A 5/4-Approximation for Two-Edge Connectivity

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Abstract

The 2-Edge-Connected Spanning Subgraph problem (2ECSS) is among the most basic survivable network design problems: given an undirected and unweighted graph, the task is to find a spanning subgraph with the minimum number of edges that is 2-edge-connected (i.e., it remains connected after the removal of any single edge). 2ECSS is an NP-hard problem that has been extensively studied in the context of approximation algorithms. The best-known approximation ratio for 2ECSS prior to this work was $1.3 + \varepsilon$, for any constant $\varepsilon > 0$ [Garg, Grandoni, Jabal-Ameli'23; Kobayashi, Noguchi'23]. In this paper, we present a 5/4-approximation algorithm. Our algorithm is also faster for small values of ε: its running time is $n^{O(1)}$ instead of $n^{O(1/\varepsilon)}$.

CCS Concepts

• Theory of computation → Routing and network design problems; **Network optimization**; **• Mathematics of computing → Approximation algorithms**; **Paths and connectivity problems**; **Combinatorial optimization**.

Keywords

approximation algorithms, 2-edge connectivity, network design, combinatorial optimization

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1 Introduction

Real-world networks are prone to failures. The fundamental goal of *survivable network design* is to build low-cost networks that provide the desired connectivity between pairs or groups of nodes despite the failure of a few edges or nodes. One of the most basic survivable network design problems is the 2-Edge-Connected Spanning Subgraph problem (2ECSS): we are given an undirected, unweighted graph $G = (V, E)$. A feasible solution is a subset of edges $S \subseteq E$ such that the subgraph $G' = (V, S)$ is 2-edge-connected (2EC)[1]. In this scenario, G' ensures connectivity among all pairs of nodes even in the presence of a single edge fault. Our goal is to find an (optimal) feasible solution $\mathsf{OPT} = \mathsf{OPT}(G)$ of minimum cardinality (or *size*) $\mathsf{opt} = \mathsf{opt}(G)$.

2ECSS is obviously NP-hard: a graph with n nodes has a Hamiltonian cycle if and only if it contains a 2EC spanning subgraph with n edges. In fact, 2ECSS is APX-hard [16, 19], which rules out the existence of a PTAS for it unless $P = NP$. A significant amount of research has been devoted to designing algorithms for 2ECSS with small approximation ratios. It is easy to compute a 2-approximation for this problem. For example, one can compute a DFS tree and augment it by picking the highest back edge for each non-root node. Khuller and Vishkin [32] found the first non-trivial 3/2-approximation algorithm. Cheriyan, Sebö, and Szigeti [13] improved the approximation ratio to 17/12. This was further improved to 4/3 in two independent (also in terms of techniques) works by Hunkenschröder, Vempala, and Vetta [30] and Sebö and Vygen [39]. In a recent work, Garg, Grandoni, and Jabal Ameli [24] obtained a $\frac{118}{89} + \varepsilon < 1.326$ approximation, based on a rather complex case analysis. Shortly afterward, Kobayashi and Noguchi [33] observed that one can replace a 2-edge cover used in [24] with a triangle-free 2-edge cover, simplifying the analysis in [24] while obtaining an improved $(1.3 + \varepsilon)$-approximation. Until this work, this was the best known approximation ratio. The resulting shortened analysis remains very complex. Our main result is as follows.

Theorem 1. *There is a deterministic* 5/4*-approximation algorithm for 2ECSS that runs in polynomial time.*

An overview of our approach is presented in Section 2.

1.1 Related Work

The 2-Vertex-Connected Spanning Subgraph problem (2VCSS) is the node-connectivity version of 2ECSS. The input is the same as in 2ECSS, and the objective remains to minimize the number of edges in the chosen subgraph G'. However, in this case, G' must be

[1] We recall that a graph is k-edge-connected (kEC) if it remains connected after the removal of an arbitrary subset of $k - 1$ edges.

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2-vertex-connected (2VC)[2]. In other words, G' does not contain any *cut node*. A 2-approximation for 2VCSS can be obtained in different ways. For example, one can compute an open ear decomposition of the input graph and remove the *trivial* ears, i.e., those consisting of a single edge. The resulting graph is 2VC and contains at most $2n-3$ edges (while the optimum solution must contain at least n edges). The first non-trivial 5/3-approximation was obtained by Khuller and Vishkin [32]. This was improved to 3/2 by Garg, Vempala, and Singla [26] (see also an alternative 3/2-approximation by Cheriyan and Thurimella [14]), and further to 10/7 by Heeger and Vygen [29]. The current best 4/3-approximation is due to Bosch-Calvo, Grandoni, and Jabal Ameli [5].

The k-Edge-Connected Spanning Subgraph problem (kECSS) is the natural generalization of 2ECSS to any connectivity $k \geq 2$ (see, e.g., [8, 14, 21, 22]).

A survivable network design problem related to kECSS is the k-Connectivity Augmentation Problem (kCAP): given a k-edge-connected graph $G = (V, E)$ and a collection of extra edges L (*links*), compute a minimum-cardinality subset of links S such that $G' = (V, E \cup S)$ is $(k+1)$-edge-connected. Multiple better-than-2 approximation algorithms are known for $k = 1$ (hence for odd k by a known reduction [17]), i.e., for the Tree Augmentation Problem (TAP) [1, 7, 11, 12, 18, 20, 28, 34–36]. The first such approximation was obtained only recently for an arbitrary k by Byrka, Grandoni, and Jabal Ameli [6] (later improved substantially in [7], see also [23]). Grandoni, Jabal Ameli, and Traub [27] presented the first better-than-2 approximation for the Forest Augmentation Problem (FAP), i.e., the generalization of TAP where the input graph G is a forest rather than a tree. Better approximation algorithms are known for the Matching Augmentation Problem (MAP), i.e., the special case of FAP when the input forest is a matching [3, 9, 10, 25]. The best known approximation ratio of 13/8 for MAP is due to Garg, Hommelsheim, and Megow [25]. Better-than-2 approximation ratios are also known for the vertex-connectivity version of TAP [2, 38].

For all the mentioned problems, one can naturally define a weighted version. Here, a general result by Jain [31] gives a 2-approximation, and this was the best known approximation ratio until very recently. In a recent breakthrough [40], Traub and Zenklusen presented a 1.694-approximation for the weighted version of TAP (later improved to $1.5 + \varepsilon$ by the same authors [41]). Partial results in this direction were achieved earlier in [1, 15, 20, 28, 37]. Even more recently, Traub and Zenklusen obtained a $(1.5 + \varepsilon)$-approximation algorithm for the weighted version of kCAP [42]. Finding a better-than-2 approximation algorithm for the weighted version of 2ECSS remains a major open problem in the area.

2 Overview of Our Approach

We use standard graph notation. Given a subset of edges $F \subseteq E$, we interchangeably use F and the corresponding subgraph $G' = (W, F)$, where $W = \{v \in V(G) \mid v \in f \text{ for some } f \in F\}$. The meaning will be clear from the context. For example, we might say that F is 2EC and denote by $|G'|$ the number $|F|$ of its edges. We sometimes denote paths and cycles as sequences of edges $e_1 \ldots e_{h-1}$ or nodes $v_1 \ldots v_h$.

[2]We recall that a graph G' is k-vertex-connected (kVC) if it remains connected after the removal of an arbitrary subset of $k-1$ nodes. In other words, G' has no vertex cut of size at most $k-1$.

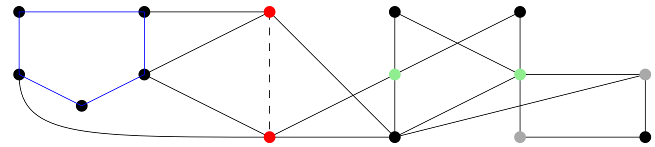

Figure 1: This figure is similar to figures in [5] and [24], where the authors also use similar notions of α-structured graphs. The subgraph induced by the blue edges is a 5/4-contractible graph. The red and green (resp. gray) pairs of nodes form a non-isolating (resp. isolating) cut. The dashed edge is irrelevant.

At a high level, our approach is similar to [24, 33] and also vaguely similar to [5, 9, 10, 25], which address related problems. The first step of our construction (details in the full version [4]) is a reduction to a special class of instances with a specific structure. We use the notion of *structured* graphs introduced in [24], with minor adaptations (see also Figure 1).

Definition 1. *Given a real number $\alpha > 1$, a graph G is α-structured if it is simple, 2VC, contains at least $\frac{4}{\alpha-1}$ nodes, and satisfies the following conditions:*

*(1) It does not contain α-*contractible* subgraphs with at most $\frac{2}{\alpha-1}$ nodes. A subgraph C is α-contractible if it is 2EC and every 2EC spanning subgraph of G contains at least $|E(C)|/\alpha$ edges of $G[V(C)]$;*
(2) It does not contain any irrelevant *edge. An edge uv is irrelevant if $\{u, v\}$ is a 2-vertex cut of G;*
(3) All its 2-vertex cuts are isolating. *A 2-vertex cut $\{u, v\}$ is isolating if $G \setminus \{u, v\}$ has exactly two connected components, one of which contains exactly 1 (isolated) node.*

In [24], it is shown how to turn an $\alpha \geq 6/5$ approximation for 2ECSS on α-structured graphs into an $(\alpha + \varepsilon)$-approximation for 2ECSS in general, for any constant $\varepsilon > 0$. We use a slightly refined reduction that allows us to obtain a clean 5/4-approximation algorithm, without an additive constant. Our reduction is also more efficient: it results in an overall running time of $n^{O(1)}$ instead of $n^{O(1/\varepsilon)}$ as in [24]. The proof of the following lemma can be found in the full version of this paper [4].

Lemma 1. *For every constant $\alpha \geq 6/5$, if there exists a deterministic polynomial-time algorithm for 2ECSS on α-structured graphs G that returns a solution of size at most $\alpha \cdot \text{opt}(G) - 2$, then there exists a deterministic polynomial-time α-approximation algorithm for 2ECSS.*

One key property of structured graphs (for any α) is the following 3-Matching Lemma, which we will also need.

Lemma 2 (3-Matching Lemma [24])**.** *Let $G = (V, E)$ be a 2VC simple graph without irrelevant edges and without non-isolating 2-vertex cuts. Consider any partition (V_1, V_2) of V such that for each $i \in \{1, 2\}$, $|V_i| \geq 3$ and if $|V_i| = 3$, then $G[V_i]$ is a triangle. Then, there exists a matching of size 3 between V_1 and V_2 in G.*

We next assume that our graph G is 5/4-structured. Recall that G contains at least 16 nodes; hence $\text{opt}(G) \geq 16$. The main property of G that we will need in this overview is that it does not contain 5/4-contractible subgraphs (along with the 3-Matching Lemma, which implicitly exploits the other properties).

The second step of our construction (details in the full version [4]) is the computation of a proper lower bound graph. Following [33], and unlike [24], we use a minimum triangle-free 2-edge cover H. A 2-edge cover H of a graph G is a subset of edges $H \subseteq E(G)$ such that every node of G is incident to at least 2 edges of H. We say that H is triangle-free if no connected component of H is a triangle[3]. For ease of exposition, in the following, we refer to a connected component of a 2-edge cover H simply as a component. Notice also that $|H| \leq \text{opt}$, because every feasible solution to 2ECSS is a 2-edge cover and is triangle-free, since it is connected and contains more than 3 nodes.

