State estimation with a remote sensor under limited communication rate

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***Abstract*—This paper deals with the problem of estimating the state of a discrete-time linear Gaussian stochastic dynamical system subject to a limitation on the communication rate from the remote sensor unit. More specifically, the attention is devoted to a 2-node network consisting of: (1) a remote node S which collects measurements of the given system, computes optimal state estimates at the full measurement rate and transmits them at a reduced communication rate; (2) a local node F that, based on received estimates, provides an estimate of the system state at the full rate. A measurement-independent strategy for deciding when transmitting estimates from S to F will be considered. Sufficient conditions for the boundedness of the state covariance at node F will be given. Further, it will be shown that the optimal communication strategy for a given rate is characterized by an upper bound on the time between consecutive transmissions.**

I. Introduction

This paper deals with the problem of estimating the state of a discrete-time linear Gaussian stochastic dynamical system:

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{w}_k \quad \text{(I.1)}$$

$$\mathbf{y}_k = \mathbf{C}\mathbf{x}_k + \mathbf{v}_k \quad \text{(I.2)}$$

subject to a limitation on the communication rate from the remote sensor unit to the state estimation unit. More specifically, the attention will be focused on the use of a *2-node network* consisting of:

- a remote *sensing node* S which collects noisy measurements $\mathbf{y}_k$ of the given system, can process them to find filtered estimates $\hat{\mathbf{x}}_{k|k}$ and transmit such estimates to the other node;
- a local *fusion node* F which receives data from S and, based on such data, should estimate, in the best possible way, the system's state.

It is assumed that the ratio

$$\frac{\text{communication rate } S \to F}{\text{measurement rate of } S} = \alpha \in (0,1)$$

where possibly $\alpha \ll 1$. Hereafter, without loss of generality, the unit rate will be fixed equal to the measurement rate so that α actually denotes the communication rate.

In particular, the objective is to devise a *communication strategy* with fixed rate α that guarantees bounded state covariance and possibly optimal estimation performance, in terms of minimum *Mean Square Error (MSE)*, at the node F.

The above scenario reflects the practical situation in which the sensing unit and the monitoring unit are remotely dislocated one with respect to each other and the communication rate between them is severely limited by energy, band and/or security concerns.

State estimation under finite communication bandwidth has been thoroughly investigated, see e.g. [1]-[6]. In the above cited references, the emphasis is on the analysis of the quantization effects due to the encoding of transmitted data into a finite alphabet of symbols as well as on the design of efficient, possibly optimal, coding algorithms. Conversely, the present work tackles the issue of communication bandwidth finiteness from a completely different viewpoint. Specifically, it is assumed that infinite-precision data are transmitted over the communication channel[1] while the bandwidth limitation is accomplished by imposing a suitable value of the communication rate α. In this context, the focus will be on the choice of a communication strategy for deciding which data transmit from S to F. In particular, measurement-independent strategies will be considered and issues related to the stability and optimality of such strategies will be investigated. Two main results will be provided. First it will be shown that boundedness of the MSE can be ensured provided that an upper bound on the time between consecutive transmissions is imposed. Secondly, the communication strategy with minimum MSE will be derived.

The rest of the paper will be organized as follows. Section II formalizes the communication strategy adopted in this work. Section III deals with the state estimation algorithm. Sections IV and V present the theoretical results concerning stability and, respectively, optimality of the communication strategy. Section VI concludes the paper.

The notations are quite standard: $\mathbb{Z}_+$ is the set of nonnegative integers; given a square matrix $\mathbf{M}$, $\mathrm{tr}(\mathbf{M})$ and $\lambda_{\max}(\mathbf{M})$ denote its trace and, respectively, its maximum eigenvalue; finally, $\mathbb{E}\{\cdot\}$ and $\mathbb{P}\{\cdot\}$ denote the expectation and, respectively, probability operators.

[1]This assumption, though incompatible with the finite bandwidth, holds in practice provided that quantization errors are negligible with respect to measurement errors.

II. Communication strategy

The aim of this section is to formalize the concept of *communication strategy (CS)* with fixed rate α. To this end, let us introduce binary variables c_k such that

$$c_k = \begin{cases} 1, & \text{if node } S \text{ transmits at time } k \\ 0, & \text{if node } S \text{ does not transmit at time } k \end{cases}$$

The communication strategy is characterized by the following two choices:

- the type of data being transmitted by the node S, either measurements $\mathbf{y}_k$ or filtered state estimates $\hat{\mathbf{x}}^S_{k|k}$;
- the mechanism by which the node S decides whether to transmit or not at time k, i.e. the mechanism of generating $c_k \in \{0, 1\}$.

