 > 0.$$

Consequently, $Ty = x$ is proved. Similarly, when $x\cdot y < 0$, $Ty = -x$ holds.

It may be said that the net remembers x correctly as an equilibrium state. Even when the intensity of noises is very strong, the net can remember x with a large stability number if n is sufficiently large. When a pattern y not satisfying (20) is given from the outside after the learning is completed, the net, being set in state y by the pattern, changes the state automatically to state x and remains in it. Thus, it may be said that the net recalls x from all the unstable patterns. In other words, the correct pattern x is recalled and reproduced from a noisy pattern y.

Let us find the number of elements which are needed for x to be recalled from all the patterns belonging to the k-neighborhood of x, i.e., from y satisfying

$$\text{dis}(x, y) \leqq k, \qquad \left(k < \frac{n}{2}\right).$$

By virtue of the identity

$$\cos(x, y) = 1 - \frac{2}{n}\,\text{dis}(x, y)$$

the above inequality is rewritten as

$$\cos(x, y) \geqq 1 - \frac{2}{n}k.$$

Since x can be recalled from all y not satisfying (20), the condition is

$$1 - \frac{2}{n}k \geqq \frac{\sigma^2}{n(1-\sigma^2)}$$

or

$$n \geqq \frac{\sigma^2}{1-\sigma^2} + 2k.$$

When the noise probability is given by

$$p = \frac{1}{4},\ p = \frac{1}{3},\quad \text{and}\quad p = \frac{2}{5},$$

the condition gives, respectively, $n \geqq 2k+3$, $n \geqq 2k+8$, and $n \geqq 2k+24$. If n and k are large, the noise effect becomes relatively small. For $k = 50$, these conditions are $n \geqq 103$, $n \geqq 108$, and $n \geqq 124$, respectively. It is rather surprising that, even when the noise disturbances are as strong as above, x can be recalled correctly in a net of such a number of elements.

B. Learning Many Patterns

When m patterns x_α ($\alpha = 1, 2, \cdots, m$) are applied repeatedly to the net x_α with relative frequency λ_α under noise disturbance, a condition that the net is organized to remember them correctly is obtained. Let σ_α^2 be the intensity of noise disturbing x_α, and put

$$\tilde{\lambda}_\alpha = (1-\sigma_\alpha^2)\lambda_\alpha \tag{21}$$

$$\sigma^2 = \sum_\alpha \lambda_\alpha \sigma_\alpha^2. \tag{22}$$

Then by learning these patterns, the weight matrix converges to

$$W = \sum_\alpha \tilde{\lambda}_\alpha x_\alpha x_\alpha^t + \sigma^2 E.$$

Let $\epsilon_{\alpha\beta}$ be the cosine of the angle between two patterns x_α and x_β,

$$\epsilon_{\alpha\beta} = \cos(x_\alpha, x_\beta).$$

Let $\bar{\epsilon}_\alpha$ be

$$\bar{\epsilon}_\alpha = \sum_{\beta\neq\alpha} \tilde{\lambda}_\beta\,|\epsilon_{\alpha\beta}|.$$

This is a weighted sum of the absolute values of the cosines of the angles betwen x_α and all the other patterns. When x_α is orthogonal to all the other patterns, $\bar{\epsilon}_\alpha = 0$.

Theorem 5: The net remembers x_α as an equilibrium state when

$$\tilde{\lambda}_\alpha > \bar{\epsilon}_\alpha - \frac{\sigma^2}{n}. \tag{23}$$

Moreover, the net recalls x_α correctly from z belonging to $N(x_\alpha, s_\alpha)$ where

$$s_\alpha = \frac{n}{2}(\tilde{\lambda}_\alpha - \bar{\epsilon}_\alpha) + \frac{\sigma^2}{2}. \tag{24}$$

Proof: Let $x_{\alpha i}$ be the ith component of x_α. Then

$$u_i(x_\alpha, x_\alpha) = x_{\alpha i}\left\{\sum_\beta \tilde{\lambda}_\beta x_{\beta i} x_\beta\cdot x_\alpha + \sigma^2 x_{\alpha i}\right\}$$
$$= \sigma^2 + n\tilde{\lambda}_\alpha + n\sum_{\beta\neq\alpha}\tilde{\lambda}_\beta \epsilon_{\alpha\beta} x_{\alpha i} x_{\beta i}$$
$$\geqq \sigma^2 + n\tilde{\lambda}_\alpha - n\bar{\epsilon}_\alpha = 2s_\alpha.$$

Therefore, when (23) is satisfied $u_i(x_\alpha, x_\alpha) > 0$ and x_α is equilibrium. Since the stability number $s(x_\alpha)$ of equilibrium x_α is not less than $[s_\alpha]$, x_α is recalled from z belonging to $N(x_\alpha, s_\alpha)$.

