Conglomerable coherent lower previsions

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Abstract Walley's theory of coherent lower previsions builds upon the former theory by Williams with the explicit aim to make it deal with conglomerability. We show that such a construction has been only partly successful because Walley's founding axiom of joint coherence does not entirely capture the implications of conglomerability. As a way to fully achieve Walley's original aim, we propose then the new theory of *conglomerable coherent lower previsions*. We show that Walley's theory coincides with ours when all conditioning events have positive lower probability, or when conditioning partitions are nested.

1 Introduction

There are two main behavioural theories of *coherent lower previsions* (these are lower expectation functionals): Walley's [3] and Williams' [4]. The main difference between them lies in the notion of *conglomerability*. This is the property that allows us to write an expectation as a mixture of conditional expectations. De Finetti discovered in 1930 that conglomerability can fail when finitely additive probabilities, as well as infinitely many conditioning events, enter the picture [1]. Walley developed his theory by modifying Williams' so as to account for conglomerability in such non-finitary setting. It is controversial that conglomerability should always be imposed; however, we have argued elsewhere [5] that this should be the case when one establishes right from the start that conditional probabilities will be used to determine future behaviour.

From this discussion, it may seem that Walley's theory should be the one to use when conglomerability is required. However, some recent research has shown that a basic procedure to construct rational models in Walley's theory

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does not fully consider the implications of conglomerability [2]. In this paper we take a closer look at this problem by analysing the core of Walley's theory: his notion of self-consistency for coherent lower previsions, which is called *joint coherence*. This can be regarded as the single axiom of Walley's theory.

To this end, we need to work with the theory of *coherent sets of desirable gambles*, which generalise coherent lower previsions. We review these theories and present some preliminary results in Section 2. We start our actual investigation in Section 3. We define the new theory of *conglomerable coherent lower previsions* based on desirable gambles and conglomerability. The founding axiom of this theory is called *conglomerable coherence*. We argue that this is the axiom one should use whenever conglomerability is required. Then we give the relationships about the several consistency notions in Walley's, Williams', and our new theory, with special regard to conglomerability. Most importantly, we show in Example 1 that Walley's joint coherence is not equivalent to conglomerable coherence, which is stronger. In our view, this implies that Walley's theory should be regarded as an approximation to the actual theory to use under conglomerability.

This approximation becomes exact in some important cases, which we discuss in Section 4: when either (i) every conditioning event has positive lower probability or (ii) we consider nested partitions, that is, partitions that are finer and finer, then joint coherence coincides with conglomerable coherence. These two outcomes are important because working with conglomerable coherence can be much more difficult than with Walley's joint coherence.

2 Coherent lower previsions and sets of desirable gambles

Given a possibility space Ω, a *gamble* f is a bounded real-valued function on Ω. $\mathcal{L}(\Omega)$ (or $\mathcal{L}$) denotes the set of all gambles on Ω, and $\mathcal{L}^+(\Omega)$ (or just $\mathcal{L}^+$) the set of so-called positive gambles: $\{f \in \mathcal{L} : f \gneq 0\}$ (where $f \gneq 0$ is a shorthand for $f \geq 0$ and $f \neq 0$). A *lower prevision* $\underline{P}$ is a real functional defined on $\mathcal{L}$. From any lower prevision $\underline{P}$ we can define an upper prevision $\overline{P}$ using conjugacy: $\overline{P}(f) := -\underline{P}(-f)$. Precise previsions, which are those for which $\underline{P}(f) = \overline{P}(f)$, are denoted by $P(f)$.

