On Conflict-Free Multi-Coloring*

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Abstract A *conflict-free coloring* of a hypergraph $H = (V, \mathcal{E})$, $\mathcal{E} \subseteq 2^V$, is a coloring of the vertices V such that every hyperedge $E \in \mathcal{E}$ contains a vertex of "unique" color. Our goal is to minimize the total number of distinct colors. In its full generality, this problem is known as the conflict-free (hypergraph) coloring problem. It is known that $\Theta(\sqrt{m})$ colors might be needed in general.
In this paper we study the relaxation of the problem where one is allowed to assign multiple colors to the same node. The goal here is to substantially reduce the total number of colors, while keeping the number of colors per node as small as possible. By a simple adaptation of a result by Pach and Tardos [2009] on the single-color version of the problem, one obtains that only $O(\log^2 m)$ colors in total are sufficient (on every instance) if each node is allowed to use up to $O(\log m)$ colors.
By improving on the result of Pach and Tardos (under the assumption $n \ll m$), we show that the same result can be achieved with $O(\log m \cdot \log n)$ colors in total, and either $O(\log m)$ or $O(\log n \cdot \log \log m) \subseteq O(\log^2 n)$ colors per node. The latter coloring can be computed by a polynomial-time Las Vegas algorithm.

1 Introduction

Consider the following scenario motivated by wireless applications. We are given a collection of n transmitters, where each transmitter can transmit at a chosen frequency. Furthermore, we are given a collection of m receivers, where each receiver receives the signal of some subset of the transmitters. Each receiver can tune to a proper frequency, and it receives any message transmitted at that frequency if precisely one transmitter in its range is transmitting at that frequency (if two or more such transmitters do this, then interferences destroy the message). We have to choose frequencies such that each receiver can receive messages, and our goal is to minimize the total number of frequencies used altogether.

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In its full generality, this problem can be naturally modelled as the following *conflict-free (hypergraph) coloring* problem. We are given a hypergraph $H = (V, \mathcal{E})$, $\mathcal{E} \subseteq 2^V$, with n nodes and m hyperedges. A coloring of H with k colors is an assignment $c : V \to \{1, \dots, k\}$ of an integer value (*color*) to each node. A coloring is *conflict-free* if for each hyperedge E there exists at least one node $v \in E$ such that $c(v) \neq c(u)$ for any other node $u \neq v$ with $u \in E$. Our goal is to find a conflict-free coloring with the minimum number of colors. The latter quantity $\chi_{\mathrm{cf}}(H)$ is the *conflict-free chromatic number* of H. Obviously, in the above scenario, nodes, hyperedges, and colors model transmitters, receivers, and frequencies, respectively.

Trivially, $\min\{n, m+1\}$ frequencies are sufficient to achieve (in general) a conflict-free coloring. This result can be improved to $\Theta(\sqrt{m})$ [17][3]. The latter result is already tight: simply consider a complete graph on n nodes; a conflict-free coloring requires $n = \Theta(\sqrt{m})$ colors.

Our results and techniques. Motivated by the large (polynomial) number of colors needed to solve conflict-free coloring in general, in this paper we study a relaxation of the problem where we are allowed to use multiple colors at each node. This models a situation in which transmitters can transmit on multiple frequencies.

More formally, we study the following *conflict-free (hypergraph) multi-coloring* problem. Given a hypergraph $H = (V, \mathcal{E})$, $\mathcal{E} \subseteq 2^V$, with n nodes and m hyperedges, a multi-coloring of H with k colors is an assignment $C : V \to 2^{\{1,\dots,k\}}$ of a subset of integer values (*colors*) to each node. A hyperedge $E \in \mathcal{E}$ is *conflict-free* if there exists at least one node $v \in E$ and one color $c(v) \in C(v)$ such that, for any other node $v \neq u \in E$ and any $c(u) \in C(u)$, one has $c(v) \neq c(u)$ (intuitively, some color appears exactly once in E). A multi-coloring is *conflict-free* if all hyperedges are conflict-free. Our goal is now two-fold: on one side, as before, we wish to minimize the total number of colors. At the same time, we would like to minimize the maximum number of colors assigned to each node. At high-level, we address the following main question: Is a small number of colors per node sufficient to drastically reduce the total number of colors?