As far as the first choice is concerned, the following options can for instance be adopted:

- **measurement transmission**: when $c_k = 1$, node S transmits $\mathbf{y}_k$ to node F;
- **estimate transmission**: when $c_k = 1$, node S transmits $\hat{\mathbf{x}}^S_{k|k}$ to node F.

In this paper, the attention will be restricted to estimate transmission, assuming that the node S has enough processing capability to update on-line the optimal state estimate $\hat{\mathbf{x}}^S_{k|k}$. Note that, with such a choice, there is no loss of information due to the finite bandwidth as the estimate $\hat{\mathbf{x}}^S_{k|k}$ represents a sufficient statistics.

As far as the decision mechanism is concerned, this can be formally defined as follows.

Definition - A decision mechanism with rate $\alpha \in (0, 1)$ is any, deterministic or stochastic, mechanism of generating c_k such that

$$\lim_{t\to\infty} \frac{1}{t} \sum_{k=1}^{t} \mathbb{E}\{c_k\} = \alpha \qquad \text{(II.1)}$$

Several decision mechanisms can clearly be devised. In this work, the attention will be restricted to *measurement-independent* strategies that, at time k, decide whether to transmit or not independently of the measurement sequence $\mathbf{y}^k \triangleq \{\mathbf{y}_0, \mathbf{y}_1, \ldots, \mathbf{y}_k\}$. More specifically, we suppose that each c_k is chosen according to some probabilistic criterion, possibly adapted only on the basis of the past transmission pattern, so as to ensure the desired communication rate. With this respect, let $n_k \geq 0$ denote, at a generic time k, the number of time instants elapsed from the last transmission, i.e.

$$c_{k-n_k} = 1 \quad \text{and} \quad c_k = c_{k-1} = \cdots = c_{k-n_k+1} = 0.$$

Note that n_k is a function of $c^k \triangleq \{c_0, c_1, \ldots, c_k\}$ and can be recursively computed as

$$n_k = (1 - c_k)\,(n_{k-1} + 1)\,.$$

It should be evident that, in the considered framework, such a quantity provides a rough estimate of how good is the estimate $\hat{\mathbf{x}}^F_{k|k}$ at node F. Then it seems reasonable to take into account the value of n_{k-1} when choosing whether to transmit or not at the generic time k. In this connection, the conditional probabilities

$$\mathbb{P}\{c_k = 1 | n_{k-1} = i\}\,, \quad i = 0, 1, \ldots$$

can be considered as design parameters in the communication strategy that can be suitably tuned to improve the performance (e.g., to reduce the MSE). This will be the subject of Sections IV and V.

III. State estimation with remote sensor

Before concentrating on the properties, i.e. boundedness and optimality, of the specific transmission strategy, this section will describe the operations that must be performed in the nodes S and F in order to recover at the node F the optimal estimate regardless of the actually employed strategy.

A. Node S operation

To enable the estimate transmission option, it is assumed that the node S has enough processing capabilities to compute the optimal state estimate. Let $\mathbf{y}^k$ be the set of measurements gathered by node S at time k. In the Bayesian approach, the solution of the estimation problem is given by the conditional PDF $p(\mathbf{x}_k|\mathbf{y}^k)$. This PDF can be obtained recursively as follows:

$$\begin{aligned} p^S(\mathbf{x}_k|\mathbf{y}^k) &\propto p(\mathbf{y}_k|\mathbf{x}_k)\, p^S(\mathbf{x}_k|\mathbf{y}^{k-1}) \\ p^S(\mathbf{x}_k|\mathbf{y}^{k-1}) &= \int_{\mathbf{x}_{k-1}} p(\mathbf{x}_k|\mathbf{x}_{k-1})\, p^S(\mathbf{x}_{k-1}|\mathbf{y}^{k-1})\, d\mathbf{x}_{k-1} \end{aligned} \qquad \text{(III.1)}$$

where the superscript S in $p^S(\mathbf{x}_k|\mathbf{y}^k)$ and $p^S(\mathbf{x}_k|\mathbf{y}^{k-1})$ indicates that these PDFs are calculated with respect to the information available at the node S. It is well known that for the linear Gaussian system (I.1)-(I.2), the above PDFs turn out to be Gaussian and the optimal estimates and covariances, fully characterizing such PDFs, are recursively provided by the Kalman Filter:

$$\begin{cases} \hat{\mathbf{x}}^S_{k|k-1} = \mathbf{A}\hat{\mathbf{x}}^S_{k-1|k-1} \\ \mathbf{P}^S_{k|k-1} = \mathbf{A}\mathbf{P}^S_{k-1|k-1}\mathbf{A}' + \mathbf{Q} \\ \mathbf{S}_k = \mathbf{R} + \mathbf{C}\mathbf{P}^S_{k|k-1}\mathbf{C}' \\ \mathbf{K}_k = \mathbf{P}^S_{k|k-1}\mathbf{C}'\mathbf{S}_k^{-1} \\ \hat{\mathbf{x}}^S_{k|k} = \hat{\mathbf{x}}^S_{k|k-1} + \mathbf{K}_k(\mathbf{y}_k - \mathbf{C}\hat{\mathbf{x}}^S_{k|k-1}) \\ \mathbf{P}^S_{k|k} = \mathbf{P}^S_{k|k-1} - \mathbf{P}^S_{k|k-1}\mathbf{C}'\mathbf{S}_k^{-1}\mathbf{C}\mathbf{P}^S_{k|k-1} \end{cases} \qquad \text{(III.2)}$$

where $\mathbf{Q}$ and $\mathbf{R}$ are the covariances of $\mathbf{w}_k$ and, respectively, $\mathbf{v}_k$. Notice that the covariance matrices of the Kalman Filter do not depend on the sequence of measurements and, hence, can be autonomously computed by the node F so that only transmission of estimates but no transmission of covariances from S to F is needed.

B. Node F operation

For the node F, the available information at time k is given by:

$$I^k = \{n_k, \mathbf{y}^{k-n_k}\}$$

It is easy to verify that in this case the PDFs $p(\mathbf{x}_k|I^k)$ at the fusion node F turn out to be Gaussian. In fact, $p(\mathbf{x}_k|I^k) = p(\mathbf{x}_k|n_k, \mathbf{y}^{k-n_k}) = p(\mathbf{x}_k|\mathbf{y}^{k-n_k})$, n_k being independent of $\mathbf{x}_k$. Clearly, $p(\mathbf{x}_k|\mathbf{y}^{k-n_k})$ turns out to be Gaussian with mean $\hat{\mathbf{x}}^S_{k|k-n_k}$ and covariance $\mathbf{P}^S_{k|k-n_k}$. Hence, the optimal estimates $\hat{\mathbf{x}}^F_{k|k}$ and covariances $\mathbf{P}^F_{k|k}$ can be recursively computed by

$$\begin{cases} \hat{\mathbf{x}}^F_{k|k-1} &= \mathbf{A}\hat{\mathbf{x}}^F_{k-1|k-1} \\ \mathbf{P}^F_{k|k-1} &= \mathbf{A}\mathbf{P}^F_{k-1|k-1}\mathbf{A}' + \mathbf{Q} \\ \hat{\mathbf{x}}^F_{k|k} &= c_k\,\hat{\mathbf{x}}^S_{k|k} + (1-c_k)\,\hat{\mathbf{x}}^F_{k|k-1} \\ \mathbf{P}^F_{k|k} &= c_k\,\mathbf{P}^S_{k|k} + (1-c_k)\,\mathbf{P}^F_{k|k-1} \end{cases} \quad \text{(III.3)}$$

where the covariance matrix $\mathbf{P}^S_{k|k}$ is updated in the node F in the same way as in the node S; see (III.2). It is important to note that, in this case, also the covariance $\hat{\mathbf{P}}^F_{k|k}$ is a random variable as it depends on c^k.

IV. Boundedness of the state covariance

In this section, the boundedness of the state covariance at node F is discussed for different choices of the transmission probabilities $\mathbb{P}\{c_k = 1|n_{k-1} = i\}$. This allows one to gain some insights on how the choice of such probabilities may significantly affect the performance of the transmission strategy.

In order to ensure the well-posedness of the state estimation problem, the following preliminary assumption is needed.

A1. $(\mathbf{A}, \mathbf{C})$ is detectable, $(\mathbf{A}, \mathbf{Q}^{1/2})$ is stabilizable, and $\mathbf{R} > 0$.