For technical reasons related to the final step of our algorithm, we impose that H contains at least one component with at least 8 nodes. This can be ensured as follows. We guess a subset F of edges from an optimal solution OPT that induces a connected graph spanning 8 nodes by considering all possible (polynomially many) options. Next, we compute a minimum triangle-free 2-edge cover H containing F. This constraint can be imposed easily as follows. We subdivide each edge $ab \in F$, replacing it with a path acb, where c is a distinct dummy node. Let G' be the resulting graph. We then compute an (unrestricted) minimum triangle-free 2-edge cover H' of G'. Notice that H' must contain all edges incident to dummy nodes since these nodes have degree 2 in G'. Finally, we obtain a triangle-free 2-edge cover H of G containing F by replacing both edges ac and cb incident to each dummy node c with the corresponding edge ab. In the final step, we exploit the fact that F induces a connected graph and that $|F| \geq 4$: this guarantees that replacing the dummy edges with F does not create components that are triangles. We also remark that all nodes spanned by F belong to the same component of H, hence this component contains at least 8 nodes. Notice that OPT induces a triangle-free 2-edge cover of G' of size $\text{opt} + |F|$. Thus, we have $|H'| \leq \text{opt} + |F|$, which implies $|H| \leq \text{opt}$, as required to obtain a valid lower bound.

It is possible to impose extra useful properties on H without increasing its size. A component of H is *complex* if it is not 2EC. A *2EC block* of H is a maximal 2EC subgraph of H.

Definition 2 (Canonical 2-edge cover)**.** *We say that a 2-edge cover H of a graph G is canonical if: (1) H is triangle-free; (2) every component of H with less than 8 edges is a cycle; (3) every 2EC block of a complex component of H contains at least 4 edges; (4) every complex component contains at least two 2EC blocks with at least 6 edges each; (5) H contains at least one component with at least 8 nodes.*

Lemma 3. *Let G be a 5/4-structured graph G, and let H be a triangle-free 2-edge cover of G containing at least one component with at least 8 nodes. Then, one can compute in polynomial time a canonical 2-edge cover H' of G such that $|H'| \leq |H|$.*

The third step of our construction consists of several iterations. We start with $S = H$, and we gradually transform S by adding and sometimes deleting edges. At the end of the process, the final S is the desired approximate solution. To control the size of S during the process, we use a *credit-based* argument. The idea is to assign a certain amount of credits $\text{cr}(S)$ to S based on its structure. Specifically, we define the following *credit-assignment scheme*:

Invariant 1 (Credit assignment scheme)**.** *Let S be a canonical 2-edge cover of a 5/4-structured graph G. Then:*

(1) Let C be a 2EC component of S with $|C| < 8$ (small component). Then $\text{cr}(C) = |C|/4$.
(2) Let C be a 2EC component of S with $|C| \geq 8$ (large component). Then $\text{cr}(C) = 2$.
(3) Let C be a complex component of S (i.e., not 2EC). Then $\text{cr}(C) = 1$.
(4) Let B be a 2EC block of a complex component C of S. Then $\text{cr}(B) = 1$.
(5) Let b be a bridge[4] of a complex component C of S. Then $\text{cr}(b) = 1/4$.

We denote by $\text{cr}(S)$ the total amount of the above credits assigned to the components, 2EC blocks, and bridges of S.

We define the *cost* of S as $\text{cost}(S) = |S| + \text{cr}(S)$. We can guarantee the following upper bound on the initial cost of S.

Lemma 4. *Let H be a canonical 2-edge cover of a 5/4-structured graph G. Then, $\text{cost}(H) \leq \frac{5}{4}|H|$.*

Throughout the iterations, we maintain the invariant that the cost of S never increases. That is, at each step, we replace S with S' ensuring that $\text{cost}(S') \leq \text{cost}(S)$. The transformation process of S consists of two main stages. The first stage, called *bridge covering*, handles all the bridges of S. Here, we essentially replicate the analysis in [24], with minor modifications, to obtain the following (details in the full version [4]).

Lemma 5 (Bridge Covering)**.** *Let S be a canonical 2-edge cover of a 5/4-structured graph G. Then, one can compute in polynomial time a canonical 2-edge cover S' of G such that all components of S' are 2EC and $\text{cost}(S') \leq \text{cost}(S)$.*

The second and final stage of our construction, called *gluing*, gradually merges the 2EC components of S into larger 2EC components. This is the most novel part of our analysis (details in Section 3). Notably, the most complicated part of the analysis in [24, 33] is a similar gluing stage. Our approach significantly simplifies this part of their analysis while reducing the approximation ratio from 13/10 to 5/4.

Recall that, by the definition of a canonical 2-edge cover and the fact that S is bridgeless, S consists of the following two types of components:

- *Small components*: A small component is a k-cycle with $k \in \{4, 5, 6, 7\}$. Each small component C has $\text{cr}(C) = k/4$ credits.
- *Large components*: A large component is a 2EC component containing at least 8 edges. A large component C has $\text{cr}(C) = 2$ credits.

The high-level idea is as follows. Consider the *component (multi)-graph* $\hat{G}_S$, that is obtained by contracting each 2EC component C of S into a single component-node $\hat{C}$ and removing any loops (parallel edges may still exist). The key idea is to compute a cycle

[3]We remark that a connected component with at least 4 nodes is allowed to contain a triangle as an induced subgraph.

[4]A *bridge* of a graph G' is an edge whose removal increases the number of connected components of G'.

F in $\hat{G}_S$ (involving nodes $\hat{C}_1, \ldots, \hat{C}_{|F|}$). We add F to S, replacing S with $S' = S \cup F$. As a result, all components C_i merge into a single (large) 2EC component C' of S'. The decrease in the solution cost (which must be non-negative) is given by

$$\mathrm{cost}(S) - \mathrm{cost}(S') = |S| - |S'| + \sum_{j=1}^{|F|} \mathrm{cr}(C_i) - \mathrm{cr}(C') = -|F| + \sum_{j=1}^{|F|} \mathrm{cr}(C_i) - 2\,.$$

In many cases, this simple approach is sufficient. For example, if all the C_i's are large (each owning 2 credits), then, for any $|F| \geq 2$, we have $\mathrm{cost}(S') \leq \mathrm{cost}(S)$, as desired. The most problematic case arises when $|F|$ is small and involves small components of S, which own fewer credits. Here, we need to delete some edges from S; otherwise, the credits of the C_i's are insufficient. A typical situation when we can perform such a deletion occurs when two edges $e, f \in F$ are adjacent to consecutive nodes u and v of some small component C_i. In this case, we can remove the edge uv from S' while keeping C' 2EC. In other words, we can set $S' \leftarrow S' \setminus \{uv\}$. In other cases, we can ensure that u and v, though not adjacent in C_i, are connected via a Hamiltonian path P that spans all nodes $V(C_i)$ of C_i. In this case, we can still save one edge by setting $S' \leftarrow (S' \setminus E(C_i)) \cup E(P)$. In some cases, we can delete multiple edges this way. A key aspect of the case analysis is proving that any increase in $|S|$ can always be compensated by a corresponding decrease in $\mathrm{cr}(S)$. Specifically, we will prove the following lemma.

Lemma 6 (Gluing). *Given a disconnected canonical 2-edge cover S of a 5/4-structured graph G such that every component of S is 2EC. One can compute in polynomial time a canonical 2-edge cover S' of G such that every component of S' is 2EC, S' has fewer components than S, and $\mathrm{cost}(S') \leq \mathrm{cost}(S)$.*

The 5/4-approximation algorithm then easily follows.

Proof of Theorem 1. By Lemma 1, it suffices to describe an algorithm that, given a 5/4-structured graph G, finds a solution of cost at most $5/4 \cdot \mathrm{opt}(G) - 2$. We guess a set F of edges belonging to some optimal solution OPT and induce a connected graph with 8 nodes. We then compute a minimum triangle-free 2-edge cover $\hat{H}$ of G that contains F, as described earlier. We transform $\hat{H}$ into a canonical 2-edge cover H of no larger size using Lemma 3. Recall that $|H| \leq \mathrm{opt}(G)$.

By first applying Lemma 5 and then iteratively applying Lemma 6 a polynomial number of times, we obtain a 2-edge cover APX consisting of a single 2EC component, which we return as the feasible solution. By construction, $\mathrm{cost}(\mathrm{APX}) \leq \mathrm{cost}(H)$. By Lemma 4, $\mathrm{cost}(H) \leq 5/4 \cdot |H| \leq 5/4 \cdot \mathrm{opt}(G)$. Since APX contains exactly one large 2EC component, we have $|\mathrm{APX}| = \mathrm{cost}(\mathrm{APX}) - \mathrm{cr}(\mathrm{APX}) = \mathrm{cost}(\mathrm{APX}) - 2$, and the claim follows. ◻

It remains to prove Lemma 6. The key insight of our approach is that it is beneficial to focus on the 2VC blocks[5] B of $\hat{G}_S$. More precisely, we will consider one such block that contains a large component of S. Recall that, by the definition of canonical, S must contain a component with at least 8 nodes and, since S is a 2-edge cover, with at least 8 edges. Thus, a large component is guaranteed to exist. A crucial tool in our analysis is the following *local* version of the 3-Matching Lemma, which applies to a single 2VC block of $\hat{G}_S$.

Lemma 7 (Local 3-Matching Lemma). *Let B be a 2VC block of $\hat{G}_S$ and let $(\hat{V}_1, \hat{V}_2)$ be a partition of the nodes of B such that $\hat{V}_1 \neq \emptyset \neq \hat{V}_2$. Let $V_i = \cup_{\hat{C}_i \in \hat{V}_i} V(C_i)$ be the set of nodes of G that correspond to $\hat{V}_i$, for $i \in \{1, 2\}$. Then, there is a matching of size 3 between V_1 and V_2 in G.*

Proof. Let $\hat{V}_i'$ be the set of nodes in $V(\hat{G}_S) \setminus V(B)$ connected to $\hat{V}_i$ by some path contained in $\hat{G}_S \setminus E(B)$, for $i \in \{1, 2\}$. Since B is a maximal 2VC subgraph of $\hat{G}_S$, $\hat{V}_1'$ and $\hat{V}_2'$ are disjoint. Moreover, nodes in $\hat{V}_1'$ are not adjacent to nodes of $\hat{V}_2'$, as that would imply they also belong to $\hat{V}_2'$. Thus, there are no edges from $\hat{V}_1'$ to $\hat{V}_2 \cup \hat{V}_2'$ or from $\hat{V}_2'$ to $\hat{V}_1 \cup \hat{V}_1'$. Let V_1' and V_2' be the set of nodes of G corresponding to $\hat{V}_1'$ and $\hat{V}_2'$, resp.

The 3-Matching Lemma (Lemma 2) guarantees that there is a matching M of size 3 between $V_1 \cup V_1'$ and $V_2 \cup V_2'$. However, the nodes in V_1' (resp., V_2') are not adjacent to those in $V_2 \cup V_2'$ (resp., $V_1 \cup V_1'$). As a result, the endpoints of M must be in $V_1 \cup V_2$. ◻

3 Gluing

This section is devoted to proving Lemma 6. Before delving into the proof, we first establish a series of structural lemmas that hold for structured graphs. In particular, many of these lemmas rely on the structure of the 2VC blocks of $\hat{G}_S$. Specifically, we begin with the following lemmas that show that small components are "well connected" to the rest of the graph.