We answer affirmatively to the above question. Indeed, one simple way to achieve a result of the above kind is via an adaptation of a result by Pach and Tardos [17] on the standard (single-color) version of the problem. Suppose that all hyperedges have size at least $2t - 1$ (for any integer $t \geq 1$). They show how to compute a conflict-free coloring with $O(tm^{1/t} \log m)$ colors in total using a simple (expected) polynomial-time Las-Vegas algorithm. The idea is to make $\Theta(\log m)$ copies of each node, and then apply the algorithm in [17]. The set of colors assigned to a given node is simply the union of the colors assigned to its copies. This way each node is assigned at most $O(\log m)$ colors, and the total number of colors is $O(\log^2 m)$. Pach and Tardos improve their result when the

[3] Note that this is an absolute upper bound on the conflict-free chromatic number, while of course some hypergraphs might need fewer colors. All upper bounds in this paper are of this type.

dependencies among hyperedges are limited, by means of a constructive version [16] of Lovász's Local Lemma (LLL) [10,19,22]. More formally, let $\Gamma \leq m-1$ denote the maximum number of different hyperedges that any hyperedge E intersects (the maximum *hyperedge degree* of H). In this case the result in [17] is refined to $O(t\Gamma^{1/t}\log\Gamma)$, and consequently one can obtain a conflict-free multi-coloring with $O(\log^2\Gamma)$ colors in total, and $O(\log\Gamma)$ colors per node.

Our main result (which might be of independent interest) is an improvement on the $O(t\Gamma^{1/t}\log\Gamma)$ upper bound, under each of the following assumptions: (i) n is sufficiently smaller than Γ (see Section 2), (ii) hyperedges have size at most $O(\log\Gamma)$ (see Section 3):

Theorem 1. *There exists a polynomial-time Las Vegas algorithm for conflict-free coloring using $O(t\Gamma^{1/t}\log n) \subseteq O(tm^{1/t}\log n)$ colors, where $2t-1$ is a lower bound on the size of any hyperedge and Γ is the maximum hyperedge degree. If the maximum hyperedge size is $O(\log\Gamma)$, the number of colors can be reduced to $O(t\Gamma^{1/t})$.*

We remark that there are ranges of values of Γ, m and n such that we reduce the upper-bound on the conflict-free chromatic number by a factor $\Omega(\sqrt{n})$. For example, consider a hypergraph on n nodes for which we choose uniformly at random $m = n^{\sqrt{n}/\ln n}$ hyperedges of size $\sqrt{n}$. Then the probability that two hyperedges E, E' intersect is at least $1/\sqrt{n}$ (the probability that a fixed node is contained in E'). Hence in expectation a given hyperedge intersects at least $(m-1)/\sqrt{n}$ other hyperedges. Therefore we can assume $\Gamma \in \Omega(m/\sqrt{n})$. Since also $\Gamma \leq m-1$, we have $\sqrt{n} \in \Theta(\log m) = \Theta(\log\Gamma)$. In this case the result of Pach-Tardos gives the (up to constant factors) trivial bound of $O(\log^2\Gamma) = O(n)$ colors, while our construction uses only $O(\log\Gamma) = O(\sqrt{n})$ colors.

Furthermore we can improve on Theorem 1 by a refined analysis in case of hypergraphs with hyperedge sizes bounded from below by $2t-1$ and from above by $O(t)$. For such almost-uniform hypergraphs we achieve a conflict-free coloring with $O(tm^{1/(t+1)})$ colors. This generalizes a result on uniform hypergraphs [14].

We next discuss the main ideas behind Theorem 1. The conflict-free coloring algorithm in [17] works as follows. There is a sequence of rounds. At round i we use a new color i and color each still uncolored node independently at random with some (fairly small) probability p. Observe that the color assigned to each node follows a geometric distribution.

Our main idea is to replace colors in the above approach with disjoint *color classes*, each one containing h colors. Then each node is independently assigned a color chosen uniformly at random in its color class. For our goal it is convenient to use a constant probability p and a large enough value of h. The rough idea is that, with large-enough probability, for each hyperdge E there is some round i where for the yet unassigned nodes $E' \subseteq E$ we have that $(h/|E'|)^{|E'|}$ is lower bounded by a polynomial in the maximum hyperedge degree Γ. Therefore with sufficiently large probability (with respect to $1/\Gamma$) some color in the i-th color class will appear only once in E' and hence in E, since color classes are disjoint.

By using node duplication as discussed before, one immediately obtains the following corollary for conflict-free multi-coloring.

Corollary 1. *There exists a polynomial-time Las Vegas algorithm for conflict-free multi-coloring using* $O(\log \Gamma \cdot \log n)$ *colors in total, and* $O(\log \Gamma)$ *colors per node, where* Γ *is the maximum hyperedge degree.*

Note that m (hence Γ) can be exponential in n. Therefore the upper bound $O(\log \Gamma)$ on the number of colors per node can be linear in n: this might be too much due to technological constraints. We were able to reduce the mentioned upper bound (for large enough Γ) by means of a more sophisticated algorithm, without increasing the total number of colors (see Section 3).