As well known, assumption A1 ensures that the a-priori state covariance $\mathbf{P}^S_{k|k-1}$ and the a-posteriori covariance $\mathbf{P}^S_{k|k}$ exponentially converge to the steady-state values $\mathbf{P}_b$ and $\mathbf{P}_a$, respectively, where $\mathbf{P}_b$ is the unique positive definite solution of the algebraic Riccati equation

$$\mathbf{P}_b = \mathbf{A}\left[\mathbf{P}_b - \mathbf{P}_b\mathbf{C}'\left(\mathbf{R} + \mathbf{C}\mathbf{P}_b\mathbf{C}'\right)^{-1}\mathbf{C}\mathbf{P}_b\right]\mathbf{A}' + \mathbf{Q} \quad \text{(IV.1)}$$

and $\mathbf{P}_a$ can be obtained as

$$\mathbf{P}_a = \mathbf{P}_b - \mathbf{P}_b\mathbf{C}'\left(\mathbf{R} + \mathbf{C}\mathbf{P}_b\mathbf{C}'\right)^{-1}\mathbf{C}\mathbf{P}_b\,. \quad \text{(IV.2)}$$

Let us first focus on the case in which c_k are chosen to be mutually independent Bernoulli random variables with parameter α, i.e.,

$$\mathbb{P}\{c_k = 1|n_{k-1} = i\} = \alpha\,, \quad i = 0, 1, \ldots\,.$$

As the covariance $\mathbf{P}^F_{k|k}$ is a random variable, in order to analyze the performance of the considered strategy, the expected value $\mathbb{E}\{\mathbf{P}^F_{k|k}\}$ has to be considered. With this respect, since c_k is supposed to be independent from c^{k-1}, it is easy to see that

$$\mathbb{E}\{\mathbf{P}^F_{k|k}\} = \alpha\,\mathbf{P}^S_{k|k} + (1-\alpha)\left[\mathbf{A}\mathbb{E}\{\mathbf{P}^F_{k-1|k-1}\}\mathbf{A}' + \mathbf{Q}\right]. \quad \text{(IV.3)}$$

Then the following result can be stated.

Proposition 1: Let c_k, $k = 0, 1, \ldots$, be mutually independent Bernoulli random variables with parameter α. Under assumption A1, the following facts hold:

1) if $\alpha > 1 - 1/\lambda^2_{\max}(\mathbf{A})$ then

$$\lim_{k\to\infty} \mathbb{E}\{\mathbf{P}^F_{k|k}\} = \mathbf{X}$$

where $\mathbf{X}$ is the unique positive definition solution of the Lyapunov equation

$$(1-\alpha)\,\mathbf{A}\,\mathbf{X}\,\mathbf{A}' - \mathbf{X} = -(1-\alpha)\,\mathbf{Q} - \alpha\mathbf{P}_a\,;$$

2) if $\alpha \leq 1 - 1/\lambda^2_{\max}(\mathbf{A})$ then

$$\lim_{k\to\infty} \mathrm{tr}\left(\mathbb{E}\{\mathbf{P}^F_{k|k}\}\right) = +\infty\,.$$

Proof: Let us rewrite (IV.3) as

$$\mathbb{E}\{\mathbf{P}^F_{k|k}\} = \mathbf{M}_\alpha\left(\mathbb{E}\{\mathbf{P}^F_{k-1|k-1}\}, \mathbf{P}^S_{k|k}\right)\,.$$

Since $\mathbf{P}^S_{k|k}$ exponentially converges to the positive definite matrix $\mathbf{P}_a$, by continuity of the matrix function $\mathbf{M}_\alpha(\cdot,\cdot)$, if the sequence $\mathbf{P}^S_{k|k}$ converges then its steady-state value $\mathbf{X}$ must be a fixed point of $\mathbf{M}_\alpha(\cdot, \mathbf{P}_a)$, i.e.,

$$\mathbf{X} = \mathbf{M}_\alpha\left(\mathbf{X}, \mathbf{P}_a\right)\,.$$

Such a condition can be rewritten as

$$\begin{aligned} \mathbf{X} \;=\; & \left((1-\alpha)^{1/2}\mathbf{A}\right)\mathbf{X}\left((1-\alpha)^{1/2}\mathbf{A}'\right) \\ & + \alpha\,\mathbf{P}_a + (1-\alpha)\mathbf{Q}\,. \end{aligned} \quad \text{(IV.4)}$$

As well known, (IV.4) admits a (unique) solution if and only if the matrix $(1-\alpha)^{1/2}\mathbf{A}$ is Schur stable, i.e., $\alpha > 1 - 1/\lambda^2_{\max}(\mathbf{A})$. The rest of the proof follows from standard arguments (see, e.g., [7]) and is omitted for the sake of brevity. □

Proposition 1 states that, when the matrix $\mathbf{A}$ is not Schur stable (i.e., $\lambda_{\max}(\mathbf{A}) \geq 1$), the expected covariance is ultimately bounded if and only if the transmission rate exceeds a critical value $\alpha_c \triangleq 1 - 1/\lambda^2_{\max}(\mathbf{A})$. It is worth noting that in [7] the existence of a similar critical value was proved for Kalman filtering with intermittent observations, which in the present framework corresponds to transmitting measurements instead of estimates. Thus, Proposition 1 shows that equipping the remote sensor with the processing capability of computing the optimal estimates is not sufficient to overcome such a drawback.

