MLE in presence of equality and inequality non-linear constraints for the ballistic target problem

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Abstract. **This paper focuses on the estimation of the impact point of a ballistic target by means of a batch processing approach which can be applied more specifically when the number of radar measurements is poor. In particular, the Maximum Likelihood (ML) estimator is proposed for the estimation of the ballistic target trajectory approximated to a pure parabolic curve; also the Cramer-Rao Lower Bound (CRLB), which gives the minimum theoretically achievable variance of the estimate, is calculated. The estimator accuracy is improved by the application of equality and inequality constraints on the trajectory characteristics such as maximum range, maximum height and trajectory plane. In the case of application of equality constraints also the CRLB can be computed and the constrained estimator improves strongly its accuracy. In real operational scenarios, a smoothed information can be expressed by means of inequality constraints. The performance of the corresponding estimator has been analyzed by means of Monte Carlo simulation, because of the unavailability of computation of the CRLB for this case.**

***Key words*: Maximum Likelihood, CRLB, ballistic target.**

I. Introduction

The problem of tracking ballistic targets is of great interest for several applications, in particular the knowledge of the whole trajectory is important to estimate both the launch and impact points. The paper focuses on the estimation of the impact point of a ballistic target. The real dynamic model of a ballistic target is very complicated because of its high non-linearity and because of the rapid change of model (e.g., the change between the burn out phase and the ballistic one) [1-2]. Moreover the target trajectory is sensitive to a large number of parameters depending on the target characteristics which are, in general, unknown. For this reason a specific tracking architecture is needed [3]. The estimation of a ballistic target trajectory by a ground based radar may be hard because the number of available radar measurements is small because of long radar scan periods (e.g., 10 s) and because of the reduced time of visibility of the target (i.e. the time the ballistic target remains in the radar coverage is short).

When the number of available radar measurements is no more than, say, 10, a batch approach may give better results than a recursive approach. In this paper, a Maximum Likelihood (ML) approach has been considered to estimate the parameters of the ballistic trajectory. The ML estimator has been applied to an approximated ballistic model, i.e. a pure parabolic trajectory. This approximation allows to evaluate theoretically the Cramer-Rao Lower Bound (CRLB) of the ML estimator. The CRLB, computed by the inversion of the Fisher Information Matrix (FIM), gives the minimum theoretically achievable variance of the estimate [4]. The ML estimation of a parabolic trajectory and the CRLB computation have already been approached in [5] and [6]. In [5] the improvement of estimator accuracy was achieved by the fusion of the measurements of few radars.

The need to improve the estimator accuracy leads to the exploitation of all possible information which is a-priori available about the target trajectory; for instance some information about the target maximum height and maximum range or the target trajectory plane. This aim can be achieved by the application of constraints on the parameters to be estimated [7-8]. In [9] the application of hard and soft constraints has been analyzed only for a simple line fitting problem. In this paper, the ML estimation accuracy of a parabolic trajectory is improved by the application of equality and inequality (hard) constraints on the trajectory characteristics such as maximum range, maximum height and trajectory plane. In the case of application of equality constraints also the CRLB can be computed [7] and it is shown that the constrained estimator improves strongly its accuracy. In real operational scenarios, the perfect knowledge of the height or the range of a trajectory is not available; on the other hand their knowledge may be available unless a given uncertainty. For example a ballistic target trajectory cannot overcome a certain height corresponding to a maximum range. This information can be expressed by means of inequality (hard) constraints. The performance of the hard-constrained estimator has been analyzed by means of Monte Carlo simulation, because of the unavailability of computation of the CRLB for this case.

The paper is organized as follows: section II describes the model and the reference geometry of the target. Section III reports the unconstrained problem of ML estimation, whose performance is taken as reference for the analysis of constrained estimators. The equality constrained estimator is described in Section IV with the pertinent computation of the CRLB; while the hard-constrained estimator is described in Section V. Finally Section VI reports the conclusion of this analysis.

II. Model of the data: reference geometry

Consider a simplified ballistic trajectory of a target on the x, y, z Cartesian reference frame centred on the radar position R: see Figure 1. The ballistic target has been modelled according to the following hypotheses:

- instantaneous thrust,
- flat Earth surface,
- gravity acceleration constant with target height,
- x, y, z is an inertial coordinate reference frame.