Theorem 2. *There exists a polynomial-time Las Vegas algorithm for conflict-free multi-coloring using* $O(\log \Gamma \cdot \log n)$ *colors in total, and* $O(\log n \cdot \log\log \Gamma) \subseteq O(\log^2 n)$ *colors per node, where* Γ *is the maximum hyperedge degree.*

The above refinement is obtained as follows: Observe that, using our result from Theorem 1, hyperedges of size $\Omega(\log \Gamma)$ can be conflict-free colored with a single color per node and $O(\log \Gamma \cdot \log n)$ colors in total. We partition the remaining hyperedges in $O(\log\log \Gamma)$ buckets of approximately uniform size. Hyperedges in each bucket are colored independently, using a novel set of colors each time. In each bucket we perform a node duplication which is strictly sufficient to achieve hyperedges of size $\Theta(\log \Gamma)$, and then apply a modified conflict-free coloring algorithm. As mentioned, due to the (approximate) uniformity of the hyperedge sizes, $O(t\Gamma^{1/t}) = O(\log \Gamma)$ colors are sufficient in each bucket (adding overall $O(\log \Gamma \cdot \log\log \Gamma) \subseteq O(\log \Gamma \cdot \log n)$ many colors to the total). For increasing value of the bucket size, on one hand the (potential) number of hyperedges increases, while on the other hand the number of node duplicates needed to reach the size $\Omega(\log \Gamma)$ decreases. The two phenomena compensate well. In particular, it is always sufficient to create $O(\log n)$ copies of each node (hence the total number of colors per node is $O(\log n \log\log \Gamma) \subseteq O(\log^2 n)$).

Our work also implies improved bounds for conflict hypergraphs induced by certain shapes in the plane. In particular, we can easily extend some known results for axis-parallel rectangles and disks to any shape with constant description complexity. Details are omitted from this extended abstract.

The following lower bounds show that our results are not very far from best possible, at least in some relevant cases.

Theorem 3. *Consider a complete* r*-uniform hypergraph on* n *nodes, with* $r < n/2$*. Then any conflict-free multi-coloring needs to use* $\Omega(\log n)$ *colors in total. Furthermore, any such coloring using polylog*(n) *colors has to use* $\Omega(\frac{\log n}{\log\log n})$ *colors on some node.*

For intuition we give a proof for $r = 2$; the complete proof will be given in the full version of the paper. We can represent the multi-coloring of each node as a 0-1 vector, where the 1's indicate the colors assigned to that node. If two nodes u and v are labelled with the same vector, then the edge uv is not conflict-free. Suppose we use h_{tot} colors in total, and at most h_{max} colors per node. Then the number of 0-1 vectors is $O(\min\{h_{tot}^{h_{max}}, 2^{h_{tot}}\})$. As a consequence, we need $h_{tot} = \Omega(\log n)$ to have n distinct vectors. Similarly, if $h_{tot} = polylog(n)$, we

Constraint	Previous Results	Our Results
$\forall E \in \mathcal{E}:\ 2t-1 \leq \lvert E\rvert$	$O(t\Gamma^{1/t}\log\Gamma)$ [17]	$O(t\Gamma^{1/t}\log n)$
$\forall E \in \mathcal{E}:\ 2t-1 \leq \lvert E\rvert \in O(\log\Gamma)$		$O(t\Gamma^{1/t})$
$\forall E \in \mathcal{E}:\ \lvert E\rvert = r$	$\Omega(\frac{rm^{2/(r+2)}}{\log m})$, $O(rm^{\frac{2}{r+2}})$ [14]	
$\forall E \in \mathcal{E}:\ 2t-1 \leq \lvert E\rvert \in O(t)$		$O(tm^{1/(t+1)})$

Table 1. Bounds on the conflict-free chromatic number of hypergraphs on n vertices, m edges and maximum hyperedge degree Γ.

need $h_{max} = \Omega(\log n/\log\log n)$ to have n distinct vectors.

For comparison, consider a hypergraph on $m = \binom{n}{r}$ hyperedges of uniform size $r \leq n/e$. Observe that $\left(\frac{n}{r}\right)^r \leq m \leq \left(\frac{n \cdot e}{r}\right)^r$, hence $r \leq r(\ln n - \ln r) \leq \ln m$. The algorithm from Theorem 2 uses only one bucket and by this refined analysis assigns $O(\log m) = O(r\log n)$ colors in total and $O(\log n)$ colors per node. Hence for small r our algorithm is not far from best possible.