Nevertheless, it turns out that the limitation expressed in Proposition 1 can be easily overcome by means of a suitable choice of the conditional probabilities $\mathbb{P}\{c_k = 1|n_{k-1} = i\}$. To see this, consider a modified strategy in which, at the generic time k, c_k is chosen to be a Bernoulli random variable with parameter $\varphi(n_{k-1})$, where the function $\varphi : \mathbb{Z}_+ \to [0,1]$ has to be chosen so that the transmission rate constraint is met. This corresponds to the assignments

$$\mathbb{P}\{c_k = 1 | n_{k-1} = i\} = \varphi(i)\,, \quad i = 0, 1, \ldots\,.$$

Moreover, suppose that $\varphi(N-1) = 1$ for some $N > 1$. This amounts to assuming that the time interval between two consecutive transmissions never exceeds N. As it will be shown in the next section, transmission strategies of this kind satisfying the communication rate constraint (II.1) always exist provided that $N > 1/\alpha$. It is also worth noting that such a class of strategies comprises periodic strategies with transmission rate equal to $1/N$ (in this case $\varphi(i) = 0$, for $i = 0, \ldots, N-2$).

It is immediate to check that for such a modified strategy the recursion provided in equation (III.3) still holds. As to the expected state covariance, equation (IV.3) does not hold since in this case c_k is correlated with c^{k-1} through n_{k-1}. However, since $\varphi(N-1) = 1$ implies that $\sum_{i=0}^{N-1} \mathbb{P}\{n_k = i\} = 1$, the expected covariance can be written as

$$\mathbb{E}\{\mathbf{P}^F_{k|k}\} = \sum_{i=0}^{N-1} \mathbb{P}\{n_k = i\}\, \mathbf{P}^S_{k|k-i}$$

where $\mathbf{P}^S_{k|k-i}$ is the covariance at the node F provided that the last transmission has occurred i time instants ago.

In order to analyze the asymptotic behavior of the expected state covariance, first note that under assumption A1 we have

$$\lim_{k\to\infty} \mathbf{P}^S_{k|k-i} = \mathbf{P}^{(i)}\,, \quad i = 0, \ldots, N-1$$

where the covariances $\mathbf{P}^{(i)}$ can be obtained by repeatedly applying the Lyapunov difference equation:

$$\left\{ \begin{array}{rcl} \mathbf{P}^{(0)} & = & \mathbf{P}_a \\ \mathbf{P}^{(i)} & = & \mathbf{A}\mathbf{P}^{(i-1)}\mathbf{A}' + \mathbf{Q}\,, \quad i = 1, \ldots, N-1\,. \end{array} \right. \tag{IV.5}$$

Moreover, note that the sequence $n_0, n_1, \ldots$ can be described by a discrete-time Markov chain characterized by the state space $\mathcal{S} = \{0, 1, \ldots, N-1\}$ and the transition matrix

$$\mathbf{\Phi} = \begin{bmatrix} \varphi(0) & \varphi(1) & \ldots & \varphi(N-2) & 1 \\ 1-\varphi(0) & 0 & \ldots & 0 & 0 \\ 0 & 1-\varphi(1) & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1-\varphi(N-2) & 0 \end{bmatrix}.$$

For the reader's convenience, the associated state transition diagram is provided in Fig. 1. It is easy to check that the Markov chain $n_0, n_1, \ldots$ is irreducible if and only if $\varphi(i) < 1$ for $i = 0, \ldots, N-2$. In this case, there exists a unique invariant distribution for the probabilities $\mathbb{P}\{n_k = i\} = p(i)\,,\quad i = 0, \ldots, N-1$, that represents also the long-run average distribution, i.e.,

$$\lim_{t\to\infty} \frac{1}{t} \sum_{k=1}^{t} \mathbb{P}\{n_k = i\} = p(i)\,. \tag{IV.6}$$

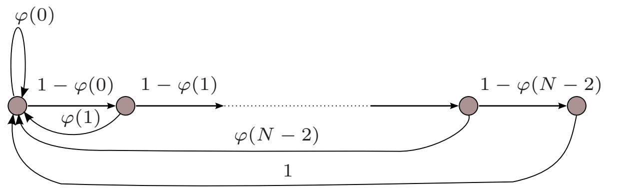

Fig. 1. State transition diagram of the Markov chain $n_0, n_1, \ldots$ for the modified strategy.