The trajectory belongs to a plane which determines an angle ϕ with the y axis. The trajectory starting point is P_0 (x_0, y_0, z_0). We have assumed, without loss of generality, that the initial height of the launch point is characterized by zero height (z_0=0). The trajectory point corresponding to the maximum height is P_M (x_M, y_M, z_M); the ending point of the trajectory is P_I (x_I, y_I, z_I) which, we suppose again, is characterized by zero height (z_I=0). The parametric expression of the trajectory as a function of time t is given by:

$$\begin{cases} x(t) = x_0 + v_x t \\ y(t) = y_0 + v_y t \\ z(t) = z_0 + v_z t - \frac{1}{2} g t^2 \end{cases} \quad (1)$$

where v_x, v_y, v_z are the absolute speed components with respect to x, y, z and g is the gravity acceleration absolute value.

The value of the maximum height is obtained by putting equal to zero the first derivative of $z(t)$:

$$\frac{dz(t)}{dt} = v_z - gt = 0 \quad (2)$$

which gives the time $t_M = v_z / g$ corresponding to the trajectory apogee; by replacing it in the parametric expression of $z(t)$ of Eq. (1), we obtain the trajectory height H:

$$H = z_0 + \frac{v_z^2}{2g} \quad (3)$$

The maximum range of the launch is characterized by a zero height, so it is given by solving the equation[1]:

$$z(t) = z_0 + v_y t - \frac{1}{2} g t^2 = 0 \quad (4)$$

whose solutions give the starting time t=0, if the starting height is zero, and the time t_I corresponding to the impact point. By replacing it in the parametric expression of the trajectory of Eq. (1), we obtain the coordinate of the impact point P_I. The maximum range G is given by the distance, computed on the x-y plane, between P_0 and P_I:

[1] In the case the height z_I corresponding to the maximum range is different from zero, eq. (4) becomes $z(t) = z_0 + v_y t - 1/2\, g t^2 = z_I$.

$$G = \sqrt{(x_0 - x_I)^2 + (y_0 - y_I)^2} = \sqrt{v_x^2 t_I^2 + v_y^2 t_I^2} = \frac{v_z + \sqrt{v_z^2 + 2 g z_0}}{g} \sqrt{v_x^2 + v_y^2} \quad (5)$$

When the starting point has zero height, the maximum range reduces to:

$$G\big|_{z_0=0} = \frac{2 v_z}{g} \sqrt{v_x^2 + v_y^2} \quad (6)$$

With reference to the geometry of Figure 1, the straight line given by the intersection of the trajectory plane and the x-y plane is:

$$\frac{y - y_0}{y_I - y_0} = \frac{x - x_0}{x_I - x_0} \quad (7)$$

The $\tan^{-1}$ of its angular coefficient represents the angle α that the straight line determines with the x axis; the trajectory plane is represented by the angle ϕ which is:

$$\phi = \alpha - 90° = arctg\left(\frac{y_I - y_0}{x_I - x_0}\right) - 90° = arctg\left(\frac{v_y}{v_x}\right) - 90° \quad (8)$$

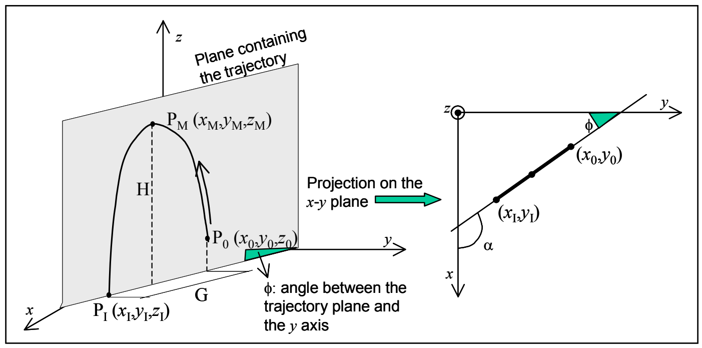

Figure 1 – Reference geometry

III. Unconstrained Estimation problem

Consider the general problem of ML estimation of a vector $\mathbf{x}$, given a set of N independent measurements:

$$\mathbf{z}_k = h(\mathbf{x}) + \mathbf{n}_k;\ k = 1,\ldots,N \quad (9)$$

where the noise samples $\mathbf{n}_k$ are white Gaussian zero-mean random variables, with covariance matrix $\mathbf{R}$. Then the estimation of the vector $\mathbf{x}$ is given by:

$$\hat{\mathbf{x}}_{unc} = arg\,min_{\mathbf{x}}\ \Lambda(\mathbf{x}) = arg\,min_{\mathbf{x}} \sum_{k=1}^{N} [\mathbf{z}_k - h(\mathbf{x})]^T \mathbf{R}^{-1} [\mathbf{z}_k - h(\mathbf{x})] \quad (10)$$

where $\Lambda(\mathbf{x})$ is the functional to be minimized. It can be shown that the estimator is unbiased and the unconstrained Cramer Rao Lower Bound (CRLB) can be computed by means of the Fisher Information Matrix (FIM) $\mathbf{J}$:

$$\begin{aligned} & E\left\{(\hat{\mathbf{x}} - \mathbf{x})(\hat{\mathbf{x}} - \mathbf{x})^T\right\} \geq \mathbf{J}^{-1} \\ & \mathbf{J}(\bar{\mathbf{x}}) = E\left\{(\nabla_{\mathbf{x}} \Lambda(\mathbf{x}))(\nabla_{\mathbf{x}} \Lambda(\mathbf{x}))^T\right\}_{\mathbf{x}=\bar{\mathbf{x}}} \\ & CRLB^{uncon} = \mathbf{J}^{-1}(\bar{\mathbf{x}}) \end{aligned} \quad (11)$$

where the mean value is computed with respect to probability density function of the measurements conditioned to the vector of parameters to be estimated $p(\mathbf{z}_1, \cdots, \mathbf{z}_k / \mathbf{x})$; the derivatives of eq. (11) are computed in correspondence of the true value $\bar{\mathbf{x}}$ of the vector to be estimated.

In the specific case of the ballistic problem, the target trajectory can be identified by the knowledge of its starting point P_0 (x_0, y_0, z_0) and of its initial speed vector whose components are v_x, v_y, v_z. If the time of launch t_0 is assumed to be known a priori, (in this case $t_0 = 0$) the whole trajectory can be estimated throughout the estimation of the following vector:

$$\mathbf{x} = \begin{bmatrix} x_0 & y_0 & z_0 & v_x & v_y & v_z \end{bmatrix}^T \quad (12)$$

In case the launch time is unknown, it may be inserted in the vector of parameters to be estimated. Consider the availability of N independent radar measurements of range ρ, azimuth θ and elevation φ collected into the vector:

$$\mathbf{z}_k = \begin{bmatrix} \rho_k & \theta_k & \varphi_k \end{bmatrix}^T + \mathbf{n}_k, k = 1, \ldots, N \quad (13)$$

affected by the measurement noise which is Gaussian, with zero-mean and covariance matrix:

$$\mathbf{R} = \begin{bmatrix} \sigma_\rho^2 & 0 & 0 \\ 0 & \sigma_\theta^2 & 0 \\ 0 & 0 & \sigma_\varphi^2 \end{bmatrix} \quad (14)$$

σ_ρ^2, σ_θ^2 and σ_φ^2 are the variances of the radar measurement errors. The estimator and the pertinent CRLB computation for this specific case has been calculated in [6]; for sake of brevity the mathematical details are not reported here. The accuracy of the unconstrained estimator will be taken as reference to evaluate the results presented in the sequel.

IV. Exploitation of a-priori knowledge: use of equality constraints

In the ballistic problem the estimation of the whole trajectory is necessary for the prediction, on the base of the radar measurements, of the target impact point. To improve the accuracy of impact point estimation any available information need to be exploited. The a-priori information can be expressed in a mathematical formulation via constraints on the parameters to be estimated. The constraints can be applied directly on some of the parameters to be estimated: for instance on the launch point coordinates (x_0, y_0, z_0) if the launch has been detected by another sensor and then it has been exploited to cue the radar which can initialise the ballistic target track. In the hypothesis of flat Earth, the launch height may be put equal to zero (i.e. z_0=0). In the following we take into account the constraints only on the main characteristics of the parabolic trajectory: the maximum range G, the maximum height H and the trajectory plane angle ϕ. These constraints can be collected in the following vector which depends non linearly on the parameters to be estimated:

$$\mathbf{f}(\mathbf{x}) = \begin{bmatrix} f_G(\mathbf{x}) \\ f_H(\mathbf{x}) \\ f_\phi(\mathbf{x}) \end{bmatrix} = \begin{bmatrix} \dfrac{v_z + \sqrt{v_z^2 + 2gz_0}}{g} \sqrt{v_x^2 + v_y^2} \\ z_0 + \dfrac{v_z^2}{2g} \\ arctg\left(\dfrac{v_y}{v_x}\right) - \dfrac{\pi}{2} \end{bmatrix} \quad (15)$$