Related work. An anonymous reviewer pointed us to the independent work of Bar-Yehuda, Goldreich and Itai [5] on the radio broadcast problem, which considers assigning time-slots to transmitters (rather than frequencies) in a periodic schedule. One can reinterpret time slots of their framework as colors in a multi-coloring, hence obtaining for our setting a randomized multi-coloring algorithm using $O(\log m \cdot \log\Delta)$ colors in total, where Δ is the maximum hyperedge size. One can infer that – using Lovász's local lemma – this can be improved to $O(\log\Gamma \cdot \log\Delta)$ colors in total and $O(\log\Gamma)$ colors per node. Considering Theorem 2, the two results differ (i) for the total number of colors if $\log\Gamma \ll \Delta \ll n$ and (ii) for the number of colors per transmitter if $\Gamma \gg n$.

Other multi-coloring models for frequency assignment problems have been considered for standard graphs (for a survey, see e.g. [1]). We already mentioned a few results about the (single-color) conflict-free hypergraph coloring problem. Pach and Tardos [17] raised the question whether it is possible to get a coloring with $\tilde{O}(tm^{1/t})$ colors even when hyperedges have size at least t (rather than $2t-1$). Kostochka et al. [14] have answered this in the negative, proving that there exists a r-uniform hypergraph H with m hyperedges (and *even* $r \leq \ln m$) such that $\chi_{\text{cf}}(H) \in \Omega(rm^{2/(r+2)}/\log m)$. They have also shown that for all r-uniform hypergraphs H, $\chi_{\text{cf}}(H) \in O(rm^{2/(r+2)})$. The known bounds on the conflict-free chromatic number are summarized in Table 1.

For obvious reasons related to the mentioned applications, it makes sense to consider the conflict-free coloring problem under geometric restrictions on the structure of the hypergraph. In particular, one can consider transmitters and receivers as points in an Euclidean space. Here each transmitter v reaches all the receivers E in a given geometric region around v (e.g., a circle or sphere centered at v). Indeed, the problem was first defined having such a geometric model in mind by Even et al. [11], and has further been studied by Smorodinsky [2,12,20], Pach [18] and Cheilaris [4,8] for various geometric hypergraphs, such as those induced by disks, rectangles or intervals. The problem has been studied in terms of approximation [13] and online algorithms [3,9]. For a comprehensive survey on this problem, see also [21].

Another recently studied conflict-free coloring problem is a chromatic variant of the art gallery problem, in which the hypergraph is induced by visibility regions of transmitters in a given polygon [6,7]. In this problem, the structure of the hypergraph depends on the placement of the transmitters, which is not prescribed, but rather can be chosen together with the coloring.

2 An improved conflict-free coloring algorithm

In this section we describe the conflict free-coloring algorithm from Theorem 1. Recall that n denotes the number of nodes, m the number of hyperedges, and $\Gamma \leq m-1$ the maximum number of hyperedges that intersect any given hyperedge E. Furthermore, the minimum hyperedge size is $2t-1$.

Our proof proceeds as follows. We start by describing a simple randomized algorithm that assigns colors independently to each node. We remark that our algorithm framework contains the algorithm by Pach and Tardos [17] as a special case, however our choice of parameters is substantially different. Let B_E denote the (bad) event that a given hyperedge E is *not* conflict-free. We will show that $\Pr[B_E] \leq \frac{1}{e\Gamma}$. Since the event B_E depends on at most Γ other bad events B_F (namely those corresponding to hyperedges F intersecting E), we conclude from Lovasz Local Lemma (LLL) that our algorithm succeeds with positive probability[4]. We can therefore use the polynomial-time Las Vegas algorithm MT by Moser and Tardos [16] to construct the desired conflict-free coloring in expected polynomial time.

A geometric color classes algorithm. Consider the following Geometric Color Classes algorithm GCC. GCC has two parameters, a probability p and a positive integer h (to be fixed later). Let $C_1, C_2, \ldots, C_{\lceil \ln n \rceil}$ be pairwise disjoint subsets of h colors each (*color classes*). Our algorithm works in two steps:

Step 1 We independently assign a color class C_i to each node as follows. At each round $i = 1, \ldots, \lceil \ln n \rceil - 1$ we consider every node v that has not been assigned any color class yet, and we independently assign color class C_i to v with probability p. At the end of the process we assign the final color class $C_{\lceil \ln n \rceil}$ to the remaining unassigned nodes.

Step 2 For each node v we choose independently and uniformly at random one of the h colors from its assigned color class.

We next set the parameters p and h, and discuss some consequences of our choices that will turn out to be useful in the analysis of the algorithm. We choose $p = 1 - \frac{1}{e}$ and $h = 48t(2e\Gamma)^{1/t}$.