As well known, $\mathbf{p} \triangleq [p(0) \quad \ldots \quad p(N-1)]'$ corresponds to the Perron-Frobenius eigenvector of the transition matrix $\mathbf{\Phi}$ and can be computed by imposing the balance equations

$$\mathbf{\Phi}\mathbf{p} = \mathbf{p} \tag{IV.7}$$

and

$$\mathbf{p}'\mathbf{1} = 1 \tag{IV.8}$$

where $\mathbf{1}$ is the column vector with all entries equal to 1. With this respect, from (IV.7) one can derive

$$p(i) = p(0) \prod_{j=0}^{i-1} (1 - \varphi(j))\,, \quad i = 0, \ldots, N-1$$

where, for the sake of compactness, we define $\prod_{j=0}^{-1}(\cdot) \triangleq 1$. Since equation (IV.8) can be rewritten as

$$p(0) \sum_{i=0}^{N-1} \prod_{j=0}^{i-1} (1 - \varphi(j)) = 1\,,$$

the invariant distribution is obtained as

$$p(i) = \left(\sum_{i=0}^{N-1} \prod_{j=0}^{i-1} (1 - \varphi(j)) \right)^{-1} \prod_{j=0}^{i-1} (1 - \varphi(j))\,, \tag{IV.9}$$

for $i = 0, \ldots, N-1$.

It is important to remark that, when $\varphi(i) = 1$ for some $i < N-1$, the considered Markov chain is not irreducible. However, in this case, there exists a unique closed communicating class corresponding to the set of states $\{0, 1, \ldots, M-1\}$, where M is the smallest integer such that $\varphi(M) = 1$. All the other states have null asymptotic probabilities. Then, it is straightforward to see that (IV.6) and (IV.9) still hold. This generalization will be useful in the next section.

The foregoing discussion leads to the following result.

Proposition 2: Let c_k, $k = 0, 1, \ldots$, be Bernoulli random variables with parameter $\varphi(n_{k-1})$. Moreover, let $\varphi(N-1) = 1$ for some $N > 1$. Then, under assumption A1, we have

$$\lim_{t\to\infty} \frac{1}{t} \sum_{k=1}^{t} \mathbb{E}\{\mathbf{P}^F_{k|k}\} = \left(\sum_{i=0}^{N-1} \prod_{j=0}^{i-1} (1 - \varphi(j)) \right)^{-1} \times \sum_{i=0}^{N-1} \prod_{j=0}^{i-1} (1 - \varphi(j))\, \mathbf{P}^{(i)}\,.$$

□

In other words, the modified strategy always provides an ultimately bounded expected state covariance regardless of the transmission rate α and of the stability of the system matrix $\mathbf{A}$.

V. Optimal transmission probabilities

This section is devoted to the synthesis of an optimized transmission strategy. In this connection, note that, for a given N, the modified strategy introduced in the previous section is not uniquely determined in that there are $N-1$ degrees of freedom, i.e., the parameters $\varphi(i)$ for $i = 0, \ldots, N-2$. To be more precise, when the transmission rate α is fixed, the degrees of freedom are only $N-2$ since the parameters have to be chosen so that the equality constraint (II.1) is satisfied. It is quite natural to think of exploiting such degrees of freedom to optimize some performance index. With this respect, a reasonable choice for the performance index is given by the cost

$$J \stackrel{\triangle}{=} \lim_{t\to\infty} \frac{1}{t} \sum_{k=1}^{t} \operatorname{tr}\left(\mathbb{E}\{\mathbf{P}^F_{k|k}\}\,\mathbf{W}\right)$$

where $\mathbf{W}$ is a positive definite weight matrix.

In the light of Proposition 2, it is immediate to write cost J as a function of the parameters $\varphi(i)$ for $i = 0, \ldots, N-2$:

$$J = \left(\sum_{i=0}^{N-1}\prod_{j=0}^{i-1}(1-\varphi(j))\right)^{-1} \times \sum_{i=0}^{N-1} \operatorname{tr}\left(\mathbf{P}^{(i)}\,\mathbf{W}\right) \prod_{j=0}^{i-1}(1-\varphi(j)) . \tag{V.1}$$

In the sequel, for the sake of compactness, we shall use the notation $w(i) \stackrel{\triangle}{=} \operatorname{tr}\left(\mathbf{P}^{(i)}\,\mathbf{W}\right)$.