If the available information is very precise the constraints can be expressed by means of an equality and the unconstrained estimator of eq. (10) becomes:

$$\begin{cases} \hat{\mathbf{x}}_{eq} = \arg\min_{\mathbf{x}} \Lambda(\mathbf{x}) = \arg\min_{\mathbf{x}} \sum_{k=1}^{N} [\mathbf{z}_k - h(\mathbf{x})]^T \mathbf{R}^{-1} [\mathbf{z}_k - h(\mathbf{x})] \\ subject\ to: \\ \mathbf{f}(\mathbf{x}) = \mathbf{c} \end{cases} \quad (16)$$

where **c** is the vector collecting the numerical values assumed by the constraint vector. The constrained minimization problem can be solved resorting to the mathematical technique of the Lagrange multipliers; then the estimator of eq. (16) reduces to the following minimization problem:

$$\hat{\mathbf{x}}_{eq} = arg\ min_{\mathbf{x},\gamma}\ \Lambda(\mathbf{x}) = arg\ min_{\mathbf{x},\gamma} \left\{ \sum_{k=1}^{N} [\mathbf{z}_k - h(\mathbf{x})]^T \mathbf{R}^{-1} [\mathbf{z}_k - h(\mathbf{x})] - \sum_{i=1}^{L} \gamma_i [f_i(\mathbf{x}) - c_i] \right\} \quad (17)$$

where L is the number of components of the constraint vector **f**. The corresponding CRLB can be computed from the unconstrained $CRLB^{uncon}$ $\mathbf{J}^{-1}$ of eq. (11) by [7]:

$$CRLB^{eq\ constr} = \mathbf{J}^{-1} - \mathbf{J}^{-1}\mathbf{F}\left(\mathbf{F}^T\mathbf{J}^{-1}\mathbf{F}\right)^{-1}\mathbf{F}^T\mathbf{J}^{-1} \quad (18)$$

Notice that the constrained CRLB is obtained by the subtraction from the unconstrained CRLB of a term which depends on the Jacobian **F** of the constraint function $\mathbf{f}(\mathbf{x})$ computed in correspondence of the true value $\bar{\mathbf{x}}$ of the vector to be estimated (for mathematical details see the Appendix):

$$\mathbf{F} = \nabla_{\mathbf{x}} \mathbf{f}^T(\mathbf{x})\Big|_{\mathbf{x}=\bar{\mathbf{x}}} \quad (19)$$

This algorithm has been applied to the ballistic target whose trajectory is represented in Figure 2 with H=260 km, G=187 km and ϕ=240°. To estimate the vector **x**, N=5 radar measurements have been exploited with the following characteristics:

- scan period T=12 s,
- range measurement standard deviation σ_ρ=50 m,
- azimuth measurement standard deviation σ_θ=0.15°,
- elevation measurement standard deviation σ_φ=0.15°.

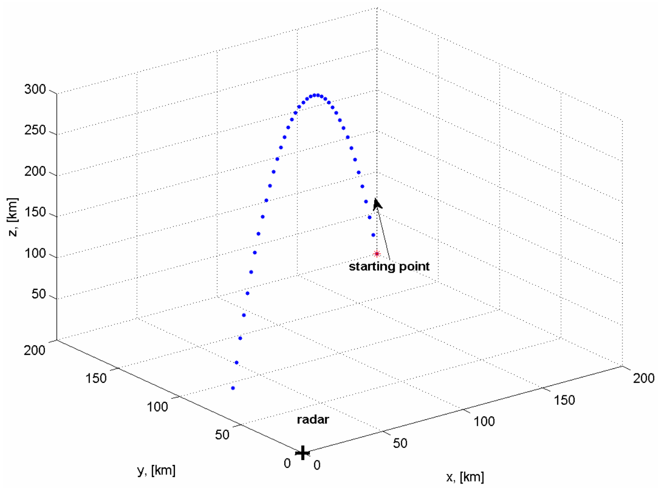

Figure 2 – Ballistic trajectory for a study case

	Constraints	Major semi-axis (99%)	
		theoretical	*simulated*
1	**Uncon.**	18.98	-
2	***H-G***	8.68	8.72
3	ϕ	5.08	5.09
4	***H-*** ϕ	3.00	3.02
5	***G-*** ϕ	1.25	1.25
6	***H-G-*** ϕ	0.75	0.77