Observation 1. *For $\alpha \geq \beta \geq 1$, a 2EC subgraph C of a 2EC graph G that is β-contractible is also α-contractible.*

Lemma 8. *Let G be a 5/4-structured graph, and let $C = x_1x_2x_3x_4x_5x_6$ be a 6-cycle in G. If there is at most one pair of distinct nodes $\{x, x'\} \subseteq \{x_1, x_3, x_5\}$ that has a x, x'-Hamiltonian path in $G[V(C)]$, then there are at least two distinct edges in $G \setminus G[V(C)]$ incident to nodes in $\{u_2, u_4, u_6\}$.*

Proof. Assume there is at most one such edge in G. We first show that there is no edge between any pair of nodes in $\{x_2, x_4, x_6\}$. W.l.o.g. assume that $x_2x_6 \in E(G)$. Then the paths $x_1x_2x_6x_5x_4x_3$ and $x_1x_6x_2x_3x_4x_5$ are Hamiltonian paths in $G[V(C)]$ between x_1, x_3 and x_1, x_5, respectively, a contradiction.

Thus, every 2EC spanning subgraph of G contains at least 5 edges of $G[V(C)]$, as it must contain at least 2 edges incident to each node in $\{x_2, x_4, x_6\}$. Therefore, C is $\frac{6}{5}$-contractible, a contradiction to the fact that G is $\frac{5}{4}$-structured. ◻

Lemma 9 (Hamiltonian Pairs Lemma). *Let G be a 5/4-structured graph, and let C be a k-cycle in G. A Hamiltonian pair $\{u, v\}$ of C consists of distinct nodes $u, v \in V(C)$ such that: (1) they are both adjacent to nodes in $V(G) \setminus V(C)$ and (2) there is a Hamiltonian u, v-path in $G[V(C)]$. If $k \leq 6$, then C has at least 2 distinct Hamiltonian pairs $\{u_1, v_1\}, \{u_2, v_2\}$, in particular $|\{u_1, v_1\} \cap \{u_2, v_2\}| \leq 1$. If $k = 7$, then C has at least one such pair.*

[5] A 2VC block B of a graph G' is a maximal subgraph that is 2VC.

Proof. Let $M = \{u_1v_1, u_2v_2, u_3v_3\}$ be a matching obtained by applying the 3-Matching Lemma 2 to the partition $(V(C), V(G) \setminus V(C))$, where $u_i \in V(C)$ and $v_i \in V(G) \setminus V(C)$ for all $i \in \{1, 2, 3\}$ (observe that the partition satisfies the conditions of Lemma 2). Notice that, for $k = 4$, at least one node u_i, say u_3, is adjacent to the other two nodes u_1 and u_2. Hence the pairs $\{u_1, u_3\}$ and $\{u_2, u_3\}$ satisfy the claim.

We next assume that $C = x_1x_2 \dots x_k$ with $k \in \{5, 6, 7\}$. We can assume w.l.o.g. that there is at most one Hamiltonian pair with both its nodes from $\{u_1, u_2, u_3\}$, otherwise we are done. In particular, $\{u_1, u_2, u_3\} \neq \{x_1, x_2, x_3\}$. Moreover, if such a pair exists, we assume w.l.o.g. that it is $\{u_1, u_2\}$ (while $\{u_1, u_3\}$ and $\{u_2, u_3\}$ are not). We now consider different cases based on the value of k:

(a) $k = 5$. We can assume w.l.o.g. that $u_1 = x_1, u_2 = x_2, u_3 = x_4$. $\{x_1, x_2\}$ is one of the desired Hamiltonian pairs. We can assume that x_3 (resp., x_5) has no neighbors in $V(G) \setminus V(C)$, otherwise the second pair is $\{x_2, x_3\}$ (resp., $\{x_1, x_5\}$). Now, the edge x_3x_5 must exist in G, otherwise every 2EC subgraph of G must contain at least 4 edges of $G[V(C)]$, 2 for x_3, and 2 for x_5, making C 5/4-contractible, a contradiction to the fact that G is 5/4-structured. Then, the path $x_1x_2x_3x_5x_4$ is a Hamiltonian x_1, x_4-path in $G[V(C)]$, and hence $\{x_1, x_4\}$ is the other desired Hamiltonian pair.

(b) $k = 6$.

(b.1) u_1 is adjacent to u_2 in C. We can assume w.l.o.g. that $u_1 = x_1, u_2 = x_2, u_3 = x_4$. $\{x_1, x_2\}$ is one of the desired pairs. Moreover, we can assume that x_3, x_5, and x_6 have no neighbors in $V(G) \setminus V(C)$, otherwise $\{x_3, x_2\}$, $\{x_5, x_4\}$, and $\{x_6, x_1\}$, respectively, is the other desired pair. Notice that x_3 must be adjacent to either x_5 or x_6 in G, otherwise every 2EC subgraph of G must contain at least 5 edges of $G[V(C)]$ (2 for x_3 and 3 in total for x_5 and x_6), making C 6/5-contractible, and hence 5/4-contractible by Observation 1, a contradiction to the fact that G is 5/4-structured. If $x_3x_5 \in E(G)$, then $x_2x_1x_6x_5x_3x_4$ is a Hamiltonian x_2, x_4-path in $G[V(C)]$, and hence $\{x_2, x_4\}$ is the other desired Hamiltonian pair. Similarly, if $x_3x_6 \in E(G)$, then $\{x_1, x_4\}$ is the other desired pair.

(b.2) u_1 is not adjacent to u_2 in C. We can assume w.l.o.g. $u_1 = x_1, u_2 = x_3, u_3 = x_5$. Moreover, we can assume that x_2, x_4, and x_6 have no neighbors in $V(G) \setminus V(C)$, otherwise the two desired pairs are $\{\{x_2, x_1\}, \{x_2, x_3\}\}$, $\{\{x_4, x_3\}, \{x_4, x_5\}\}$, and $\{\{x_6, x_5\}, \{x_6, x_1\}\}$, respectively. Now, applying Lemma 8, either there must be two Hamiltonian pairs with nodes from $\{x_1, x_3, x_5\}$ and we are done or there must exist edges xy where $x \in \{x_2, x_4, x_6\}$ and $y \notin V(C)$, a contradiction.

(c) $k = 7$. In this case, we only need to prove the existence of one Hamiltonian pair. Hence, we can assume that u_1, u_2, and u_3 are not pairwise adjacent in C. Thus, w.l.o.g., we can assume $u_1 = x_1$, $u_2 = x_3$, and $u_3 = x_5$. Moreover, we can assume that x_2, x_4, x_6, and x_7 do not have neighbors in $V(G) \setminus V(C)$, otherwise the pair $\{x_2, x_1\}$, $\{x_4, x_5\}$, $\{x_6, x_5\}$, and $\{x_7, x_1\}$, resp., satisfies the claim. If $E(G[\{x_2, x_4, x_6, x_7\}]) = \{x_6x_7\}$, then every 2EC subgraph of G must use at least 7 edges of $G[V(C)]$ (2 for node x_2, 2 for node x_4, and 3 in total for the nodes x_6 and x_7 together). Then C is 1-contractible, hence 5/4-contractible by Observation 1, contradicting the fact that G is 5/4-structured. Thus there is at least another edge besides x_6x_7 between two nodes in $\{x_2, x_4, x_6, x_7\}$. If $x_2x_4 \in E(G)$, then $x_3x_4x_2x_1x_7x_6x_5$ is a Hamiltonian x_3, x_5-path in $G[V(C)]$, and thus, $\{x_3, x_5\}$ is the desired pair. If $x_2x_6 \in E(G)$, then $x_1x_7x_6x_2x_3x_4x_5$ is a Hamiltonian x_1, x_5-path in $G[V(C)]$, and thus, $\{x_1, x_5\}$ is the desired pair. If $x_2x_7 \in E(G)$, then $x_1x_2x_7x_6x_5x_4x_3$ is a Hamiltonian x_1, x_3-path in $G[V(C)]$, and thus, $\{x_1, x_3\}$ is the desired pair. The remaining cases follow symmetrically. □

In the following, we assume we are given a canonical 2-edge cover S of G, and we focus on the 2VC blocks of the component graph $\hat{G}_S$. First we introduce some terminology. We say that a component C of S is *local* if $\hat{C}$ belongs to exactly one 2VC block of $\hat{G}_S$; otherwise, it is *non-local*. That is, a component C of S is non-local if and only if $\hat{C}$ is a cut vertex of $\hat{G}_S$.

Given two components C_1, C_2 of S and a node $u_1 \in V(C_1)$, we use the notation $u_1\hat{C}_2$ to indicate an edge between u_1 and some $u_2 \in V(C_2)$ (specifically, when identifying such a u_2 is not required in our arguments). In this case, we also say that u_1 is adjacent to $\hat{C}_2$. Similarly, we use the notation $\hat{C}_1\hat{C}_2$ to indicate an edge between some $u_1 \in V(C_1)$ and some $u_2 \in V(C_2)$.

Given a 2VC block B of $\hat{G}_S$, and $\hat{C} \in V(B)$, we say that C is a component of B. Furthermore, we say that we apply the local 3-Matching Lemma 7 to C to mean that we apply it to B and to the partition $(\{\hat{C}\}, V(B) \setminus \{\hat{C}\})$.

We now present some lemmas that allow us to find cycles in $\hat{G}_S$ with convenient properties.

Lemma 10. *Let B be a 2VC block of $\hat{G}_S$ and let C_1 and C_2 be two distinct components of B. Given edges $u_1\hat{X}_1, u_2\hat{X}_2$, where $u_1 \in V(C_1), u_2 \in V(C_2), \hat{X}_1 \in V(B) \setminus \{\hat{C}_1\}, \hat{X}_2 \in V(B) \setminus \{\hat{C}_2\}$, one can compute in polynomial time a cycle F in B containing $\hat{C}_1$ and $\hat{C}_2$, such that F is incident on u_1 and another node in C_1, and incident on u_2 and another node in C_2.*

Proof. We start by computing a cycle F in B that contains the component-nodes $\hat{C}_1, \hat{C}_2$, which can be done efficiently since B is 2VC and both $\hat{C}_1$ and $\hat{C}_2$ belong to B. Let F be the union of 2 internally vertex-disjoint paths P_v, P_w between $\hat{C}_1$ and $\hat{C}_2$, where P_v is incident to $v_i \in V(C_i)$ and P_w is incident to $w_i \in V(C_i)$, for $i \in \{1, 2\}$. We remark that we might have $v_i = w_i$ for some $i \in \{1, 2\}$. Let $P_v = \hat{C}^v_1\hat{C}^v_2 \dots \hat{C}^v_{k_v}$ and $P_w = \hat{C}^w_1\hat{C}^w_2 \dots \hat{C}^w_{k_w}$, with $\hat{C}^v_1 = \hat{C}^w_1 = \hat{C}_1$ and $\hat{C}^v_{k_v} = \hat{C}^w_{k_w} = \hat{C}_2$.

We first show that we can assume that $v_1 = u_1$. Assume this is not the case, i.e., $u_1 \notin \{v_1, w_1\}$. Since B is 2VC, we can find in polynomial time a path P_{X_1} in $\hat{G}_S$ from $\hat{X}_1$ to a node in $V(F) \subseteq V(B)$ not going through $\hat{C}_1$. Assume w.l.o.g. that P_{X_1} has $\hat{C}^v_i, 1 < i \leq k_v$, as an endpoint. Then the path $\{u_1\hat{X}_1\} \cup P_{X_1} \cup \hat{C}^v_i\hat{C}^v_{i+1} \dots \hat{C}^v_{k_v}$ is a path between $\hat{C}_1$ and $\hat{C}_2$, internally vertex-disjoint with P_w and incident to u_1 in C_1.