Given $B \subseteq \Omega$, the real value $\underline{P}(f|B)$ denotes the lower prevision of f conditional on B. Given a partition $\mathcal{B}$ of Ω, then we shall represent by $\underline{P}(f|\mathcal{B})$ the gamble on Ω that takes the value $\underline{P}(f|\mathcal{B})(\omega) = \underline{P}(f|B)$ iff $\omega \in B$. The functional $\underline{P}(\cdot|\mathcal{B})$ is called a *conditional lower prevision*. We say that it is *separately coherent* (or just a *linear prevision*, in the precise case) when for all $B \in \mathcal{B}$, $\underline{P}(f|B)$ is the lower envelope of the expectations obtained from a set of finitely additive probabilities. We shall also use the notations $G(f|B) := B(f - \underline{P}(f|B))$ and $G(f|\mathcal{B}) := \sum_{B\in\mathcal{B}} G(f|B) = f - \underline{P}(f|\mathcal{B})$ for all $f \in \mathcal{L}$ and all $B \in \mathcal{B}$ (note how B is used also as the indicator function of event B). In the case of an unconditional lower prevision $\underline{P}$, we shall let $G(f) := f - \underline{P}(f)$ for any gamble f in its domain.

Definition 1. Let $\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$ be separately coherent conditional lower previsions. They are called *(jointly) coherent* if for every $f_i \in \mathcal{L}, i = 0, \dots, m$, $j_0 \in \{1, \dots, m\}, B_0 \in \mathcal{B}_{j_0}$ and given $H_1 := \sum_{i=1}^m G(f_i|\mathcal{B}_i)$ and $H_2 := G(f_0|B_0)$, it holds that $\sup_{\omega \in B}[H_1 - H_2](\omega) \geq 0$ for some $B \in \cup_{i=1}^m S_i(f_i) \cup \{B_0\}$, where $S_i(f_i) := \{B_i \in \mathcal{B}_i : B_i f_i \neq 0\}$.

A number of weaker conditions are of interest for this paper.

Definition 2. Under the above conditions, $\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$ are said to:

- *avoid partial loss* when $\sup_{\omega \in B} H_1(\omega) \geq 0$ for some $B \in \cup_{i=1}^m S_i(f_i)$;
- be *weakly coherent* when $\sup_{\omega \in \Omega}[H_1 - H_2](\omega) \geq 0$;
- be *Williams-coherent* when the coherence condition holds for the particular case when $S_i(f_i)$ is finite for $i = 1, \dots, m$.

Weak and strong coherence are equivalent in the particular case of two conditional lower previsions, if we assume in addition a positivity condition:

Lemma 1. *If $\underline{P}(\cdot|\mathcal{B}_1)$ and $\underline{P}(\cdot|\mathcal{B}_2)$ are weakly coherent with some coherent lower prevision $\underline{P}$ such that $\underline{P}(B_2) > 0\ \forall B_2 \in \mathcal{B}_2$ and $\underline{P}(B_1) > 0\ \forall B_1 \in \mathcal{B}_1$ different from a given $B_1' \in \mathcal{B}_1$, then $\underline{P}(\cdot|\mathcal{B}_1)$ and $\underline{P}(\cdot|\mathcal{B}_2)$ are coherent.*

We also have the following characterisation of weak coherence:

Lemma 2. *$\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$ are weakly coherent if and only if there is some coherent lower prevision $\underline{P}$ such that for all $j = 1, \dots, m$, it holds that*

$$\underline{P}(G(f|B_j)) = 0 \text{ and } \underline{P}(G(f|\mathcal{B}_j)) \geq 0\ \forall f \in \mathcal{L}, B_j \in \mathcal{B}_j.$$

The equality $\underline{P}(G(f|B_j)) = 0$ is called the *Generalised Bayes Rule* (GBR). Condition $\underline{P}(G(f|\mathcal{B}_j)) \geq 0$ represents a condition of conglomerability of $\underline{P}$ with respect to the conditional lower prevision $\underline{P}(\cdot|\mathcal{B}_j)$. More precisely, we have the following:

Definition 3. Let $\underline{P}$ be a coherent lower prevision, and $\mathcal{B}$ a partition of Ω. We say that $\underline{P}$ is *$\mathcal{B}$-conglomerable* if whenever $f \in \mathcal{L}$ and $B_1, B_2, \dots,$ are different sets in $\mathcal{B}$ such that $\underline{P}(B_n) > 0$ and $\underline{P}(B_n f) \geq 0$ for all $n \geq 1$, it holds that $\underline{P}(\sum_{n=1}^{\infty} B_n f) \geq 0$.