Table 1: major semi-axis of ellipse containing 99 % of estimated impact points for unconstrained estimator and constrained estimator (CRLB and simulation)

	Constraints	$3\sigma_x$, [km]		$3\sigma_y$, [km]		$3\sigma_z$, [km]	
		theor.	*simul.*	*theor.*	*simul.*	*theor.*	*simul.*
1	**Uncon.**	13.89	-	13.80	-	13.83	-
2	***H-G***	4.20	4.41	7.59	7.53	2.27	2.56
3	ϕ	4.41	4.42	2.55	2.54	13.83	13.07
4	***H-*** ϕ	2.58	2.61	1.53	1.53	2.85	2.80
5	***G-*** ϕ	1.02	1.04	0.75	0.73	7.89	7.86
6	***H-G-*** ϕ	0.54	0.55	0.54	0.55	1.83	1.81

Table 2: accuracies of the prediction of impact point for unconstrained estimator and constrained estimator (CRLB and simulation)

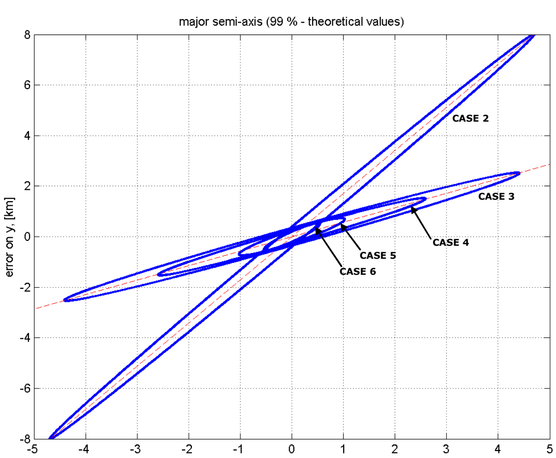

Figure 3 – Ellipses containing 99 % of estimated impact points via constrained estimators (simulation)

The CRLB (i.e. the covariance matrix of the unbiased estimator $\hat{\mathbf{x}}_{eq\,constr}$) has been computed in several cases. To analyze the resulting theoretical accuracies we have resorted to the prediction of the impact point on the base of the acquired N radar measurements, computing the accuracy of the prediction of the target impact point in terms of (i) the major semi-axis of the uncertainty ellipse at the 99% confidence level; (ii) three times the standard deviation of the (x, y, z) coordinates estimates. These parameters have been computed by the prediction of the target state ahead in time after the radar data acquisition until the target reaches the Earth surface, i.e. z_I=0; the corresponding covariance matrix characterizes the desired accuracies and the uncertainty ellipse, allowing the computation of its major semi-axis, reported in Table 1. The ellipses are depicted in Figure 3. In the unconstrained case, the major semi-axis of the uncertainty ellipse at 99 % is 18.98 km. By the comparison of the rows of Table 1, it is evident that the exploitation of equality constraints reduces strongly the major semi-axis. The theoretical values are compared, for the constrained cases, with the results of a Monte Carlo simulation performed with 500 independent trials. Table 2 reports the accuracies pertinent to impact point coordinate estimation. From both theoretical and simulation results, it can be noticed that the application of the equality constraint on the trajectory plane angle only (see line 3 on Table 2) improves strongly the accuracies of x and y impact point coordinates estimation, while doesn't give any contribution to the z coordinate estimation because the constraint on ϕ of eq. (15) doesn't depend on the parameters z_0 and v_z. The same happens for the application of the constraint on G (joint to ϕ, see line 5 of Table 2), because it doesn't depend on the parameter z_0. The estimation of the z coordinate can be improved by the application of the constraint on the maximum height H (see cases 2, 4 and 6).