Remark 1. The assignment of nodes to color classes follows a *truncated* geometric distribution. In more detail, the probability that a node v is assigned to the

[4] Here we consider the refined version of LLL given by Shearer [19], however this is not crucial for us modulo updating a few constants.

color class C_i is $p(1-p)^{i-1}$ for $i < \ln n$, and $(1-p)^{\lceil \ln n \rceil - 1}$ for $i = \lceil \ln n \rceil$. In particular, the number X of nodes assigned to $C_{\lceil \ln n \rceil}$ in expectation is $\mathbb{E}[X] = n \cdot (1/e)^{\lceil \ln n \rceil - 1} \leq e$.

Remark 2. Consider h as a function $h(t)$ of t. Note that we can restrict the domain of $h(t)$ to $t \leq \ln \Gamma$: Since $t\Gamma^{1/t}$ achieves its minimum for $t = \ln \Gamma$, and since any hyperedge with more than $2\ln \Gamma - 1$ nodes has size at least $2 \ln \Gamma - 1$, it is enough to show the claimed bound of $O(t\Gamma^{1/t} \log n)$ colors for $t \leq \ln \Gamma$. Over this domain $h(t)$ is monotonically decreasing because for $t \leq \ln \Gamma$ we have $h'(t) = 48(2e\Gamma)^{1/t} \cdot (t - \ln \Gamma - \ln 2 - 1)/t < 0$. We will make use of the fact that $t \leq t' \leq \ln \Gamma$ implies $h(t) \geq h(t')$.

Existence of a good coloring. In this section we will show that GCC, using $O(t\Gamma^{1/t} \log n)$ colors, finds a conflict-free coloring with positive probability. To this end, we prove that LLL is applicable to the randomized coloring given by GCC. Recall that in LLL one considers a set of (bad) events, each one happening with probability at most p, where each event is independent of all the others except for at most d of them. Then, if $epd \leq 1$, there is a nonzero probability that none of the events occur [19]. In our case the bad events are $\{B_E\}_{E \in \mathcal{E}}$, where B_E denotes the event that the hyperedge E is not conflict-free. Since E intersects at most Γ other hyperedges and colors are assigned independently, B_E is independent from all but Γ other events B_F (i.e., $d = \Gamma$). By LLL it is sufficient to show that $\Pr[B_E] \leq \frac{1}{e\Gamma}$.

Consider any given hyperedge E of size s. Recall that by assumption $s \geq 2t - 1$. We distinguish between the case that E is *small*, i.e., $s \leq 24 \ln \Gamma$, and the case that E is *large*, i.e., $s > 24 \ln \Gamma$.

Case of small hyperedges. Let E be a (small) hyperedge with $s = |E| \leq 24 \ln \Gamma$ nodes. We can upper bound $\Pr[B_E]$ by means of the following coupling argument. Suppose that a node v is assigned the j-th color of the i-th color class C_i. Then we reassign to v the j-th color of C_1. Clearly this reassignment can only decrease the probability that each given hyperedge E is conflict-free. Therefore it is sufficient to upper bound $\Pr[B_E]$ under the assumption that all nodes in E are assigned to the same color class. Lemma 1 follows Kostochka et al. [14]:

Lemma 1. *Let E be a hyperedge of size s and let its nodes be colored uniformly at random with h colors. Then the probability that there is no unique color in E is $Pr[B_E] \leq \left(\frac{2s}{h}\right)^{\lceil s/2 \rceil}$.*

Lemma 2. *For any small hyperedge E, $Pr[B_E] \leq \frac{1}{2e\Gamma}$.*

Proof. By definition one has $s = |E|$ with $2t - 1 \leq s \leq 24 \ln \Gamma$. First note that by Lemma 1 (and the mentioned coupling argument) we have

$$\Pr[B_E] \leq \left(\frac{2s}{h}\right)^{\lceil s/2 \rceil} = \left(\frac{2s}{48t(2e\Gamma)^{1/t}}\right)^{\lceil s/2 \rceil}. \qquad (1)$$

Now we distinguish between two cases, depending on whether the size s of E is relatively close to t or not, i.e. whether $s \leq 24t$ or $t < \frac{s}{24}$.
Case 1: $s \leq 24t$. One has $2s \leq 48t$ and $\lceil s/2 \rceil \geq t$. Hence the right-hand side of (1) is bounded by

$$\left(\frac{2s}{48t(2e\Gamma)^{1/t}}\right)^{\lceil s/2 \rceil} \leq \left(\frac{1}{(2e\Gamma)^{1/t}}\right)^{\lceil s/2 \rceil} \leq \left(\frac{1}{(2e\Gamma)^{1/t}}\right)^{t} = \frac{1}{2e\Gamma}.$$