We now show that we can assume $v_1 = u_1$ and $w_1 \neq u_1$. To get a contradiction, assume $v_1 = u_1 = w_1$. By the local 3-Matching Lemma (Lemma 7), there must exist an edge $u\hat{X}$, where $u \in V(C_1) \setminus \{u_1\}$ and $\hat{X} \in V(B) \setminus \{\hat{C}_1\}$. Since B is 2VC, we can find in polynomial time a path P_X in $\hat{G}_S$ from $\hat{X}$ to a node in $V(F) \subseteq V(B)$ not going through $\hat{C}_1$. Assume w.l.o.g. that P_X has $\hat{C}^w_i, 1 < i \leq k_w$, as an endpoint. Then the path $\{u\hat{X}\} \cup P_X \cup \hat{C}^w_i\hat{C}^w_{i+1} \dots \hat{C}^w_{k_w}$ is a path between $\hat{C}_1$ and $\hat{C}_2$, incident to $u \neq u_1$ in C_1, and internally vertex-disjoint with P_v which is incident to u_1 in C_1.

We have shown that $v_1 = u_1$ and $w_1 \neq u_1$. Furthermore, we now show that we can assume either $v_2 = u_2$ or $w_2 = u_2$. Assume this is not the case, i.e., $u_2 \notin \{v_2, w_2\}$. Then, since B is 2VC, we can find in polynomial time a path P_{X_2} in $\hat{G}_S$ from $\hat{X}_2$ to a node in $V(F) \subseteq V(B)$ not going through $\hat{C}_2$. Assume first that P_{X_2} has $\hat{C}^w_i, 1 \leq i < k_w$, as an endpoint, and if $i = 1$, then P_{X_2} is not incident to u_1. Then the path $\{u_2\hat{X}_2\} \cup P_{X_2} \cup \hat{C}^w_i\hat{C}^w_{i-1}\dots\hat{C}^w_1$ is a path between $\hat{C}_2$ and $\hat{C}_1$, incident to u_2 in C_2, not incident to u_1 in C_1, and internally vertex-disjoint with P_v. Since P_v is incident to u_1 in C_1 the claim follows. Assume now P_{X_2} has $\hat{C}^v_i, 1 \leq i < k_v$, as an endpoint, and if $i = 1$, then P_{X_2} is incident to u_1. Then the path $\{u_2\hat{X}_2\} \cup P_{X_2} \cup \hat{C}^v_i\hat{C}^v_{i-1}\dots\hat{C}^v_1$ is a path between $\hat{C}_2$ and $\hat{C}_1$, incident to u_1 in C_1, incident to u_2 in C_2, and internally vertex-disjoint with P_w. Since P_w is not incident to u_1 in C_1, the claim follows.

Finally, we can now assume that $v_1 = u_1$, $w_1 \neq u_1$, and $u_2 = v_2 = w_2$, otherwise we are done. By the local 3-Matching Lemma 7, there must exist an edge $u\hat{X}$ with $u \in V(C_2) \setminus \{u_2\}$ and $\hat{X} \in V(B) \setminus \{\hat{C}_2\}$. Since B is 2VC, we can find in polynomial time a path P_X in $\hat{G}_S$ from $\hat{X}$ to a node in $V(F) \subseteq V(B)$ not going through $\hat{C}_2$. Assume first that P_X has $\hat{C}^w_i, 1 \leq i < k_w$, as an endpoint, and if $i = 1$, then P_X is not incident to u_1. Then the path $\{u\hat{X}\} \cup P_X \cup \hat{C}^w_i\hat{C}^w_{i-1}\dots\hat{C}^w_1$ is a path between $\hat{C}_2$ and $\hat{C}_1$, not incident to u_1 in C_1, not incident to u_2 in C_2, and internally vertex-disjoint with P_v. Since P_v is incident to u_1 in C_1 and u_2 in C_2, the lemma follows. Assume now that P_X has $\hat{C}^v_i, 1 \leq i < k_v$, as an endpoint, and if $i = 1$, then P_X is incident to u_1. Then the path $\{u\hat{X}\} \cup P_X \cup \hat{C}^v_i\hat{C}^v_{i-1}\dots\hat{C}^v_1$ is a path between $\hat{C}_2$ and $\hat{C}_1$, incident to u_1 in C_1, not incident to u_2 in C_2, and internally vertex-disjoint with P_w. Since P_w is incident to $w_1 \neq u_1$ in C_1 and u_2 in C_2, the lemma follows. □

Corollary 1. *Let B be a 2VC block of $\hat{G}_S$ and let C_1, C_2 be two different components of B. Given an edge $u_1\hat{X}$, $u_1 \in V(C_1)$, $\hat{X} \in V(B) \setminus \{\hat{C}_1\}$, one can compute in polynomial time a cycle F in B containing $\hat{C}_1$ and $\hat{C}_2$ such that F is incident on u_1 and another node in C_1 and $|F| \geq \min\{3, |V(B)|\}$.*

Proof. Let F' be the cycle in B found by applying Lemma 10 to C_1 and the edge $u_1\hat{X}$ and C_2 and an arbitrary edge in B incident on C_2. Let u_1, v_1 be the distinct nodes of C_1 incident with F'. We can assume that $|F'| < \min\{3, |V(B)|\}$, otherwise we are done. Thus, $|F'| = 2$ and $|V(B)| \geq 3$, and $F' = \{u_1u_2, v_1v_2\}$, where $u_2, v_2 \in V(C_2)$. Since B is 2VC and $|V(B)| \geq 3$, it must contain an edge $w_1\hat{C}$, where $w_1 \in V(C_1)$ and $\hat{C} \in V(B) \setminus \{\hat{C}_1, \hat{C}_2\}$. Because B is 2VC, there must exist a path P in B from C_2 to C not going through C_1. If $w_1 \neq u_1$, let $F := P \cup \{u_1u_2, w_1\hat{C}\}$. This way, F is a cycle in B incident to distinct nodes u_1, w_1 of C_1, and $|F| \geq 3$. Otherwise, if $w_1 = u_1$, $F := P \cup \{v_1v_2, w_1\hat{C}\}$ is the desired cycle. □

Lemma 11. *Let B be a 2VC block of $\hat{G}_S$ and let C_1, C_2 be two different components of B such that C_1 is 4-cycle or a local 5-cycle. Given an edge $u_2\hat{X}$, where $u_2 \in V(C_2)$ and $\hat{X} \in V(B) \setminus \{\hat{C}_2\}$, one can compute in polynomial time a cycle F in B incident to distinct nodes $u_i, v_i \in V(C_i)$ for $i \in \{1, 2\}$ such that there is a Hamiltonian u_1, v_1-path in $G[V(C_1)]$.*

Proof. We start by computing a cycle F incident to distinct nodes of C_1 and to u_2, v_2 in C_2 via Lemma 10, where v_2 is a node in $V(C_2) \setminus \{u_2\}$. Let $C_1 = x_1x_2\dots x_k, k \in \{4, 5\}$. We can assume that F is incident to x_1 and x_3, and there is no Hamiltonian x_1, x_3-path in $G[V(C)]$, otherwise we are done. Let P_u and P_v be the two internally vertex-disjoint paths from C_1 to C_2 such that their union is F, they are incident to x_1 and x_3 in C_1, and to u_2 and v_2 in C_2. Let $P_u = \hat{C}^u_1\hat{C}^u_2\dots\hat{C}^u_{k_u}$ and $P_v = \hat{C}^v_1\hat{C}^v_2\dots\hat{C}^v_{k_v}$, where $\hat{C}^u_1 = \hat{C}^v_1 = \hat{C}_1$ and $\hat{C}^u_{k_u} = \hat{C}^v_{k_v} = \hat{C}_2$.

First, assume that there is an edge $y\hat{Y}$, where $y \in V(C_1) \setminus \{x_1, x_3\}$ and $\hat{Y} \in V(B) \setminus \{\hat{C}_1\}$ such that there is a Hamiltonian y, x_i-path in $G[V(C_1)]$ for each $i \in \{1, 3\}$. Since B is 2VC, we can find a path P_Y in $\hat{G}_S$ from $\hat{Y}$ to a node in $V(F) \subseteq V(B)$ not going through $\hat{C}_1$. Assume first that $\hat{C}^u_i, 1 < i \leq k_u$, is an endpoint of P_Y, and if $i = k_u$, then P_Y is incident to u_2. Then, the path $\{y\hat{Y}\} \cup P_Y \cup \hat{C}^u_i\hat{C}^u_{i+1}\dots\hat{C}^u_{k_u}$ is incident to y in C_1, incident to u_2 in C_2, and internally vertex-disjoint with P_v. Since P_v is incident to x_3 in C_1 and to v_2 in C_2, the lemma follows. We can now assume $\hat{C}^v_i, 1 < i \leq k_v$, is an endpoint of P_Y, and if $i = k_v$, then P_Y is not incident to u_2. Then, the path $\{y\hat{Y}\} \cup P_Y \cup \hat{C}^v_i\hat{C}^v_{i+1}\dots\hat{C}^v_{k_v}$ is incident to y in C_1, not incident to u_2 in C_2, and internally vertex-disjoint with P_u. Since P_u is incident to x_1 in C_1 and to u_2 in C_2, the lemma follows.

We can now assume that there is no such edge. In particular, this implies that there is no edge $x_2\hat{X}_2$, where $\hat{X}_2 \in V(B) \setminus \{\hat{C}_1\}$. Then, by the local 3-Matching Lemma 7, there must be an edge $y\hat{Y}$ such that $y \in V(C_1) \setminus \{x_1, x_2, x_3\}$. If C_1 is a 4-cycle $y = x_4$, then, since $x_1x_4, x_3x_4 \in E(C_1)$, $x_4\hat{Y}$ is an edge of the excluded type. Thus, it must be that C_1 is a 5-cycle, and by symmetry, we can assume w.l.o.g. $y = x_4$. Thus, there is an edge $x_4\hat{X}_4$, where $\hat{X}_4 \in V(B) \setminus \{\hat{C}_1\}$. By our previous assumption, there is no Hamiltonian x_1, x_4-path in $G[V(C_1)]$. Since there is no edge $x_2\hat{X}_2$, where $\hat{X}_2 \in V(B) \setminus \{\hat{C}_1\}$, by the Hamiltonian pairs lemma (Lemma 9), and since C_1 is local by assumption of the lemma, there must also exist an edge $x_5\hat{X}_5$, where $\hat{X}_5 \in V(B) \setminus \{\hat{C}_1\}$.

Since B is 2VC, we can find in polynomial time a path P_{X_4} in $\hat{G}_S$ from $\hat{X}_4$ to a node in $V(F) \subseteq V(B)$ not going through $\hat{C}_1$. Assume first P_{X_4} has $\hat{C}^u_i, 1 < i \leq k_u$, as an endpoint, and if $i = k_u$, then it is incident to u_2. Then $\{x_4\hat{X}_4\} \cup P_{X_4} \cup \hat{C}^u_i\hat{C}^u_{i+1}\dots\hat{C}^u_{k_u}$ is incident to x_4 in C_1, incident to u_2 in C_2, and internally vertex-disjoint with P_v. Since P_v is incident to x_3 in C_1 and to v_2 in C_2, the lemma follows. We can then assume that P_{X_4} has $\hat{C}^v_i$ as an endpoint for some $1 < i \leq k_v$, and if $i = k_v$, then P_{X_4} is not incident to u_2. Similarly, we define P_{X_5}, and by a similar argument it has $\hat{C}^u_j$ as an endpoint for some $1 < j \leq k_u$, and if $j = k_u$, then P_{X_5} is incident to u_2.