V. Exploitation of a-priori knowledge: use of non linear hard constraints

In real operational scenarios, the perfect knowledge of the height or the range of a trajectory is not available; on the other hand their knowledge may be available unless a given uncertainty. This kind of a-priori information can be expressed by means of inequality constraints. For example, a ballistic target cannot overcome a certain height corresponding to a maximum range. This can be modeled by means of inequality constraints on H and G. Also the trajectory plane, identified by the angle ϕ, may be known with a certain degree of accuracy based on the location of the possible targets and launch point. In [9] hard constraints versus soft constraints are compared for a general line-fitting problem. In the ballistic case the application of hard constraints on the trajectory characteristics estimation is analyzed. The hard constrained estimator becomes:

$$\begin{cases} \hat{\mathbf{x}} = \arg\min_{\mathbf{x}} \Lambda(\mathbf{x}) = \arg\min_{\mathbf{x}} \sum_{k=1}^{N} [\mathbf{z}_k - h(\mathbf{x})]^T \mathbf{R}^{-1} [\mathbf{z}_k - h(\mathbf{x})] \\ subject\ \ to: \mathbf{f}_{\min} \le \mathbf{f}(\mathbf{x}) \le \mathbf{f}_{MAX} \end{cases} \quad (20)$$

with $\mathbf{f}_{min} = [G_{min} \quad H_{min} \quad \phi_{min}]^T$ and $\mathbf{f}_{MAX} = [G_{MAX} \quad H_{MAX} \quad \phi_{MAX}]^T$. The CRLB of the estimator of eq. (20) cannot be computed in a closed form and, thus, the estimator performance can be analyzed only by means of a Monte Carlo simulation. It can be argued that the benefits introduced by the hard constraints are significant only if the constraint bounds of eq. (20) are small if compared with the measurement uncertainty.

A way to quantify the tightness" of the constraints bounds w.r.t. the measurements accuracy is to quantify the distribution of the estimates of the maximum height $\hat{H}$, maximum range $\hat{G}$ and flight-plane $\hat{\phi}$, i.e.:

$$\begin{cases} \hat{G} = \dfrac{\hat{v}_z + \sqrt{\hat{v}_z^2 + 2g\hat{z}_0}}{g} \sqrt{\hat{v}_x^2 + \hat{v}_y^2} \\ \hat{H} = \hat{z}_0 + \dfrac{\hat{v}_z^2}{2g} \\ \hat{\phi} = arctg\left(\dfrac{\hat{v}_y}{\hat{v}_x}\right) - \dfrac{\pi}{2} \end{cases} \tag{21}$$

where $\hat{\mathbf{x}} = [\hat{x}_0 \quad \hat{y}_0 \quad \hat{z}_0 \quad \hat{v}_x \quad \hat{v}_y \quad \hat{v}_z]^T$ results by the solution of the unconstrained optimization problem. Figures 4-6 show the distributions of the estimates of $\hat{H}$, $\hat{G}$ and $\hat{\phi}$ relatively to the same study case described in section IV; the corresponding standard deviations are reported in Table 3. An alternative to extrapolate the trajectory plane angle may be the use of two radar position measurements expressed in Cartesian coordinates $[x_1 \quad y_1 \quad z_1]^T$ and $[x_2 \quad y_2 \quad z_2]^T$, affected by the measurement noise with zero mean and variances: σ_x^2, σ_y^2, σ_z^2, σ_{xy}, σ_{xz} and σ_{yz}. By two position measurements, a speed estimation is obtained by:

$$v_x = \frac{x_2 - x_1}{\Delta T}, v_y = \frac{y_2 - y_1}{\Delta T} \tag{22}$$

where ΔT is time interval between the two measurements (i.e., a multiple of radar scan period). The variances of the speed components estimation are given by:

$$\sigma_{Vx}^2 = \frac{2\sigma_x^2}{\Delta T^2}, \sigma_{Vy}^2 = \frac{2\sigma_y^2}{\Delta T^2}, \sigma_{VxVy} = \frac{2\sigma_{xy}}{\Delta T^2} \tag{23}$$

The accuracy of the estimation of the trajectory plane angle depends on the accuracies in eq. (23). The error $\Delta\phi$ on the angle estimation depends on the error on the speed components Δv_x and Δv_y:

$$\Delta\phi = \frac{\partial\phi}{\partial v_x}\Delta v_x + \frac{\partial\phi}{\partial v_y}\Delta v_y = \frac{-v_y}{v_x^2 + v_y^2}\Delta v_x + \frac{v_x}{v_x^2 + v_y^2}\Delta v_y \tag{24}$$

where ϕ is given by eq. (8). The accuracy becomes:

$$\sigma_\phi^2 = E\{\Delta\phi^2\} = \frac{1}{v_x^2 + v_y^2}\left(v_y^2\sigma_{Vx}^2 + v_x^2\sigma_{Vy}^2 - 2v_y v_x \sigma_{VxVy}\right) \tag{25}$$

The values computed by eq. (25) are very close to the standard deviation reported in Table 3 and represented in Figure 6.