In order to satisfy the transmission rate constraint (II.1), the parameters $\varphi(i)$ for $i = 0, \ldots, N-2$ have to be chosen so that

$$\lim_{t\to\infty} \frac{1}{t} \sum_{k=1}^{t} \sum_{i=0}^{N-1} \varphi(i)\,\mathbb{P}\{n_{k-1} = i\} = \alpha . \tag{V.2}$$

By applying (IV.6) and (IV.9), one can rewrite (V.2) as

$$\left(\sum_{i=0}^{N-1}\prod_{j=0}^{i-1}(1-\varphi(j))\right)^{-1} \sum_{i=0}^{N-1}\prod_{j=0}^{i-1}(1-\varphi(j))\,\varphi(i) \; = \alpha . \tag{V.3}$$

Moreover, since $\varphi(N-1) = 1$ implies the algebraic identity

$$\sum_{i=0}^{N-1}\prod_{j=0}^{i-1}(1-\varphi(j))\,\varphi(i) \; = 1 ,$$

condition (V.3) turns out to be equal to

$$\sum_{i=0}^{N-1}\prod_{j=0}^{i-1}(1-\varphi(j)) \; = \; 1/\alpha . \tag{V.4}$$

Note that one can exploit such a constraint and, instead of J, consider the simplified cost

$$J' \stackrel{\triangle}{=} \sum_{i=0}^{N-1} w(i) \prod_{j=0}^{i-1}(1-\varphi(j)) .$$

Summing up, the following constrained optimization problem can be stated that leads to the synthesis of an optimized strategy.

Problem 1: For a given $N > 1$, find the optimal parameters $\varphi^\circ(i)$, $i = 0, \ldots, N-2$ that solve the constrained minimization problem

$$\begin{aligned} \text{minimize} \quad & \sum_{i=0}^{N-1} w(i) \prod_{j=0}^{i-1}(1-\varphi(j)) \\ \text{subject to} \quad & \sum_{i=0}^{N-1}\prod_{j=0}^{i-1}(1-\varphi(j)) \; = \; 1/\alpha , \\ & 0 \le \varphi(i) \le 1 , \quad i = 0, \ldots, N-2 . \end{aligned} \tag{V.5}$$

□

Since the cost function J' is multilinear with respect to the parameters $\varphi(i)$, Problem 1 is not convex. However, it is possible to recast it as a Linear Programming (LP) problem by means of the change of variables

$$z(i) = \prod_{j=0}^{i-1}(1-\varphi(j)) , \quad i = 1, \ldots, N-1 . \tag{V.6}$$

More specifically, cost J' becomes

$$J' = w(0) + \sum_{i=1}^{N-1} w(i)\, z(i)$$

and the transmission rate constraint (V.4) can be rewritten as

$$1 + \sum_{i=1}^{N-1} z(i) \; = \; 1/\alpha . \tag{V.7}$$

Finally, the feasibility constraints $0 \le \varphi(i) \le 1$, $i = 0, \ldots, N-2$, turn out to be equivalent to

$$\begin{aligned} & z(i) \ge 0 , && i = 1, \ldots, N-1 , \\ & z(1) \le 1 , && \\ & z(i+1) \le z(i) , && i = 1, \ldots, N-2 . \end{aligned}$$

Thus, Problem 1 admits the equivalent LP formulation:

$$\begin{aligned} \text{minimize} \quad & w(0) + \sum_{i=1}^{N-1} w(i)\, z(i) \\ \text{subject to} \quad & 1 + \sum_{i=1}^{N-1} z(1) \; = \; 1/\alpha \\ & z(i) \ge 0 , && i = 1, \ldots, N-1 , \\ & z(1) \le 1 , && \\ & z(i+1) \le z(i) , && i = 1, \ldots, N-2 . \end{aligned} \tag{V.8}$$

In order to derive the optimal solution of problem (V.8), and consequently of Problem 1, the following lemma is needed.

Lemma 1: Under assumption A1, for any $\mathbf{W} > 0$ the weights $w(i)$ are monotonically increasing, i.e.,

$$w(0) \leq w(1) \leq \ldots \leq w(N-1)\,. \tag{V.9}$$

Proof: The proof can be given by induction. To this end, first note that equations (IV.1)-(IV.2) and (IV.5) imply $\mathbf{P}^{(1)} = \mathbf{P}_b$. Thus, from (IV.2), it follows that $\mathbf{P}^{(0)} \leq \mathbf{P}^{(1)}$. Suppose now that, for a given i, we have $\mathbf{P}^{(i-1)} \leq \mathbf{P}^{(i)}$. Then

$$\begin{aligned}\mathbf{P}^{(i+1)} - \mathbf{P}^{(i)} &= \mathbf{A}\mathbf{P}^{(i)}\mathbf{A}' + \mathbf{Q} - \mathbf{A}\mathbf{P}^{(i-1)}\mathbf{A} - \mathbf{Q} \\ &= \mathbf{A}(\mathbf{P}^{(i)} - \mathbf{P}^{(i-1)})\mathbf{A}' \geq 0\,.\end{aligned}$$

and consequently, by induction,

$$\mathbf{P}^{(0)} \leq \mathbf{P}^{(1)} \leq \ldots \leq \mathbf{P}^{(N-1)}\,.$$

This, in turn, implies also (V.9). □

In the light of Lemma 1 and of the LP reformulation (V.8), the following result can now be stated.