If the paths P_{X_4}, P_{X_5} are not internally vertex-disjoint, then we can efficiently compute a path P'_{X_4} in $\hat{G}_S$ from $\hat{X}_4$ to a node in $V(F) \subseteq V(B)$ having $C^u_j, 1 < j \leq k_u$, as an endpoint and satisfying that if $j = k_u$, then P'_{X_4} is incident to u_2, so by the same arguments as above the claim follows. Otherwise, that is, P_{X_4} and P_{X_5} are internally vertex-disjoint, the paths $\{x_4\hat{X}_4\} \cup P_{X_4} \cup \hat{C}^v_i\hat{C}^v_{i+1}\dots\hat{C}^v_{k_v}$ and $\{x_5\hat{X}_5\} \cup P_{X_5} \cup \hat{C}^u_j\hat{C}^u_{j+1}\dots\hat{C}^u_{k_u}$ are internally vertex-disjoint and they have endpoints x_4 and x_5 in C_1 and w_2 and u_2 in C_2, where $w_2 \in V(C_2) \setminus \{u_2\}$. Thus the lemma follows also in this case. □

Finally, the following lemma is crucial to deal with non-local components of S.

Lemma 12. *Let F be a 2EC graph and let C be a cycle of F. Assume that there are paths $P_{u_1v_1}$ and $P_{u_2v_2}$ in $F \setminus C$ between nodes u_i and $v_i \in V(C)$ for $i \in \{1,2\}$. Assume $u_1v_i \in C$ for some $i \in \{1,2\}$. Let $d_{C'}(u,v)$ denote the length of the u,v-path in $C' := C \setminus \{u_1v_i\}$, for every $u,v \in V(C)$. If $d_{C'}(u_1,u_i) \leq d_{C'}(u_1,v_1)$, then $F \setminus \{u_1v_i\}$ is 2EC.*

Proof. Let $F' := F \setminus \{u_1v_i\}$. Assume to get a contradiction F' contains a bridge e. If $e \notin C$, then let C_1, C_2 be the two connected components of $F' \setminus \{e\}$ such that $C \setminus \{u_1v_i\} \subseteq C_1$. Since the edge u_1v_i has both endpoints in $V(C_1)$, e is also a bridge in F, a contradiction.

Thus, we can assume that $e \in C$. Since C' is a u_1,v_i-path, there are nodes $u,v \in V(C)$ such that $e = uv$ and $d_{C'}(v_i,u) = d_{C'}(v_i,v) + 1$. Let P_j be the u_j,v_j-path in C' for $j \in \{1,2\}$. If $e \in P_j$ for some $j \in \{1,2\}$, then e belongs to the closed trail $P_j \cup P_{u_jv_j}$ in F', a contradiction to the fact that e is a bridge of F'. Thus it must be that $e \notin P_1 \cup P_2$. Since $e \notin P_1$ and C' is a u_1,v_i-path, one has that $d_{C'}(u_1,v_1) \leq d_{C'}(u_1,u)$. Since $e \notin P_i$ and C' is a u_1,v_i-path, one has that $d_{C'}(u_i,v_i) \leq d_{C'}(v_i,v)$. It holds that:

$$d_{C'}(u_1,u_i) \leq d_{C'}(u_1,v_1) \leq d_{C'}(u_1,u) = d_{C'}(u_1,v_i) - d_{C'}(v_i,u) < d_{C'}(u_1,v_i) - d_{C'}(v_i,v) \leq d_{C'}(u_1,v_i) - d_{C'}(u_i,v_i) = d_{C'}(u_1,u_i),$$

a contradiction. The first inequality above follows by the assumption of the lemma, the second inequality follows from having $d_{C'}(u_1,v_1) \leq d_{C'}(u_1,u)$, and the last inequality follows from having $d_{C'}(u_i,v_i) \leq d_{C'}(v_i,v)$. The two equalities follow from the fact that C' is a u_1,v_i-path, and the strict inequality follows from having $d_{C'}(v_i,u) = d_{C'}(v_i,v) + 1$. □

Corollary 2. *Let F be a 2EC graph and let C be a 5-cycle in F. Assume that there are paths $P_{u_1v_1}$ and $P_{u_2v_2}$ in $F \setminus C$ between nodes $u_i, v_i \in V(C)$ such that $u_i \neq v_i$ for $i \in \{1,2\}$. If $u_2 \notin \{u_1,v_1\}$, then we can find in polynomial time an edge $e \in C$ such that $F \setminus \{e\}$ is 2EC.*

Proof. If u_j is adjacent to v_j in C for some $j \in \{1,2\}$, then $F \setminus \{u_jv_j\}$ is 2EC by Lemma 12 (with $u_1 := u_j, v_1 := v_j, u_2 := u_{3-j}, v_2 := v_{3-j}$). Otherwise, since C is a 5-cycle, u_1 is not adjacent to v_1, and $u_2 \notin \{u_1,v_1\}$, therefore u_2 is adjacent to either u_1 or v_1, say w.l.o.g. to v_1. Also because C is a 5-cycle, u_1 is not adjacent to v_1 and $u_2 \notin \{u_1,v_1\}$, it holds that the u_1,u_2-path in $C \setminus \{u_2v_1\}$ has length at most 2. Thus, $F \setminus \{u_2v_1\}$ is 2EC follows from Lemma 12 (with $u_1 := u_2, v_1 := v_2, u_2 := u_1, v_2 := v_1$) by observing that $d_{C\setminus\{u_2v_1\}}(u_2,u_1) \leq 2 \leq d_{C\setminus\{u_2v_1\}}(u_2,v_2)$, where the second inequality follows from the fact that u_2 is not adjacent to v_2 in C. □

Corollary 3. *Let F be a 2EC graph and let C be a 6-cycle or a 7-cycle in F. Assume that there are paths $P_{u_1v_1}$ and $P_{u_2v_2}$ in $F \setminus C$ between nodes $u_i, v_i \in V(C)$ such that $u_i \neq v_i$ for $i \in \{1,2\}$. If $u_2 \notin \{u_1,v_1\}$ and u_2 is in a shortest u_1,v_1-path in C, then we can find in polynomial time an edge $e \in C$ such that $F \setminus \{e\}$ is 2EC.*

Proof. If u_j is adjacent to v_j in C for some $j \in \{1,2\}$, then $F \setminus \{u_jv_j\}$ is 2EC by Lemma 12 (with $u_1 := u_j, v_1 := v_j, u_2 := u_{3-j}, v_2 := v_{3-j}$). Otherwise, since C is a 6-cycle or a 7-cycle, $u_2 \notin \{u_1,v_1\}$ and u_2 is in a shortest u_1,v_1-path in C, it must be that u_2 is adjacent in C to either u_1 or v_1. Say w.l.o.g. u_2 is adjacent to v_1 in C. Since C is a 6-cycle or a 7-cycle and u_2 is in a shortest u_1,v_1-path in C, it holds that the u_1,u_2-path in $C \setminus \{u_2v_1\}$ has length at most 2. Then, $F \setminus \{u_2v_1\}$ is 2EC follows from Lemma 12 (with $u_1 := u_2, v_1 := v_2, u_2 := u_1, v_2 := v_1$) by observing that $d_{C\setminus\{u_2v_1\}}(u_2,u_1) \leq 2 \leq d_{C\setminus\{u_2v_1\}}(u_2,v_2)$ where, the second inequality follows from the fact that u_2 is not adjacent to v_2 in C. □

3.1 Proof of Lemma 6

We are now ready to prove Lemma 6, which we restate next.

Lemma 6 (Gluing). *Given a disconnected canonical 2-edge cover S of a 5/4-structured graph G such that every component of S is 2EC. One can compute in polynomial time a canonical 2-edge cover S' of G such that every component of S' is 2EC, S' has fewer components than S, and $\mathrm{cost}(S') \leq \mathrm{cost}(S)$.*

By the definition of canonical, S contains a large component C. Let B be any 2VC block of $\hat{G}_S$ that contains C (if there are multiple such blocks, we choose B arbitrarily). We separate the proof into a few lemmas.

Lemma 13. *Let C_1 and C_2 be two components of B such that $|C_1| \leq 7$ and $C_1 \neq C$. If there are edges $u_1\hat{C}$ and $v_1\hat{C}_2$ such that $u_1, v_1 \in V(C_1)$ and there is a Hamiltonian u_1,v_1-path in $G[V(C_1)]$, then one can compute in polynomial time a canonical 2-edge cover S' of G such that every component of S' is 2EC, S' has fewer components than S, and $\mathrm{cost}(S') \leq \mathrm{cost}(S)$.*

Proof. Let F be the cycle in B formed by the edges $u_1\hat{C}, v_1\hat{C}_2$, and a path in B from C_2 to C not going through C_1. Note that such a path exists because B is 2VC. Let P be the Hamiltonian u_1,v_1-path in $G[V(C_1)]$ (which can be found in polynomial time because $|C_1| \leq 7$). We set $S' := (S \setminus C_1) \cup F \cup P$. In S' there is a large component C' spanning the nodes of all the components of B incident on F. Moreover, every such component yields at least 1 credit, while C yields 2 credits, so we collect at least $|F| + 1$ credits from these components. One has $\mathrm{cost}(S) - \mathrm{cost}(S') \geq |S| - |S'| + |F| + 1 - \mathrm{cr}(C') = -|F| + 1 + |F| + 1 - 2 = 0$. We remark that S' is canonical and it contains fewer components than S, so the lemma follows. □

Lemma 14. *Let C_1 be a component of B that is either a 4-cycle or a local 5-cycle. Then, in polynomial time one can compute a canonical 2-edge cover S' of G such that every component of S' is 2EC, S' has fewer components than S, and $\mathrm{cost}(S') \leq \mathrm{cost}(S)$.*

Proof. Apply Lemma 11 with $C_1 := C_1, C_2 := C$ and with $u_2\hat{X}$ being an arbitrary edge of B incident to C. This way one can find in polynomial time a cycle F in B incident to C and to nodes u_1, v_1 of C_1 such that there is a Hamiltonian u_1,v_1-path P in $G[V(C_1)]$. Let $S' := (S \setminus C_1) \cup F \cup P$. In S' there is a large component C' spanning the nodes of all components of B incident on F. Also, every such component yields at least 1 credit, while C yields 2 credits, so we collect at least $|F| + 1$ credits from these components. Thus, one has $\mathrm{cost}(S) - \mathrm{cost}(S') \geq |S| - |S'| + |F| + 1 - \mathrm{cr}(C') = -|F| + 1 + |F| + 1 - 2 = 0$. We remark that S' is canonical and it contains fewer components than S, so the lemma follows. □

Lemma 15. *Let C_1 be a component of B that is a non-local 5-cycle. Then, one can compute in polynomial time a canonical 2-edge cover S'*

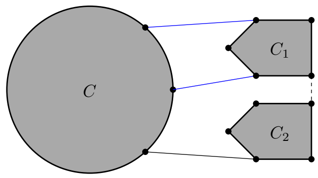

Figure 2: An illustration of Case (b.1). The solid blue edges are the matching M, while the dashed edge is the edge $w_1\hat{X}$. In the figure, one has that $X = C_2$.

of G such that every component of S' is 2EC, S' has fewer components than S, and $\mathrm{cost}(S') \leq \mathrm{cost}(S)$.