A coherent lower prevision $\underline{P}$ is $\mathcal{B}$-conglomerable if and only if there is a separately coherent conditional lower prevision $\underline{P}(\cdot|\mathcal{B})$ such that $\underline{P}, \underline{P}(\cdot|\mathcal{B})$ are (jointly) coherent (see [3, Theorem 6.8.2(a)]).

The above theory of coherent lower previsions is generalised by the theory of *coherent sets of desirable gambles*, which we summarise next. Given $\mathcal{R} \subseteq \mathcal{L}$, let us denote $\mathrm{posi}(\mathcal{R}) := \{\sum_{k=1}^n \lambda_k f_k \colon f_k \in \mathcal{R}, \lambda_k > 0, n \geq 1\}$.

Definition 4. A set $\mathcal{R} \subseteq \mathcal{L}$ is called *coherent* when $\mathcal{R} = \text{posi}(\mathcal{R} \cup \mathcal{L}^+)$ and $0 \notin \mathcal{R}$. It is said to *avoid partial loss* when it is included in a coherent set, and given a partition $\mathcal{B}$ of Ω, the set $\mathcal{R}$ is said to be $\mathcal{B}$-conglomerable when for any gamble f, $Bf \in \mathcal{R} \cup \{0\}$ for all $B \in \mathcal{B}$ implies that $f \in \mathcal{R} \cup \{0\}$.

Given $\mathcal{R} \subseteq \mathcal{L}$ that avoids partial loss, its smallest coherent superset is called its *natural extension*, and its smallest coherent and $\mathcal{B}$-conglomerable superset (provided it exists), its *$\mathcal{B}$-conglomerable natural extension*. The interior $\underline{\mathcal{R}}$ of a coherent set $\mathcal{R}$ (in the topology of uniform convergence) is called a set of *strictly desirable gambles*. A set of gambles induces a conditional lower prevision by

$$\underline{P}(f|B) := \sup\{\mu : B(f-\mu) \in \mathcal{R}\}. \tag{1}$$

The set of strictly desirable gambles induced by $\underline{P}(f|B)$ is the smallest coherent set of gambles on B that induces $\underline{P}(f|B)$. This allows us to establish the following characterisation.

Theorem 1 ([2, Theorem 3]). *Let $\mathcal{R}$ be a coherent set of desirable gambles, and let $\underline{P}$ be the coherent lower prevision it induces by means of Eq.* (1)*. Then $\underline{P}$ is $\mathcal{B}$-conglomerable if and only if $\underline{\mathcal{R}}$ is $\mathcal{B}$-conglomerable.*

3 Conglomerable coherent lower previsions

A separately coherent conditional lower prevision $\underline{P}(\cdot|\mathcal{B}_i)$ induces the following sets of gambles:

$$\mathcal{R}_i^{B_i} := \{G(f|B_i) + \varepsilon B_i : f \in \mathcal{L}(\Omega), \varepsilon > 0\} \cup \{f \in \mathcal{L}(\Omega) : f = B_i f \gneq 0\}, \tag{2}$$

where $B_i \in \mathcal{B}_i$. Similarly, a collection of separately coherent conditional lower previsions $\underline{P}(\cdot|\mathcal{B}_1), \ldots, \underline{P}(\cdot|\mathcal{B}_m)$ induces the set of desirable gambles $\cup_{i=1}^m \cup_{B_i \in \mathcal{B}_i} \mathcal{R}_i^{B_i}$. Using this set, we can re-formulate one of Williams' basic results [4, Section 1.1] in our language, where lower previsions are conditional on partitions:

Theorem 2. *Let $\underline{P}(\cdot|\mathcal{B}_1), \ldots, \underline{P}(\cdot|\mathcal{B}_m)$ be separately coherent conditional lower previsions. Consider $\mathcal{E} := \text{posi}(\mathcal{L}^+(\Omega) \cup (\cup_{i=1}^m \cup_{B_i \in \mathcal{B}_i} \mathcal{R}_i^{B_i}))$.*

1. *If $\underline{P}(\cdot|\mathcal{B}_1), \ldots, \underline{P}(\cdot|\mathcal{B}_m)$ are Williams-coherent, then $\mathcal{E}$ is coherent.*
2. *$\mathcal{E}$ induces $\underline{P}(\cdot|\mathcal{B}_1), \ldots, \underline{P}(\cdot|\mathcal{B}_m)$ by means of* (1)*.*

In contrast to Williams', here we are concerned with the additional requirement of conglomerability. This is the motivation behind the following notions, which modify some of the consistency conditions in [3, Chapter 7].

Definition 5. Let $\mathcal{R}$ be a set of desirable gambles and $\mathcal{B}$ a partition of Ω. We say that it *avoids $\mathcal{B}$-conglomerable partial loss* if it has a $\mathcal{B}$-conglomerable coherent superset.

Definition 6. Let $\mathcal{B}_1, \dots, \mathcal{B}_m$ be partitions of Ω. A set of desirable gambles that is conglomerable with respect to all the partitions $\mathcal{B}_1, \dots, \mathcal{B}_m$, shall be called $\mathcal{B}_{1:m}$-*conglomerable*.

Definition 7. Conditional lower previsions $\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$ are called *conglomerable coherent* if there is a $\mathcal{B}_{1:m}$-conglomerable coherent set of desirable gambles that induces them. They are said to *avoid conglomerable partial loss* if they have dominating conglomerable coherent extensions.

Let us illustrate the relationships between the notions of avoiding (conglomerable) partial loss for desirable gambles and coherent conditional lower previsions.

Theorem 3. *Let $\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$ be separately coherent conditional lower previsions. Let $\mathcal{R} := \cup_{i=1}^m \cup_{B_i \in \mathcal{B}_i} \mathcal{R}_i^{B_i}$, where the sets of gambles $\mathcal{R}_i^{B_i}$ are determined by Eq.* (2)*.*

1. *If $\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$ avoid partial loss, then $\mathcal{R}$ avoids partial loss.*
2. *$\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$ avoid conglomerable partial loss if and only if the conglomerable natural extension $\mathcal{F}$ of $\mathcal{R}$ exists. Moreover, the smallest dominating conglomerable coherent extensions are induced by the conglomerable natural extension $\mathcal{F}$ of $\mathcal{R}$.*

Now we move on to characterise the different forms of coherence. We start by a preliminary result: we detail how the coherence properties of a set of desirable gambles affect those of the conditional lower previsions it induces.

Theorem 4. *Let $\mathcal{R}$ be a coherent set of desirable gambles, and for every $i = 1, \dots, m$, $B_i \in \mathcal{B}_i$, let $\underline{P}(\cdot|\mathcal{B}_i)$ denote the conditional lower prevision it induces by* (1)*.*

1. *$\underline{P}(\cdot|\mathcal{B}_i)$ is separately coherent for all $i = 1, \dots, m$.*
2. *$\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$ are Williams-coherent.*
3. *[3, Appendix F3] If $\mathcal{R}$ is in addition $\mathcal{B}_{1:m}$-conglomerable, then the conditional lower previsions $\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$ are coherent.*