Theorem 1: Problem 1 admits solution if and only if $N \geq 1/\alpha$. Moreover, under assumption A1, for any $\mathbf{W} > 0$ the optimal solution is given by

$$\begin{aligned}&\varphi^\circ(i) = 0\,, && i = 0, \ldots, M-3\,, \\ &\varphi^\circ(M-2) = M - 1/\alpha\,, && \\ &\varphi^\circ(M-1) = 1\,, &&\end{aligned} \tag{V.10}$$

where M is the smallest integer such that $M \geq 1/\alpha$.

Proof: Since $z(i) \leq 1$ for $i = 1, \ldots, N-1$, constraint (V.7) can be satisfied only if $N \geq 1/\alpha$. Conversely, when $N \geq 1/\alpha$, it is immediate to see that the choice

$$\begin{aligned}&z(i) = 1\,, && i = 1, \ldots, M-2\,, \\ &z(M-1) = 1/\alpha - M + 1\,, && \\ &z(i) = 0\,, && i = M, \ldots, N-1\,,\end{aligned} \tag{V.11}$$

satisfies all the constraints of problem (V.8). Such a choice is also optimal being the weights $w(i)$ monotonically increasing. Then, in order to conclude the proof, it is sufficient to note that (V.10) is equivalent to (V.11) by virtue of relationships (V.6). □

A few remarks about Theorem 1 are in order. First note that, when $1/\alpha$ is integer, the optimal transmission strategy is periodic and consists of transmitting the estimate $\hat{\mathbf{x}}^S_{k|k}$ once every $1/\alpha$ instants. The same does not hold true if $1/\alpha$ is not integer. However, in the case in which $1/\alpha$ is rational, i.e., $1/\alpha = p/q$ with $p, q \in \mathbb{Z}_+$ and coprime, it is still possible to devise a periodic strategy with period q equivalent to the optimal one (i.e., with the same cost). Such a strategy consists of transmitting the estimate $\hat{\mathbf{x}}^S_{k|k}$ p times every q instants in such a way that the maximum time between consecutive transmissions is minimized.

VI. Conclusions

The paper has addressed the state estimation problem assuming that: (1) estimates are required at a distant location from the sensor connected via a communication link; (2) a limitation on the communication rate is imposed; (3) the sensor node has enough processing capability to compute the optimal estimate. Measurement-independent strategies for deciding which data transmit have been investigated. It has been shown that: (1) boundedness of the mean square error is guaranteed provided that an upper bound on the inter-transmission time is enforced; (2) a periodic communication strategy providing the minimum mean square error can be devised whenever the communication rate is rational. Future work will concern the extension to measurement-dependent transmission strategies and multisensor networks.

References

[1] X. Li and W.S. Wong: "State estimation with communication constraints", *Systems & Control Letters*, vol. 28, pp. 49-54, 1996.

[2] W.S. Wong and R.W. Brockett: "Systems with finite communication bandwidth constraints - Part I: state estimation problem", *IEEE Trans. on Automatic Control*, vol. 42, pp. 1294-1299, 1997.

[3] X. Li and W.S. Wong: "Constrained state estimation for systems with finite communcation bandwidth", *Proc. 37th IEEE Conf. on Decision and Control*, pp. 257-262, Tampa, USA, 1998.

[4] G.N. Nair and R.J. Evans: "State estimation under bit-rate constraints", *Proc. 37th IEEE Conf. on Decision and Control*, pp. 251-256, Tampa, USA, 1998.

[5] G.N. Nair and R.J. Evans: "A finite-dimensional coder-estimator for rate-constrained state estimation", *Proc. of the 14th IFAC World Congress*, pp. 19-24, Beijing, China, 1999.

[6] G.N. Nair and R.J. Evans: "Structural results for finite bit-rate state estimation", *Proc. 38th IEEE Conf. on Decision and Control*, pp. 47-52, Adelaide, Australia, 1999.

[7] B. Sinopoli, L. Schenato, M. Franceschetti, K. Poolla, M. Jordan, and S. Sastry, "Kalman filtering with intermittent observations," *IEEE Trans. on Automatic Control*, vol. 49, pp. 1453-1464, 2004.