As λ_α becomes large, or $\bar{\epsilon}_\alpha$ becomes small, or σ_α^2 becomes small, x_α can be remembered better. When the x_α are orthogonal to one another, $\bar{\epsilon}_\alpha = 0$ holds. Hence all the patterns are remembered in this case. If their relative frequencies are the same, x_α can be recalled from a pattern belonging to $N(x, s_\alpha)$, where

$$s_\alpha = \frac{(1-\sigma_\alpha^2)n}{2m} + \frac{\sigma^2}{2}.$$

Obviously s_α becomes small as m becomes large. By the mutual interactions of patterns, the domain of patterns from which x_α is recalled decreases, as the number of remembered patterns increases. This shows that the number of patterns remembered with positive stability numbers at the same time is at most n.

This explains the capacity of a type I net as well as the associatron [11]. The net can remember a set of n mutually orthogonal patterns. One of these patterns has n bits of information amount. However, the second pattern is restricted to be orthogonal to the first, and hence it has $n-1$ degrees of freedom, carrying $n-1$ bits of information. Similarly, the third pattern is restricted to be orthogonal to the first and second patterns, and hence it carries $n-2$ bits of information, and so on. Therefore, a set of n orthogonal patterns has

$$i = \tfrac{1}{2}n(n+1)$$

bits of information. This is the information capacity of a type I net.

It should be noted that the number of independent synaptic weights is $\frac{1}{2}n(n+1)$, because, in a type I net, $w_{ij}=w_{ji}$ holds. It is noteworthy that the net has the same bits of information capacity as the number of its independent synaptic weights.

V. Learning Patterns and Pattern Sequences by Type II Net

A. Learning Pattern Sequences

When a sequence of patterns $B=\{x_1, \cdots, x_m, x_{m+1}\}$ is applied for many times to a type II net as stimuli, weight matrix W of the net is expected to converge to

$$W = \frac{1}{m}\sum_{i=1}^{m} x_{i+1}x_i^t.$$

Noise disturbances are disregarded in this section for simplicity's sake, although it is easy to evaluate the effect of noises in a similar manner.

Let $B_\alpha=\{x_1^\alpha, x_2^\alpha, \cdots, x^\alpha_{m_\alpha+1}\}$, $\alpha=1, 2, \cdots, k$, be k pattern sequences, B_α being of length $m_\alpha+1$. Some of B_α may be cycles of length m_α, for which $x_{m_\alpha+1}=x_1^\alpha$ holds. When these sequences are applied to the net, B_α with relative frequency λ_α, W is expected to converge to

$$W = \sum_\alpha \frac{\lambda_\alpha}{m_\alpha}\sum_{i=1}^{m_\alpha} x_{i+1}^\alpha x_i^{\alpha t}. \tag{25}$$

A condition, that the net remembers these B_α at the same time as state-transition sequences or cycles is studied. Put

$$\epsilon_{ij}^{\alpha\beta} = \cos(x_i^\alpha, x_j^\beta)$$

and

$$\epsilon_j^\alpha = \sum_{\beta, i} \frac{\lambda_\beta}{m_\beta}\,|\epsilon_{ij}^{\beta\alpha}| - \frac{\lambda_\alpha}{m_\alpha}.$$

ϵ_j^α is the average of the absolute values of cosines of x_j^α to all the other patterns.

Theorem 6: By applying k sequences, B_α with relative frequency λ_α, the net remembers B_α as a state-transition sequence or cycle, when

$$\lambda_\alpha > m_\alpha \epsilon_j^\alpha \tag{26}$$

for all $j=1, 2, \cdots, m_\alpha+1$. The stability number $s_B(x_j^\alpha)$ of x_j^α in B_α is not less than

$$s_j^\alpha = \left[\frac{n}{2}\left(\frac{\lambda_\alpha}{m_\alpha} - \epsilon_j^\alpha\right)\right].$$