Case 2: $t < \frac{s}{24}$. Recall that by assumption we have $s \leq 24 \ln \Gamma \Leftrightarrow s = 24d \ln \Gamma$ for some $d \leq 1$. Thus we can write $t < d \ln \Gamma \leq \ln \Gamma$. By Remark 2, $h(t)$ is monotonically decreasing for $t \leq \ln \Gamma$ and thus $h = h(t) > h(d \ln \Gamma)$. Furthermore, $\lceil s/2 \rceil \geq 12d \ln \Gamma$. Putting everything together, the right-hand side of (1) is bounded by

$$\left(\frac{2s}{48t(2e\Gamma)^{1/t}}\right)^{\lceil s/2 \rceil} \leq \left(\frac{48d \ln \Gamma}{48d \ln \Gamma \cdot (2e\Gamma)^{1/(d \ln \Gamma)}}\right)^{12d \ln \Gamma} = \frac{1}{(2e\Gamma)^{12}} \leq \frac{1}{2e\Gamma}. \quad \square$$

Case of large hyperedges. Let E be a (large) hyperedge with $s = |E| > 24 \ln \Gamma$ nodes. To upper bound $\Pr[B_E]$, we show that with large enough probability E contains a subset of nodes $E' \subsetneq E$ of size $2t - 1 \leq |E'| \leq 24 \ln \Gamma$, whose nodes are assigned colors not appearing in $E \setminus E'$. This allows us to reuse the analysis for the case of small hyperedges (a coloring that is conflict-free on E' will also be conflict-free on E).

In more detail, consider the color classes assigned to the nodes of E by GCC, and denote them by $C'_1, C'_2, \ldots, C'_k$ (in the order given by the algorithm.) Recall that these color classes are pairwise disjoint. Denote by E_j the subset of nodes with color class C'_j. We show that there is either a *single* subset E_j of small size or that there is a *union* $E_{>k-l} := \bigcup_{j=0}^{l-1} E_{k-j}$ of the last l subsets, for some l, that has a small size. Depending on which case applies, we will use either $E' = E_j$ or $E' = E_{>k-l}$. Let us formally define these two events, for which we use mnemonic identifiers S (single color class) and U (union of color classes):

- $S =$ "There is an index j, $1 \leq j \leq l$, such that $2t - 1 \leq |E_j| \leq 24 \ln \Gamma$."
- $U =$ "There is an index l, $0 \leq l < k$, such that $2t - 1 \leq |E_{>k-l}| \leq 24 \ln \Gamma$."

Lemma 3. *For the events S and U as defined, we have $Pr[\neg S \wedge \neg U] \leq \frac{1}{2e} \frac{1}{\Gamma}$.*

Proof. Assume neither S nor U occurs. Recall that $E_{>k-l} = \bigcup_{j=0}^{l-1} E_{k-j}$. Since by assumption U does not occur, there exists a unique l with $0 \leq l < k$ such that $E' := E_{>k-l}$ has a comparatively very small size $|E'| < 2t - 1$, while $E'' = E_{>k-l-1} = E' \cup E_{k-l}$ already has a large size $|E''| := a \ln \Gamma$ for some $a > 24$.

Since S does not occur, we must have $|E_{k-l}| > 24 \ln \Gamma$. Recall that by Remark 2 we can restrict ourselves to the case $t \leq \ln \Gamma$ and hence $|E'| < 2t - 1 < 2 \ln \Gamma$. Thus $|E_{k-l}| = |E'' \setminus E'| > (a - 2) \ln \Gamma$. We can conclude that

$$\Pr[\neg S \wedge \neg U] \leq \Pr[|E_{k-l}| \geq \ln \Gamma \cdot \max\{24, a - 2\}].$$

We distinguish two cases, depending on whether C'_{k-l} is the last color class $C_{\lceil \ln n \rceil}$ ever assigned by the algorithm or not.
First let us implicitly condition on the event $C'_{k-l} = C_{\lceil \ln n \rceil}$. By Remark 1, the number of all nodes X with assigned color class $C_{\lceil \ln n \rceil}$ is a sum of independent Bernoulli random variables with expectation $\mathbb{E}[X] \leq e$. Thus we can apply Chernoff bounds (see, e.g., [15]) to get

$$\Pr\left[|E_{k-l}| \geq 24 \ln \Gamma\right] \leq \Pr\left[X \geq 24 \ln \Gamma\right] \leq 2^{-24 \ln \Gamma} \leq \tfrac{1}{2e\Gamma}.$$