Let us take now the inverse path, where we start from separately coherent conditional lower previsions $\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$. For every $i = 1, \dots, m, B_i \in \mathcal{B}_i$, let $\mathcal{R}_i^{B_i}$ be given by Eq. (2). The $\mathcal{B}_i$-conglomerable natural extension of the sets $\mathcal{R}_i^{B_i}$ ($B_i \in \mathcal{B}_i$) is the smallest $\mathcal{B}_i$-conglomerable coherent set of desirable gambles that extends the originating sets, and is given by $\mathcal{F}_i := \mathcal{L} \cap \{\sum_{B_i \in \mathcal{B}_i} B_i f_i : B_i f_i \in \mathcal{R}_i^{B_i} \cup \{0\}\} \setminus \{0\}$ (see [5, Proposition 4]). Obviously, it need not be $\mathcal{B}_j$-conglomerable for another partition $\mathcal{B}_j$, and we can show the following:

Lemma 3. *$\mathcal{F}_i$ is $\mathcal{B}_j$-conglomerable iff $\mathcal{R}_i^{B_i}$ is $\mathcal{B}_j$-conglomerable $\forall B_i \in \mathcal{B}_i$.*

The natural extension of the union of the related sets $\mathcal{F}_1, \dots, \mathcal{F}_m$ is equal to $\mathcal{F}_1 \oplus \dots \oplus \mathcal{F}_m := \{\sum_{i=1}^m f_i : f_i \in \mathcal{F}_i, i = 1, \dots, m\}$, taking into account that all of these sets are coherent. We shall denote by $\mathcal{F}$ the $\mathcal{B}_{1:m}$-conglomerable natural extension of $\cup_{i=1}^m \cup_{B_i \in \mathcal{B}_i} \mathcal{R}_i^{B_i}$, provided that it exists. It can be checked that this set $\mathcal{F}$ is also the $\mathcal{B}_{1:m}$-conglomerable natural extension of $\cup_{i=1}^m \mathcal{F}_i$.

Theorem 5. *1. $\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$ are conglomerable coherent if and only if the $\mathcal{B}_{1:m}$-conglomerable natural extension $\mathcal{F}$ of $\cup_{i=1}^m \mathcal{F}_i$ exists and it induces them by means of Eq.* (1).
2. If $\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$ are conglomerable coherent, then $\mathcal{F}_i$ is $\mathcal{B}_j$-conglomerable for all i, j in $\{1, \dots, m\}$, and $\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$ are coherent.

At this point we have characterised some important relationships between coherence and conglomerable coherence. Yet, we have not addressed the most important issue: whether or not these two notions are equivalent. The next example settles the problem showing that they are not, and hence—using Theorem 5—that conglomerable coherence is indeed stronger than coherence.

Example 1. Consider $\Omega := \mathbb{N}$, and a coherent lower prevision $\underline{P}$ which is not $\mathcal{B}$-conglomerable for some partition $\mathcal{B}$ of Ω but such that there exists a dominating $\mathcal{B}$-conglomerable linear prevision with $P(B) > 0$ for all $B \in \mathcal{B}$ (one such $\underline{P}$ is given in [2, Example 5]).

Let us define $\Omega_1 := \Omega \cup -\Omega$, and the partitions of Ω_1 $\mathcal{B}_1 := \{\Omega, -\Omega\}$ and $\mathcal{B}_2 := \{B \cup -B : B \in \mathcal{B}\}$. Define $\underline{P}(\cdot|\mathcal{B}_1)$ on $\mathcal{L}(\Omega_1)$ by $\underline{P}(f|\Omega) := \underline{P}(f_1)$ and $\underline{P}(f|-\Omega) := \underline{P}(f_2)$, where

$$f_1 : \Omega \to \mathbb{R} \quad \omega \hookrightarrow f(\omega) \qquad \text{and} \qquad f_2 : \Omega \to \mathbb{R} \quad \omega \hookrightarrow f(-\omega). \tag{3}$$

It follows from the coherence of $\underline{P}$ that $\underline{P}(\cdot|\mathcal{B}_1)$ is separately coherent.