Proof: From

$$Wx_j^\alpha = \sum_\beta \frac{\lambda_\beta}{m_\beta}\sum_h x_{h+1}^\beta(x_h^\beta \cdot x_j^\alpha)$$

$$\begin{aligned} u_i(x_j^\alpha, x_{j+1}^\alpha) &= x_{j+1}^\alpha{}_{,i}\left(\sum_{\beta,h}\frac{\lambda_\beta}{m_\beta}\, x_{h+1}^\beta{}_{,i}\, n\epsilon_{hj}^{\beta\alpha}\right) \\ &\geqq n\left\{\frac{\lambda_\alpha}{m_\alpha} - \left(\sum_{\beta,h}\frac{\lambda_\beta}{m_\beta}\,|\epsilon_{hj}^{\alpha\beta}| - \frac{\lambda_\alpha}{m_\alpha}\right)\right\} \\ &= n\left(\frac{\lambda_\alpha}{m_\alpha} - \epsilon_j^\alpha\right) \end{aligned}$$

is obtained, where $x_j^\alpha{}_{,i}$ is the ith component of x_j^α. Hence, when (26) is satisfied, $u_i(x_j^\alpha, x_{j+1}^\alpha)>0$ and $Tx_j^\alpha = x_{j+1}^\alpha$. Moreover, the stability number of x_j^α is not less than s_j^α.

Corollary: When the patterns constituting the sequences are orthogonal to one another, all the sequences are remembered, and the stability number of B_α is

$$s_\alpha = \left[\frac{n\lambda_\alpha}{2m_\alpha}\right].$$

After the remembrance of B_α, if a pattern $z \in N(x_j^\alpha, s_j^\alpha)$ is given as a stimulus pattern, the net, being in state z in the beginning, changes the state successively to x_{j+1}^α, $x_{j+2}^\alpha, \cdots$, and $x^\alpha_{m_\alpha+1}$. The net may be said to recall successively the part of B_α that follows x_j^α, from cue pattern z resembling x_j^α. The net does not recall the part before x_j^α from it. It is interesting that this property of sequential recalling resembles that of our memory in the brain (cf. [17]).

When $\lambda_\alpha < m_\alpha\epsilon_j^\alpha$ holds for x_j^α in B_α, sequence B_α is not necessarily remembered, and Tx_j^α may be unequal to x_{j+1}^α. In this case, the sequence is cut apart at x_j^α, and it may be absorbed in some other sequence. Even after B_α is remembered, if another sequence B_β is repeatedly applied, relative frequency λ_α decreases, so that B_α may be cut apart and absorbed in a stronger sequence.

B. Formation of an Equilibrium State from a Set of Patterns

When a number of stimulus patterns are randomly applied to a type II net, these patterns, organizing the net, are expected to form a common equilibrium state, as has been shown in the four-layer perceptron [5]. Let $S_X=\{x_1, x_2, \cdots, x_m\}$ be a set of patterns, and assume that all the patterns are applied equally likely, irrespective of the patterns applied previously. In this case, the weight matrix converges to

$$W = \frac{1}{m^2} \sum_{i,j} x_i x_j^t.$$

If we put

$$X = \frac{1}{m} \sum_i x_i$$

W is rewritten as

$$W = XX^t.$$

Since the components of X are not necessarily equal to ± 1, the following pattern

$$\mathrm{x} = \operatorname{sgn} X, \qquad (x_i = \operatorname{sgn} X_i) \tag{27}$$

is defined, where every component X_i of X is assumed not to be equal to 0.[7]

Theorem 7: When pattern set S_X is applied, x becomes an equilibrium state of the net, and

$$T\mathrm{x}_i = \mathrm{x}$$

holds for all i, provided $\mathrm{x}_i \cdot X > 0$.

Proof: Calculation of $u_i(\mathrm{x}, \mathrm{x})$ yields

$$u_i(\mathrm{x}, \mathrm{x}) = x_i X_i (X \cdot \mathrm{x}) = |X_i| \sum_j |X_j| > 0.$$

Therefore, x is equilibrium. Moreover,

$$u_i(\mathrm{x}_k, \mathrm{x}) = x_i X_i (X \cdot \mathrm{x}_k) = |X_i| (X \cdot \mathrm{x}_k).$$

Hence, if $X \cdot \mathrm{x}_k > 0$, $\min (0)\{u_i\} > 0$, and $T\mathrm{x}_k = \mathrm{x}$ holds (if $X \cdot \mathrm{x}_k < 0$, $T\mathrm{x}_k = -\mathrm{x}$).