Inequality constraints on H, G and ϕ are thus not significant if their size is comparable or larger than to the standard deviations reported in Table 3 which are directly derived from the unconstrained estimator. Therefore, we argue that the application of such constraints cannot provide any improvement because of the absence of any relevant a-priori information. Since the accuracy of the unconstrained estimator increases with the number of measurements available for estimation, the improvement deriving by exploiting inequality constraints is meaningful when few measurements are available.

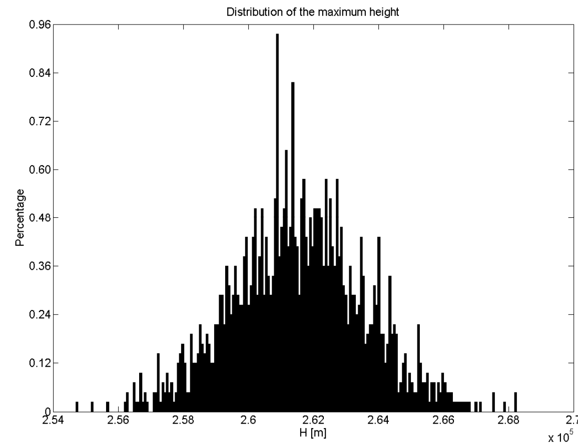

Figure 4 – Distribution of maximum height estimates over 500 independent Monte Carlo trials employing N=5 radar plots

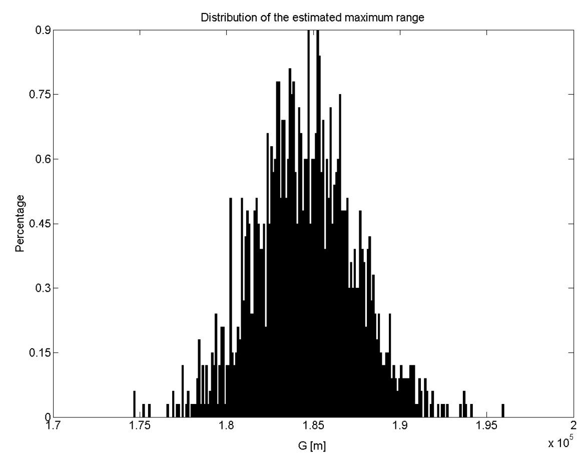

Figure 5 – Distribution of maximum range estimates over 500 independent Monte Carlo trials employing N=5 radar plots

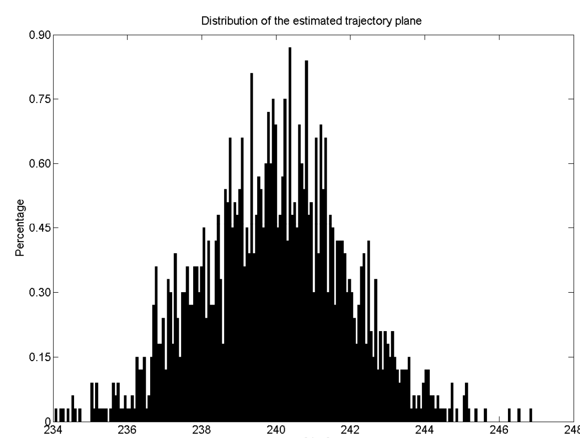

Figure 6 – Distribution of the angle ϕ of the trajectory plane estimates over 500 independent Monte Carlo trials employing N=5 radar plots

	Standard deviation
$\widehat{H}$	$\sigma_H = 2032\ m$
$\hat{G}$	$\sigma_G = 2899\ m$
$\hat{\phi}$	$\sigma_\phi = 1.97\ \deg$

Table 3: standard deviations of the maximum height, maximum range, and trajectory plane estimation resulting from the unconstrained estimation

Monte Carlo simulations have been performed considering the following inequality constraints

$$\begin{cases} -k\sigma_G \le G_{\min} \le G \le G_{MAX} = k\sigma_G \\ -k\sigma_H \le H_{\min} \le H \le H_{MAX} = k\sigma_H \\ -k\sigma_\phi \le \phi_{\min} \le \phi \le \phi_{MAX} = k\sigma_\phi \end{cases} \tag{26}$$

for $k = 0.1,\ 0.5,\ 1,\ 2$. The assumption that the constraints are centered at the true values of H, G and ϕ corresponds to the worst case [9]. The mathematical results of the simulations, obtained for the cases $k = 0.1,\ 0.5,\ 1,\ 2$ are summarized in Table 4 and compared to the unconstrained case.