Proof. We assume that the conditions of Lemmas 13 and 14 do not hold, otherwise we are done. In particular, we can assume B contains no 4-cycle nor local 5-cycle. We have the following cases:
(a) $|V(B)| = 2$. We show by contradiction that this case is not possible. It must be that B contains exactly 2 components, C and C_1. Since C_1 is a 5-cycle, by applying the local 3-matching lemma (Lemma 7) to C_1 in B, one can find a pair of edges $u_1\hat{C}, v_1\hat{C}$ such that $u_1v_1 \in E[C_1]$. This is a contradiction to the fact that the conditions of Lemma 13 do not hold (with $C_1 := C_1, C_2 := C$).
(b) $|V(B)| = 3$. Let C_2 be the component of B other than C and C_1.
(b.1) C_2 is a 5-cycle. We show by contradiction that this case is not possible. See Figure 2 for an illustration. Apply the local 3-matching lemma (Lemma 7) to C and B to see that there must exist a matching M of size 2 between C and either C_1 or C_2. W.l.o.g. assume that $M = \{uu_1, vv_1\}$, where $u, v \in V(C)$ and $u_1, v_1 \in V(C_1)$. Notice that it must be that u_1 and v_1 are not adjacent in C_1, otherwise the edges uu_1, vv_1 imply that the conditions of Lemma 13 hold (with $C_1 := C_1, C_2 := C$), a contradiction. Now, apply the local 3-matching lemma (Lemma 7) to C_1 and B to see that there must exist an edge $w_1\hat{X}$, where $w_1 \in V(C_1) \setminus \{u_1, v_1\}$ and $\hat{X} \in V(B) \setminus \{\hat{C}_1\}$. But then, since C_1 is a 5-cycle, $w_1 \notin \{u_1, v_1\}$ and u_1, v_1 are not adjacent in C_1, either u_1 or v_1 is adjacent to w_1 in C_1. Say w.l.o.g. u_1 is adjacent to w_1 in C_1. Then, the edges $uu_1, w_1\hat{X}$ imply that C_1 and X satisfy the conditions of Lemma 13 (with $C_1 := C_1, C_2 := X, u_1 := v_1$ and $v_1 := w_1$), a contradiction.
(b.2) Otherwise $|C_2| \geq 6$. Apply Corollary 1 with $C_1 := C_1, C_2 := C$ and with $u_1\hat{X}$ being an arbitrary edge of B incident to some $u_1 \in V(C_1)$. This way one can find in polynomial time a cycle F in B incident to distinct nodes u_1, v_1 of C_1 and such that $|F| = 3$. Thus, $V(F) = \{\hat{C}, \hat{C}_1, \hat{C}_2\}$. Now, since $|C_2| \geq 6$, $\mathrm{cr}(C_2) \geq 6/4$. Also, since $\mathrm{cr}(C) = 2$ and $\mathrm{cr}(C_1) = 5/4$, one has that $\sum_{\hat{X} \in V(F)} \mathrm{cr}(X) \geq |F| + 7/4$. In the following, only the fact that F is a cycle in B incident to C and to distinct nodes u_1, v_1 of C_1 such that $\sum_{\hat{X} \in V(F)} \mathrm{cr}(X) \geq |F| + 7/4$ is needed. This will be useful to avoid duplicated arguments in later cases.

Since C_1 is a non-local 5-cycle, it also belongs to some 2VC block B' of $\hat{G}_S$ distinct from B. Apply the local 3-matching lemma (Lemma 7) to C_1 in B' to find an edge $w_1\hat{C}'_1$ with $\hat{C}'_1 \in V(B')$ and $w_1 \in V(C_1) \setminus \{u_1, v_1\}$. Consider now the following subcases:
(b.2.i) C'_1 is a 4-cycle. We illustrate this case in Figure 3. Apply Lemma 11 with $C_1 := C'_1, C_2 := C_1$ and with $u_2\hat{X} := w_1\hat{X}$. This way one can find in polynomial time a cycle F' in B' incident to

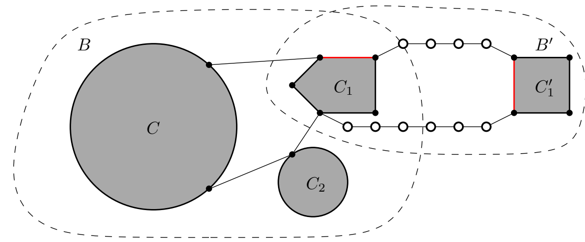

Figure 3: An illustration of Case (b.2.i). The red edges are removed from S. The red edge in C_1 is removed through the application of Corollary 2.

nodes w_1, x_1 in C_1 and to nodes u'_1, v'_1 in C'_1 such that there is a Hamiltonian u'_1, v'_1-path P in $G[V(C'_1)]$. Here x_1 is a node of C_1 distinct from w_1. Set $S'' := (S \setminus C'_1) \cup F \cup F' \cup P$. In S'' there is a large component C'' spanning the nodes of all components of S incident on F or F'. Notice that $|S''| = |S| + |F| + |F'| - 1$. One has that $\sum_{\hat{X} \in V(F)} \mathrm{cr}(X) \geq |F| + 7/4$, and every component of B' incident with F' yields at least 1 credit. Since B and B' are both 2VC blocks of $\hat{G}_S$, the only component present in both B and B' is C_1. Thus, the components of S incident on either F or F' yield at least $|F| + 7/4 + |F'| - 1$ credits. Thus, $\mathrm{cr}(S'') \leq \mathrm{cr}(S) - |F| - |F'| - 3/4 + \mathrm{cr}(C'') = \mathrm{cr}(S) - |F| - |F'| + 5/4$.

Observe that $C'' \setminus C_1$ contains a u_1, v_1-path via F and a w_1, x_1-path via F'. Since $w_1 \notin \{u_1, v_1\}$, $u_1 \neq v_1$, and $w_1 \neq x_1$, C'' and C_1 satisfy the conditions of Corollary 2 (with $F := C''$ and $C := C_1$). Therefore, one can find an edge $e \in C_1$ such that $C'' \setminus \{e\}$ is 2EC. Set $S' := S'' \setminus \{e\}$. In S' there is a large component C' spanning the nodes of all components of S incident on F or F'. Moreover, $\mathrm{cr}(S) = \mathrm{cr}(S'')$, so one has $\mathrm{cost}(S) - \mathrm{cost}(S') = |S| - |S'| + \mathrm{cr}(S) - \mathrm{cr}(S') \geq -|F| - |F'| + 2 + |F| + |F'| - 5/4 > 0$. We remark that S' is canonical and it contains fewer components than S, so this subcase follows.
(b.2.ii) Otherwise $|C'_1| \geq 5$.

We illustrate this case in Figure 4. We have $\mathrm{cr}(C'_1) \geq 5/4$. Apply Lemma 10 with $C_1 := C_1, C_2 := C'_1$, $u_1\hat{X}_1 := w_1\hat{X}$ and $u_2\hat{X}_2$ being an arbitrary edge of B incident on C'_1. This way, one can find in polynomial time, a cycle F' in B' incident on distinct nodes w_1 and x_1 in C_1 and on C'_1. Set $S'' := S \cup F \cup F'$. In S'' there is a large component C'' spanning the nodes of all components of S incident on F or F'. Notice that $|S''| = |S| + |F| + |F'|$. One has that $\sum_{\hat{X} \in V(F)} \mathrm{cr}(X) \geq |F| + 7/4$, and every component of B' incident on F' yields at least 1 credit, while C'_1 yields at least 5/4 credits. Since B and B' are both 2VC blocks of $\hat{G}_S$, the only component present in both B and B' is C_1. Thus, the components of S incident on either F or F' yield at least $|F| + 7/4 + (|F'| - 2) + 5/4 = |F| + |F'| + 1$ credits. Thus, $\mathrm{cr}(S'') \leq \mathrm{cr}(S) - |F| - |F'| - 1 + \mathrm{cr}(C'') = \mathrm{cr}(S) - |F| - |F'| + 1$.

Observe that $C'' \setminus C_1$ contains a u_1, v_1-path via F and a w_1, x_1-path via F'. Since $w_1 \notin \{u_1, v_1\}$, $u_1 \neq v_1$, and $w_1 \neq x_1$, C'' and C_1 satisfy the conditions of Corollary 2 (with $F := C''$ and $C := C_1$). Thus, we can find an edge $e \in C_1$ such that $C'' \setminus \{e\}$ is 2EC. Set $S' := S'' \setminus \{e\}$. In S' there is a large component C' spanning the nodes of all components of S incident on F or F'. Moreover, $\mathrm{cr}(S) = \mathrm{cr}(S'')$, so one has $\mathrm{cost}(S) - \mathrm{cost}(S') = |S| - |S'| + \mathrm{cr}(S) - \mathrm{cr}(S') \geq$

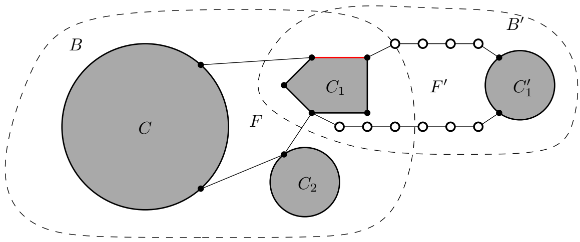

Figure 4: An illustration of Case (b.2.ii). The red edge in C_1 is removed through the application of Corollary 2.

$-|F| - |F'| + 1 + |F| + |F'| - 1 = 0$. We remark that S' is canonical and it contains fewer components than S, so this subcase follows.

(c) $|V(B)| \geq 4$. Since B is 2VC and $|V(B)| > 3$ one can find in polynomial time a cycle F in B containing $\hat{C}$ and $\hat{C}_1$ such that $|F| \geq 3$. We first show that we can assume that $|F| \geq 4$. Assume this is not the case, i.e., $|F| = 3$. Since $|V(B)| \geq 4$, there exists an edge $\hat{X}\hat{Y} \in B$ such that $\hat{X} \in V(F), \hat{Y} \in V(B) \setminus V(F)$. Since B is 2VC, we can find in polynomial time a path P from $\hat{Y}$ to some node $\hat{X}' \in V(F) \setminus \{\hat{X}\}$ not going through $\hat{X}$. The cycle $(F \setminus \{\hat{X}\hat{X}'\}) \cup P \cup \{\hat{X}\hat{Y}\}$ has length at least 4 and contains the nodes C and C_1 (because it contains every node in $V(F)$), so it is our desired cycle.

We now show that we can assume that $|F| = 4$ and every component of B incident on F other than C is a non-local 5-cycle. Indeed, every component of B incident on F yields at least 5/4 credits if it is a 5-cycle, and at least 6/4 otherwise (while C yields 2 credits). Therefore, if $|F| \geq 5$ or $|F| = 4$ and contains at least one component with at least 6 edges, one has that $\sum_{\hat{X} \in V(F)} \text{cr}(X) \geq |F| + 2$. Hence, in $S' := S \cup F$ there is a large component C' spanning the nodes of all components of B incident on F, and $\text{cost}(S) - \text{cost}(S') = |S| - |S'| + \text{cr}(S) - \text{cr}(S') \geq -|F| + |F| + 2 - \text{cr}(C') = 0$. Clearly, S' is canonical and has fewer components than S, so it satisfies the claim of the lemma.