Consider the case where two sets of patterns are applied. Let $S_X = \{\mathrm{x}_1, \mathrm{x}_2, \cdots, \mathrm{x}_m\}$ and $S_Y = \{y_1, y_2, \cdots, y_k\}$ be two sets of patterns, and let X and Y be, respectively,

$$X = \frac{1}{m} \sum_i \mathrm{x}_i$$

$$Y = \frac{1}{k} \sum_j y_i.$$

When a pattern is applied, the probability of applying a pattern in the same set at the next time is assumed to be nearly equal to 1. Inside a set, all patterns are equally likely. Then, after organization, the weight matrix converges to

$$W = \tfrac{1}{2}(XX^t + YY^t) \tag{28}$$

where terms relating to small probability are ignored.

Assume that X_i and Y_i never vanish, and put

$$\mathrm{x} = \operatorname{sgn} X$$

$$y = \operatorname{sgn} Y. \tag{29}$$

It is also assumed that $\mathrm{x} \neq \pm y$. In general $\mathrm{x} \cdot X = \sum |X_i|$ is considered to be much larger than $|\mathrm{x} \cdot Y|$. Let ϵ_x and ϵ_y be, respectively,

$$\epsilon_x = \min_i \tfrac{1}{4}\{ |X_i(\mathrm{x} \cdot X)| + x_i Y_i (\mathrm{x} \cdot Y)\} \tag{30}$$

$$\epsilon_y = \min_i \tfrac{1}{4}\{ |Y_i(y \cdot Y)| + y_i X_i (y \cdot X)\}. \tag{31}$$

The following theorem shows a condition that two equilibrium states are separately formed, each corresponding to S_X and S_Y. The result is similar to that of the four-layer perceptron [5]. The present net may be regarded as a model for concept formation.

Theorem 8: A necessary and sufficient condition for x to be equilibrium is

$$\epsilon_x > 0.$$

When x_j satisfies the condition

$$\min ([\epsilon_x])\{ |X_i(\mathrm{x}_j \cdot X)| + x_i Y_i(\mathrm{x}_j \cdot Y)\} > 0$$

$$T^2 \mathrm{x}_j = \pm \mathrm{x}. \tag{32}$$

Similar results hold for y.

Proof: From

$$u_i(\mathrm{x}, \mathrm{x}) = \tfrac{1}{2}\{ |X_i| (\mathrm{x} \cdot X) + x_i Y_i (\mathrm{x} \cdot Y)\}$$

$$\min (0)\{u_i\} = 2\epsilon_x$$

is obtained. Therefore, x is equilibrium iff $\epsilon_x > 0$ and $s_1(\mathrm{x}) = [\epsilon_x]$. When $\mathrm{x}_j \cdot X > 0$

$$u_i(\mathrm{x}_j, \mathrm{x}) = \tfrac{1}{2}\{ |X_i| (\mathrm{x}_j \cdot X) + x_i Y_i (\mathrm{x}_j \cdot Y)\}$$

is obtained. Hence, when (32) holds, $\min (s_1(\mathrm{x}))\{u_i(\mathrm{x}_j, \mathrm{x})\} > 0$ and $T\mathrm{x}_j \in N\{\mathrm{x}; s_1(\mathrm{x})\}$, so that

$$T^2 \mathrm{x}_j = \mathrm{x}.$$

When $\mathrm{x}_j \cdot X < 0$, $T^2 \mathrm{x}_j = -\mathrm{x}$.

VI. Conclusions

Stability of state transition, especially of an equilibrium state and cycle, in autonomous nets of threshold elements is elucidated by defining the stability numbers of state transition. By presuming that patterns or pattern sequences are remembered in a self-organizing net of threshold elements in a form of equilibrium states or state-transition sequences, the condition for many patterns or pattern sequences to be remembered correctly at the same time is shown. By calculating stability numbers, the stability of remembrance is shown and the domain of cue patterns from which a remembered pattern or pattern sequence is recalled, is presented. One of the features is that patterns or pattern sequences are remembered and recalled correctly even under strong noise disturbances. A condition that a representative pattern is formed by self-organization from a given set of stimulus patterns is also given.

Self-organizing nets of threshold elements seem to be more capable of interesting information processing. The present paper touches upon only one facet and there remain many interesting features of nets of threshold elements to be studied further.

[7] When m is odd, no component of X can vanish.

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