Constraints	$3\sigma_x$, [km]	$3\sigma_y$, [km]	$3\sigma_z$, [km]
Uncon.	13.89	13.80	13.83
H-G- ϕ***, k=2***	13.18	13.04	13.16
H-G- ϕ***, k=1***	9.28	9.89	10.78
H-G- ϕ***, k=0.5***	5.26	6.36	7.84
H-G- ϕ***, k=0.1***	1.21	1.74	5.40

Table 4: accuracies of the prediction of impact point for constrained and unconstrained estimators

From Table 4, it can be noticed that the advantage, in terms of accuracy, of using inequality constraints becomes meaningful only when the constraint bounds are tight.

VI. CONCLUSIONS

The paper has discussed the exploitation of a-priori information to improve the estimation accuracy. A simplified ballistic estimation problem has been considered as study case and two types of a-priori information have been taken into account, i.e. equality and inequality constraints. In the former case, the accuracy improvement deriving by the constraints exploitation has been evaluated theoretically by deriving the CRLB and, then, these theoretical results have been validated by simulation. In the inequality constraints case, since no theoretical bound is available, the benefits introduced by the constraints has only been evaluated by simulation. The results show how much the constraint bounds have to be tight to improve the estimation accuracy.

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APPENDIX

The elements of Jacobian matrix of eq. (19) are computed as follows:

$$F_{11} = \frac{\partial f_G(\mathbf{x})}{\partial x_0} = F_{21} = \frac{\partial f_G(\mathbf{x})}{\partial y_0} = 0$$

$$F_{31} = \frac{\partial f_G(\mathbf{x})}{\partial z_0} = \frac{\sqrt{v_x^2 + v_y^2}}{\sqrt{v_z^2 + 2gz_0}}$$

$$F_{41} = \frac{\partial f_G(\mathbf{x})}{\partial v_x} = \frac{v_x\left(v_z + \sqrt{v_z^2 + 2gz_0}\right)}{g\sqrt{v_x^2 + v_y^2}}$$

$$F_{51} = \frac{\partial f_G(\mathbf{x})}{\partial v_y} = \frac{v_y\left(v_z + \sqrt{v_z^2 + 2gz_0}\right)}{g\sqrt{v_x^2 + v_y^2}}$$

$$F_{61} = \frac{\partial f_G(\mathbf{x})}{\partial v_z} = \sqrt{v_x^2 + v_y^2}\left(1 + \frac{v_z}{\sqrt{v_z^2 + 2gz_0}}\right)$$

$$F_{12} = \frac{\partial f_H(\mathbf{x})}{\partial x_0} = F_{22} = \frac{\partial f_H(\mathbf{x})}{\partial y_0} = 0$$

$$F_{32} = \frac{\partial f_H(\mathbf{x})}{\partial z_0} = 1$$

$$F_{42} = \frac{\partial f_H(\mathbf{x})}{\partial v_x} = F_{52} = \frac{\partial f_H(\mathbf{x})}{\partial v_y} = 0$$

$$F_{62} = \frac{\partial f_H(\mathbf{x})}{\partial v_z} = \frac{v_z}{g}$$

$$F_{13} = \frac{\partial f_\phi(\mathbf{x})}{\partial x_0} = F_{23} = \frac{\partial f_\phi(\mathbf{x})}{\partial y_0} = F_{33} = \frac{\partial f_\phi(\mathbf{x})}{\partial z_0} = 0$$

$$F_{43} = \frac{\partial f_\phi(\mathbf{x})}{\partial v_x} = -\frac{v_y}{v_x^2 + v_y^2}$$

$$F_{53} = \frac{\partial f_\phi(\mathbf{x})}{\partial v_y} = \frac{v_x}{v_x^2 + v_y^2}$$

$$F_{63} = \frac{\partial f_\phi(\mathbf{x})}{\partial v_z} = 0$$

The previous partial derivatives of the Jacobian must be computed in correspondence of the true value $\bar{\mathbf{x}}$ of the vector to be estimated.