By our above assumptions, we can assume w.l.o.g. that $F = \{\hat{C}\hat{C}_2, \hat{C}\hat{C}_3, \hat{C}_1\hat{C}_2, \hat{C}_1\hat{C}_3\}$, where C_2 and C_3 are components of B that are non-local 5-cycles. We will show how to find a cycle F' in B such that F' is incident to C and to distinct nodes u_1, v_1 of C_1, and $|F'| \geq 4$. This is enough to prove the claim of the lemma, because if $|F'| \geq 4$, since Lemma 14 does not hold, every component of B incident with F' contains at least 5 edges and thus yields at least 5/4 credits, while C yields 2 credits, then $\sum_{\hat{X} \in V(F)} \text{cr}(X) \geq |F| + 7/4$. Therefore, we can apply the exact same arguments as in Case (b.2) above.

Assume F does not satisfy these properties. Since $|F| = 4$ and is incident to C and C_1, the edges $\hat{C}_1\hat{C}_2$ and $\hat{C}_1\hat{C}_3$ must be both incident to the same node in C_1, say u_1. By applying the local 3-matching lemma (Lemma 7) to C_1 in B, and since F is incident to u_1, there always exists a matching $\{u_1\hat{X}_1, u_2\hat{X}_2, u_3\hat{X}_3\}$ in B such that $u_2, u_3 \in V(C_1)$ and $\hat{X}_i \in V(B) \setminus V(C_1)$, for $i \in \{1, 2, 3\}$. If, for some $i \in \{2, 3\}$ $X_i = C_2$, then the cycle $F' := (F \setminus \{u_1\hat{C}_2\}) \cup \{u_i\hat{C}_2\}$ has the desired properties. A symmetric argument works when $X_i = C_3$ for some $i \in \{2, 3\}$.

Suppose now $X_2 = X_3 = C$. Then, there is a matching of size 2 between C and C_1, and by the same arguments as in Case (b.1), this leads to a contradiction to the fact that Lemma 13 does not hold.

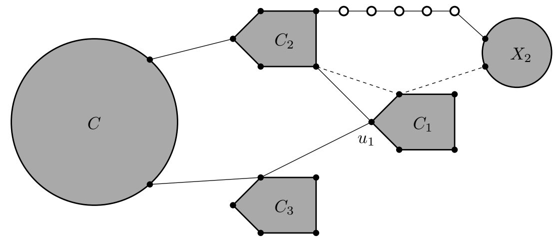

Figure 5: We illustrate how to compute a cycle of size at least 4 that is incident to distinct nodes of C_1. In the figure, any of the dashed edges yields a cycle of the desired type.

Thus, we can assume w.l.o.g. that $X_2 \notin \{C, C_1, C_2, C_3\}$. Since B is 2VC, one can find in polynomial time a path P in B from X_2 to a node in $V(F)$ not going through C_1. If P has C_2 as an endpoint, the cycle $F' := \{\hat{C}\hat{C}_2, \hat{C}\hat{C}_3, u_1\hat{C}_3, u_2\hat{X}_2\} \cup P$ has the desired properties. A symmetric argument works if P has C_3 as an endpoint. Finally, if P has C as an endpoint, the cycle $F' := \{\hat{C}\hat{C}_2, u_1\hat{C}_2, u_2\hat{X}_2\} \cup P$ has the desired properties (see Figure 5). □

Lemma 16. *Let $|V(B)| = 2$ and let C_1 be the component of B distinct from C. If $|C_1| \geq 6$, then one can compute in polynomial time a canonical 2-edge cover S' of G such that every component of S' is 2EC, S' has fewer components than S, and $\text{cost}(S') \leq \text{cost}(S)$.*

Proof. If $|C_1| \geq 8$, then $\text{cr}(C_1) = 2$. In this case we simply set $S' := S \cup \{e_1, e_2\}$, where e_1 and e_2 are two distinct edges between $V(C)$ and $V(C_1)$, which are guaranteed by the local 3-matching lemma (Lemma 7) applied to C and B. In S' there is a large 2EC component C' spanning the nodes of C and C_1. One has $\text{cost}(S) - \text{cost}(S') = |S| - |S'| + \text{cr}(C) + \text{cr}(C_1) - \text{cr}(C') = -2 + 2 + 2 - 2 = 0$.

From now on we assume that C_1 is a 6-cycle or a 7-cycle, as otherwise we will be immediately done from the previous lemmas. Assume first that there are edges $u_1\hat{C}$ and $v_1\hat{C}$, such that there exists a u_1, v_1-Hamiltonian path in $G[V(C_1)]$. By this assumption, C_1 satisfies the conditions of Lemma 13 (with $C_2 := C$), so we can construct the desired S'. Therefore, we can assume that this is not the case.

Apply the local 3-matching lemma (Lemma 7) to C_1 and B to find a matching $\{u_1\hat{C}, v_1\hat{C}, w_1\hat{C}\}$ between $\{u_1, v_1, w_1\} \subseteq V(C_1)$ and $V(C)$ in G. By the above argument, we can assume w.l.o.g. that u_1, v_1, and w_1 are pairwise non-adjacent in C_1, otherwise, there would be a Hamiltonian path in $G[V(C_1)]$ between them, a contradiction to the assumption above. Moreover, it must be that there is no Hamiltonian path in $G[V(C_1)]$ between any pair in $\{u_1, v_1, w_1\}$. By the Hamiltonian pairs lemma (Lemma 9), there must exist an edge $x_1\hat{X}$ in G such that $x_1 \in V(C_1) \setminus \{u_1, v_1, w_1\}$ and $\hat{X} \in V(B')$, where B' is a 2VC block of $\hat{G}_S$. Again, by the above argument, it must be that $B' \neq B$. Now, Since C_1 is a 6-cycle or a 7-cycle and u_1, v_1 and w_1 are pairwise non-adjacent, we can assume w.l.o.g. that x_1 is part of the shortest u_1, v_1-path in C_1. Let $F := \{u_1\hat{C}, v_1\hat{C}\}$ and let C_2 be the component of B' with the least number of edges and distinct from C_1.

We now consider two cases:

(a) C_2 is a 4-cycle. This case is illustrated in Figure 6. Apply Lemma 11 to find a cycle F' in B' incident to distinct nodes x_1, y_1 in

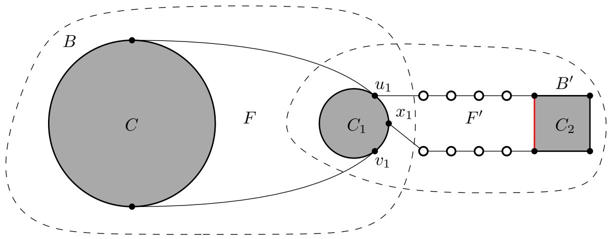

Figure 6: Illustration of Case (a). The red edge is removed from S. An additional edge of C_1 is removed from S through the application of Corollary 3.

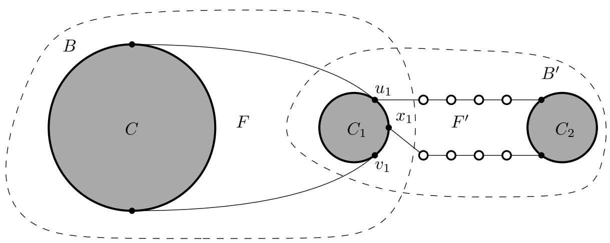

Figure 7: An illustration of the construction for Case (b). In most cases this construction will be enough, although additional steps are needed in a few subcases. An edge from C_1 is removed from S through the application of Corollary 3.

C_1 and to distinct nodes u_2, v_2 in C_2 such that there is a Hamiltonian u_2, v_2-path P in $G[V(C_2)]$. Set $S'' := (S \setminus C_2) \cup F \cup F' \cup P$. In S'' there is a large component C'' spanning the nodes of all components of S incident on F or F'. Every such component yields at least 1 credit, while C yields 2 credits and C_1 yields at least 6/4 credits. Since B and B' are 2VC blocks of $\hat{G}_S$, C_1 is the only component present in both F and F', $\mathrm{cr}(S'') \le \mathrm{cr}(S) - (|F| - 2) - (|F'| - 1) - \mathrm{cr}(C) - \mathrm{cr}(C_1) + \mathrm{cr}(C'') = \mathrm{cr}(S) - (|F| - 2) - (|F'| - 1) - 2 - 6/4 + 2 < \mathrm{cr}(S) - |F| - |F'| + 2$. Notice that $|S''| = |S| + |F| + |F'| - 1$.

Observe that $C'' \setminus C_1$ contains a u_1, v_1-path and a x_1, y_1-path. Since x_1 is in the shortest u_1, v_1-path in C_1, $u_1 \ne v_1$, and $x_1 \ne y_1$, C'' and C_1 satisfy the conditions of Corollary 3. Therefore, we can find an edge $e \in E[C_1]$ such that $C'' \setminus \{e\}$ is 2EC. Set $S' := S'' \setminus \{e\}$, where we remark that $\mathrm{cr}(S') = \mathrm{cr}(S'')$. One has $\mathrm{cost}(S) - \mathrm{cost}(S') = |S| - |S'| + \mathrm{cr}(S) - \mathrm{cr}(S') > -|F| - |F'| + 2 + |F| + |F'| - 2 = 0$.

(b) Otherwise $|C_2| \ge 5$. Apply Corollary 1 to find a cycle F' in B' incident to distinct nodes x_1, y_1 in C_1 and to C_2 such that $|F'| \ge \min\{3, |V(B')|\}$. Set $S'' := S \cup F \cup F'$. In S'' there is a large component C'' spanning the nodes of all components of S incident on F or F'. Since Case (a) does not hold, it must be that every component of B' has at least 5 edges. Thus, every component of B' incident with F' yields at least 5/4 credits. Since B and B' are 2VC blocks of $\hat{G}_S$, C_1 is the only component of S present in both F and F'. Since $V(F) = \{\hat{C}, \hat{C}_1\}$, it holds that:

$$\mathrm{cr}(S) - \mathrm{cr}(S'') \ge \mathrm{cr}(C) + \mathrm{cr}(C_1) + \mathrm{cr}(C_2) + \frac{5}{4}(|F'| - 2) - \mathrm{cr}(C'')$$
$$= \mathrm{cr}(C_1) + \mathrm{cr}(C_2) + \frac{5}{4}(|F'| - 2).$$

Observe that $C'' \setminus C_1$ contains a u_1, v_1-path via F and a x_1, y_1-path via F' (see Figure 7). Since x_1 is on the shortest u_1, v_1-path in C_1, $u_1 \ne v_1$, and $x_1 \ne y_1$, C'' and C_1 satisfy the conditions of Corollary 3 (with $F := C''$ and $C := C_1$). Therefore, we can find in polynomial time an edge $e \in C_1$ such that $C'' \setminus \{e\}$ is 2EC. Set $S' := S'' \setminus \{e\}$, where we note that $\mathrm{cr}(S') = \mathrm{cr}(S'')$. Using that $|F| = 2$ one has:

$$\begin{aligned}\mathrm{cost}(S) - \mathrm{cost}(S') &= |S| - |S'| + \mathrm{cr}(S) - \mathrm{cr}(S') \\ &\ge |S| - |S'| + \mathrm{cr}(C_1) + \mathrm{cr}(C_2) + \frac{5}{4}(|F'| - 2) \\ &= -|F| - |F'| + 1 + \mathrm{cr}(C_1) + \mathrm{cr}(C_2) + \frac{5}{4}|F'| - \frac{10}{4} \\ &= \mathrm{cr}(C_1) + \mathrm{cr}(C_2) + \frac{1}{4}|F'| - \frac{14}{4}.\end{aligned}$$

Consider now the following subcases:

(b.1) $|C_1| = 7$. Then one has that $\mathrm{cr}(C_1) \ge 7/4$ and $\mathrm{cr}(C_2) \ge 5/4$. Thus, $\mathrm{cost}(S) - \mathrm{cost}(S') \ge \mathrm{cr}(C_1) + \mathrm{cr}(C_2) + |F'|/4 - 14/4 \ge 7/4 + 5/4 + 2/4 - 14/4 = 0$. The last inequality follows from the fact that $|F'| \ge 2$.