In the above inequalities we used the fact that $X \geq |E_{k-l}|$ and that Γ is sufficiently large.
Next we implicitly condition on the event $C'_{k-l} \neq C_{\lceil \ln n \rceil}$. Each of the nodes in E'' is chosen into E_{k-l} independently with probability $p = 1 - \frac{1}{e}$. Thus $|E_{k-l}|$ is a sum of independent Bernoulli random variables with expectation $\mathbb{E}\left[|E_{k-l}|\right] = p \cdot |E''| = \frac{(e-1)a}{e} \ln \Gamma$. Hence we can apply Chernoff bounds to get

$$\begin{aligned}\Pr\left[|E_{k-l}| \geq (a-2)\ln \Gamma\right] &= \Pr\left[|E_{k-l}| \geq \left(1 + \tfrac{a-2e}{(e-1)a}\right) \tfrac{(e-1)a}{e} \ln \Gamma\right] \\ &\leq e^{-\frac{(e-1)a}{e} \ln \Gamma \cdot \left(\frac{a-2e}{(e-1)a}\right)^2 \cdot \frac{1}{3}} = e^{-\ln \Gamma \frac{(a-2e)^2}{3e(e-1)a}} \leq e^{-\ln \Gamma \frac{(24-2e)^2}{3e(e-1)\cdot 24}} \leq \tfrac{1}{2e\Gamma}.\end{aligned}$$

In the above inequalities we used the fact that $\frac{(a-2e)^2}{3e(e-1)a}$ is monotonically increasing in a for $a \geq 24$ and that Γ is sufficiently large. □

Lemma 4. *For any large hyperedge E, $Pr[B_E] \leq \frac{1}{e\Gamma}$.*

Proof. Using previous notation we get$\Pr[B_E] \leq \Pr\left[\neg S \wedge \neg U\right] + \Pr[B_E \mid S \vee U]$. By Lemma 3, $\Pr\left[\neg S \wedge \neg U\right] \leq \frac{1}{2e\Gamma}$. Given the event $S \vee U$, let E' be a corresponding subset of nodes of E. We recall that by definition $2t-1 \leq |E'| \leq 24 \ln \Gamma$, and no color used for nodes in E' is also used for nodes in $E \setminus E'$. By the same analysis as in Lemma 2, the event $B_{E'}$ that there is no unique color among nodes E' has probability at most $\Pr[B_{E'}] \leq \frac{1}{2e\Gamma}$. Furthermore, when there is a unique color in E', then there is a unique color also in E, hence $\Pr[\neg B_{E'}] \leq \Pr[\neg B_E \mid S \vee U]$. Consequently, $\Pr[B_E \mid S \vee U] \leq \Pr[B_{E'}] \leq \frac{1}{2e\Gamma}$ and $\Pr[B_E] \leq \frac{2}{2e\Gamma} \leq \frac{1}{e\Gamma}$. □

By Lemmas 2 and 4, and applying LLL, we obtain the following result.

Lemma 5. *Algorithm GCC computes a coloring using at most $O(t\Gamma^{1/t} \log n)$ colors. This coloring is conflict-free with positive probability.*

Computing a conflict-free coloring. The probability that GCC computes a conflict-free coloring might be very small. For this reason, we rather use the Las Vegas algorithm ML in [16], adapted to our setting. In more detail, we start by coloring nodes according to GCC. Then, while there is some hyperedge E that is not conflict-free, we recolor the nodes in E using GCC (by resampling from the same product probability space as before, restricted to E). By the analysis in [16], this new algorithm GCC$^+$ computes a conflict-free coloring in expected time polynomial in n and m, provided that there exists a conflict-free coloring among the ones that can be returned by GCC. The latter condition holds by Lemma 5. The main part of Theorem 1 immediately follows.

3 A refined multi-coloring algorithm

In this section we prove Theorem 2. This is achieved in two steps. First we describe and analyze a refined conflict-free coloring algorithm for the case that hyperedges have sizes in a small range. Then we present a non-trivial multi-coloring algorithm that exploits the new (and also the previous) coloring algorithm as a subroutine.

Hyperedges with upper bounded size. Suppose that every hyperedge has size at most $k \cdot \ln \Gamma$, with $k \in o(\log n)$. Then we can modify the parameters in GCC to achieve an improved upper bound of $O(t\Gamma^{1/t}k)$ colors. In particular, for constant k the number of colors needed in the single-color case is $O(t\Gamma^{1/t})$ only.

In more detail, we set $p = 1$ and $h = 2kt(3e\Gamma)^{1/t}$ (i.e., we use only one color class of a size depending linearly on k). We denote this algorithm by $1C$. In order to prove that we have a conflict-free coloring with sufficiently large probability, it is sufficient to slightly adapt the proof of Lemma 2. In particular, in the case distinction we distinguish between hypedges of size $s \leq k\,t$ and hyperedges of size $k\,t < s \leq k \ln \Gamma$. The rest of the analysis is the same. We can also similarly modify the algorithm to make it run in expected polynomial time using the approach in [16]: let $1C^+$ denote this variant. Hence we obtain the following lemma, which shows the second part of Theorem 1.