(b.2) $|C_2| \ge 6$. Then one has that $\mathrm{cr}(C_2) \ge 6/4$, and by assumption of the lemma, $\mathrm{cr}(C_1) \ge 6/4$. Thus, $\mathrm{cost}(S) - \mathrm{cost}(S') \ge \mathrm{cr}(C_1) + \mathrm{cr}(C_2) + |F'|/4 - 14/4 \ge 6/4 + 6/4 + 2/4 - 14/4 = 0$. The last inequality follows from the fact that $|F'| \ge 2$.

(b.3) $|F'| \ge 3$. By assumption of the lemma, $\mathrm{cr}(C_1) \ge 6/4$, and $\mathrm{cr}(C_2) \ge 5/4$. Thus, $\mathrm{cost}(S) - \mathrm{cost}(S') \ge \mathrm{cr}(C_1) + \mathrm{cr}(C_2) + |F'|/4 - 14/4 \ge 6/4 + 5/4 + 3/4 - 14/4 = 0$. The last inequality follows from this case assumption.

(b.4) Otherwise. By exclusion of the previous cases and the assumption of the lemma, it must be that $|C_1| = 6, |C_2| = 5$ and $|F'| = 2$. Since $|F'| \ge \min\{3, |V(B')|\}$, it must be that $V(B') = \{\hat{C}_1, \hat{C}_2\}$. In this case we show how to compute an alternate S'. Let $C_1 = a_1a_2a_3a_4a_5a_6$, and assume w.l.o.g. $u_1 = a_1, v_1 = a_3, w_1 = a_5$, and $x_1 = a_2$. We recall that x_1 is adjacent to some component-node in $V(B') \setminus \{\hat{C}_1\}$, and since $V(B') = \{\hat{C}_1, \hat{C}_2\}$, it must be that there exists an edge x_1b_1 for some $b_1 \in V(C_2)$.

(b.4.i) There exists an edge a_ib_2 for some $b_2 \in V(C_2)$ and $i \in \{2, 4, 6\}$ such that $a_ib_2 \ne a_2b_1$. If $i = 4$ then let $S' := (S \setminus \{a_1a_2, a_3a_4\}) \cup \{a_1\hat{C}, a_3\hat{C}, a_2b_1, a_4b_2\}$. In S' there is a large component C' spanning the nodes of C, C_1, and C_2. Indeed, C' consists of the components C and C_2 joined by the paths $\hat{C}a_1a_6a_5a_4\hat{C}_2$ and $\hat{C}a_3a_2\hat{C}_2$ (see Figure 8). One has $\mathrm{cost}(S) - \mathrm{cost}(S') = |S| - |S'| + \mathrm{cr}(C) + \mathrm{cr}(C_1) + \mathrm{cr}(C_2) - \mathrm{cr}(C') = -2 + 2 + 6/4 + 5/4 - 2 > 0$. The case where $i = 6$ is symmetric. Assume now $i = 2$, so since $a_ib_2 \ne a_2b_1$ it must be that $b_1 \ne b_2$. By the local 3-matching lemma (Lemma 7) and since $V(B') = \{\hat{C}_1, \hat{C}_2\}$, there is a matching M of size 3 between $V(C_1)$ and $V(C_2)$.

Claim 1. *There exists a matching $\{a_2b, ab'\}$ such that $a \in V(C_1)$ and $bb' \in E[C_2]$.*

Proof. Since C_2 is a 5-cycle, $|M| = 3$, and $b_1 \ne b_2$, there is an edge $e \in M$ incident to a node in C_2 that is adjacent to either b_1 or b_2. Let b' be the endpoint of e in C_2 and $b \in \{b_1, b_2\}$ be the node of C_2 such that $bb' \in E[C_2]$. If e is not incident on a_2 in C_1 then $\{a_2b, e\}$ is the desired matching. Assume this is not the case, so that there exist edges a_2b and $a_2b' = e$.

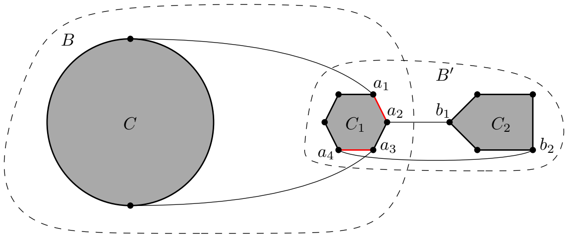

Figure 8: Illustration of Case (b.4.i), when $i = 4$. The red edges are removed from S.

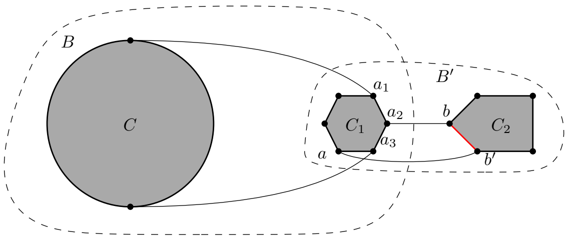

Figure 9: Illustration of Case (b.4.i), when $i = 4$. The red edge is removed from S. An additional edge of C_1 is removed from S through the application of Corollary 3.

Since $|M| = 3$, $b \neq b'$, and C_2 is a 5-cycle, there exist an edge $e' \in M \setminus \{e\}$ incident on a node adjacent in C_2 to either b or b'. Assume w.l.o.g. e' is incident on a node in C_2 adjacent to b. Since M is a matching and e is incident to a_2 in C_1, e' is not incident on a_2 in C_1 and thus $\{e', a_2b\}$ is the desired matching. The claim follows. ◻

Let $\{a_2b, ab'\}$ be the matching obtained from Claim 1. Let $S'' := (S \setminus \{bb'\}) \cup \{a_1\hat{C}, a_3\hat{C}, a_2b, ab'\}$. In S'' there is a large 2EC component C'' spanning the nodes of C, C_1, and C_2 (see Figure 9). Notice that $C'' \setminus C_1$ contains a a_1, a_3-path and a a_2, a-path. Since a_2 is in the shortest a_1, a_3-path in C_1 and $a_2 \neq a$, C'' and C_1 satisfy the conditions of Corollary 3 (with $F := C''$ and $C := C_1$). Therefore, in polynomial time we can find an edge $e \in C_1$ such that $C'' \setminus \{e\}$ is 2EC. Set $S' := S'' \setminus \{e\}$ and let C' be the large 2EC component of S' spanning the nodes of C, C_1, and C_2. One has $\text{cost}(S) - \text{cost}(S') = |S| - |S'| + \text{cr}(C) + \text{cr}(C_1) + \text{cr}(C_2) - \text{cr}(C') = -2 + 2 + 6/4 + 5/4 - 2 > 0$.

(b.4.ii) Otherwise. This case is illustrated in Figure 10. Recall that there are no Hamiltonian paths in $G[V(C_1)]$ between any pair in $\{u_1, v_1, w_1\} = \{a_1, a_3, a_5\}$. Moreover, by exclusion of Case **(b.4.i)**, there is only one edge between $\{a_2, a_4, a_6\}$ and $V(C_2)$. Thus, by Lemma 8, there must exist an edge $a_i\hat{C}_3$, $i \in \{2, 4, 6\}$, with $C_3 \neq C_2$. Since $a_i \notin \{u_1, v_1, w_1\}$, by symmetry with C_2, it must be that $\hat{C}_3 \in V(B'')$, where B'' is a 2VC block of $\hat{G}_S$ distinct from B and B' such that $V(B'') = \{\hat{C}_1, \hat{C}_3\}$ and C_3 is a 5-cycle.

By the local 3-Matching lemma (Lemma 7) and since $V(B') = \{\hat{C}_1, \hat{C}_2\}$, $V(B'') = \{\hat{C}_1, \hat{C}_3\}$, there is a matching of size 3 between $V(C_1)$ and $V(C_2)$ and between $V(C_1)$ and $V(C_3)$. Since C_ℓ is a 5-cycle, for $\ell \in \{2, 3\}$, there are edges $y_{1\ell}\hat{C}_1$ and $y_{2\ell}\hat{C}_1$ such that $y_{1\ell}y_{2\ell} \in E[C_\ell]$, for $\ell \in \{2, 3\}$. Set $S' := (S \setminus \{y_{12}y_{22}, y_{13}y_{23}\}) \cup \{y_{12}\hat{C}_1, y_{22}\hat{C}_1, y_{13}\hat{C}_1, y_{23}\hat{C}_1\}$. In S' there is a large 2EC component

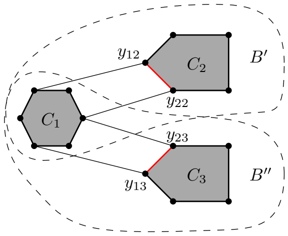

Figure 10: Illustration of Case (b.4.ii). The red edges are removed from S. Notice that, in this case, the components are not merged together with the component C.

C' spanning the nodes of C_1, C_2 and C_3. We compute $\text{cost}(S) - \text{cost}(S') = |S| - |S'| + \text{cr}(C_1) + \text{cr}(C_2) + \text{cr}(C_3) - \text{cr}(C') = -2 + 6/4 + 5/4 + 5/4 - 2 = 0$.

Notice that in every case, S' is canonical. ◻

Finally, we prove the main gluing lemma.

Proof of Lemma 6. We can assume that the conditions of Lemmas 14, 15, and 16 do not hold; otherwise we are done. Since the conditions of Lemmas 14 and 15 do not hold, it must be that every component of B has at least 6 edges. Since the condition of Lemma 16 does not hold, it must be that $|V(B)| \geq 3$. Then, since B is 2VC, one can find in polynomial time a cycle F in B containing $\hat{C}$ and such that $|F| \geq 3$. Set $S' := S \cup F$. In S' there is a large 2EC component C' spanning the nodes of every component of B incident on F. Every such component has at least 6 edges, and thus yields at least 6/4 credits, while C yields 2 credits. Therefore, we collect at least $6/4(|F| - 1) + 2$ credits from those components. One has $\text{cost}(S) - \text{cost}(S') = |S| - |S'| + \text{cr}(S) - \text{cr}(S') \geq -|F| + 6/4(|F| - 1) + 2 - \text{cr}(C') \geq 0$, where the last inequality follows from the fact that $|F| \geq 3$. Since S' is canonical and it contains strictly less connected components than S, the lemma follows. ◻

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