Lemma 6. *There is a polynomial-time Las Vegas algorithm for conflict-free coloring using $O(t\Gamma^{1/t}k)$ colors, assuming that hyperedges have size at least $2t - 1$ and at most $k \ln \Gamma$.*

For $\Gamma \in \Theta(m)$, Lemma 6 yields a conflict-free coloring using $O(tm^{1/t}k)$ colors. Suppose that, additionally, all the hyperedges have size at most $k \cdot t$, with k a constant. Then we can improve the upper bound to $O(tm^{1/(t+1)})$ by using a deterministic preprocessing of the hypergraph similarly to [14]. Though this is not needed for our multi-coloring algorithm, we briefly present this result since it might be of some interest. Indeed, this generalizes the bound in [14] from uniform hypergraphs to hypergraphs with a constant factor gap between the minimum and maximum hyperedge size. The proof is omitted from this extended abstract.

Lemma 7. *There is a polynomial-time Las Vegas algorithm for conflict-free coloring using $O(tm^{1/(t+1)})$ colors, assuming that hyperedges have size at least $2t-1$ and at most $O(t)$.*

A bucketing multi-coloring algorithm. We consider the following refined conflict-free multi-coloring algorithm. Let $q = \lceil \log_2(\ln \Gamma) \rceil$. Note that we have $q \in O(\log \log \Gamma)$. We partition the hyperedges into subsets $\mathcal{E}_0, \ldots, \mathcal{E}_q$, where for $i < q$ the subset $\mathcal{E}_i$ contains all hyperedges of size in $\left[2^i, 2^{i+1}\right)$, while the last subset $\mathcal{E}_q$ contains all the remaining hyperedges (which have size $\geq 2^q \geq \ln \Gamma$). Then there is a sequence of rounds $i = 0, \ldots, q$. In round i the algorithm considers the sub-hypergraph induced by $\mathcal{E}_i$ (containing only the nodes V_i spanned by $\mathcal{E}_i$). If $i = q$, the algorithm colors nodes in V_i using algorithm GCC^+ from Theorem 1.

Otherwise, the algorithm splits each node in V_i into $\lceil \ln \Gamma_i / 2^i \rceil$ copies, and colors such copies using algorithm 1C$^+$ from Lemma 6. Here $\Gamma_i \leq |\mathcal{E}_i| - 1$ denotes the value of Γ in the considered sub-hypergraph. The algorithm uses a novel set of colors in each round. The final assignment of colors to a node v is simply the union of the colors assigned to any copy of v in any round. We next analyze this refined algorithm, proving Theorem 2.

Proof. (of Theorem 2) Consider the above Las Vegas algorithm. Its expected running is trivially polynomial. In each round i the algorithm obtains a conflict-free multi-coloring of hyperedges $\mathcal{E}_i$. Since each round uses different colors, the overall multi-coloring is conflict-free, too.

It remains to bound the total number of colors and the maximum number of colors per node. By Theorem 1, in round $i = q$ the algorithm uses one color per node and $O(\log \Gamma \log n)$ colors in total. In round $i < q$, the algorithm considers an instance with $m_i = |\mathcal{E}_i|$ hyperedges of size $\Theta(\log \Gamma_i)$ each (after node duplication). Applying Lemma 6 with $t = \Theta(\log \Gamma_i)$ and $k = O(1)$, the total number of colors used is $O(t\Gamma_i^{1/t}k) = O(\log \Gamma_i) \subseteq O(\log \Gamma)$. Furthermore, the number of extra colors used for each node is at most $O(\log(\Gamma_i)/2^i) = O(\log(m_i)/2^i) = O(\log(n^{2^{i+1}})/2^i) = O(\log n)$. Here we used the fact that hyperedges in $\mathcal{E}_i$ have size at most 2^{i+1}, hence there can be at most $O(n^{2^{i+1}})$ such hyperedges. Altogether, in rounds $i = 0, \ldots, q-1$ the algorithm uses $O(\log \Gamma \log \log \Gamma) \subseteq O(\log \Gamma \log n)$ colors in total and $O(\log n \log \log \Gamma) \subseteq O(\log^2 n)$ colors per node. The claim follows. □

Remark 3. In case hyperedges have size at most $O(\log \Gamma)$, the above algorithm (with a slight adaptation of q) uses only $O(\log \Gamma \cdot \log \log \Gamma)$ colors in total.

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