From the linear prevision P on $\mathcal{L}$ considered above we can derive a linear prevision P_1 on $\mathcal{L}(\Omega_1)$ by $P_1(f) := P(f_1)$, where f_1 is given by Eq. (3). Then P_1 is a linear prevision satisfying $P_1(B \cup -B) = P(B) > 0$ for any $B \in \mathcal{B}$, and moreover $P_1(\Omega) = 1$. Define $P_1(\cdot|\mathcal{B}_2)$ by GBR. Then it can be checked that $P_1, P_1(\cdot|\mathcal{B}_2)$ are coherent. On the other hand, consider $\underline{P}_1(\cdot|-\Omega) := \underline{P}(\cdot|-\Omega)$ and define $P_1(\cdot|\Omega)$ from P_1 by GBR. Then $P_1, \underline{P}_1(\cdot|\mathcal{B}_1)$ are coherent, and applying Lemma 1 we deduce that $\underline{P}_1(\cdot|\mathcal{B}_1), P_1(\cdot|\mathcal{B}_2)$ are coherent.

Similarly, if we consider the linear prevision P_2 on $\mathcal{L}(\Omega_1)$ given by $P_2(f) := P(f_2)$, we can repeat the above reasoning and define $P_2(\cdot|\mathcal{B}_2)$ and $P_2(\cdot|-\Omega)$ by GBR, and let $\underline{P}_2(\cdot|\Omega)$ be equal to $\underline{P}(\cdot|\Omega)$ and we conclude that $\underline{P}_2(\cdot|\mathcal{B}_1), P_2(\cdot|\mathcal{B}_2)$ are coherent. By taking lower envelopes, we obtain coherent $\underline{Q}(\cdot|\mathcal{B}_1), \underline{Q}(\cdot|\mathcal{B}_2)$ (see [3, Theorem 7.1.6]), and the above construction implies that $\underline{Q}(\cdot|\mathcal{B}_1) = \underline{P}(\cdot|\mathcal{B}_1)$.

Now, assume ex-absurdo that $\underline{P}(\cdot|\mathcal{B}_1), \underline{Q}(\cdot|\mathcal{B}_2)$ are conglomerable coherent. Then Theorem 5(2) implies that the set $\mathcal{F}_1$ induced by $\underline{P}(\cdot|\mathcal{B}_1)$ is $\mathcal{B}_2$-conglomerable, and Lemma 3 implies then that $\mathcal{R}_1^{\Omega}$ is $\mathcal{B}_2$-conglomerable. But

$$\mathcal{R}_1^{\Omega} = \{G(f|\Omega) + \varepsilon\Omega : f \in \mathcal{L}(\Omega_1), \varepsilon > 0\} \cup \{f \in \mathcal{L}(\Omega_1) : f = \Omega f \gneq 0\}$$

is in a one-to-one correspondence with the set of strictly desirable gambles induced by $\underline{P}$. From Theorem 1, since $\underline{P}$ is not $\mathcal{B}$-conglomerable its associated set of strictly desirable gambles is not $\mathcal{B}$-conglomerable; from this we can deduce that $\mathcal{R}_1^{\Omega}$ is not $\mathcal{B}_2$-conglomerable, whence neither is $\mathcal{F}_1$ and as a consequence $\underline{P}(\cdot|\mathcal{B}_1), \underline{Q}(\cdot|\mathcal{B}_2)$ cannot be conglomerable coherent. ♦

This finding is important because it tells us that Walley's notion of coherence does not entirely capture the implications of conglomerability (as they would follow, for example, from the axioms in [3, Appendix F1]), and in this sense it is an approximation to the theory of conglomerable coherent lower previsions. In the next section we show that such an approximation becomes exact in some important special cases.

4 Particular cases

From Lemma 2, if $\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$ are weakly coherent, then there is an unconditional lower prevision $\underline{P}$ that is pairwise coherent with them. This connects weak coherence and conglomerability:

Theorem 6. *$\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$ are weakly coherent if and only if there are coherent sets $\mathcal{R}, \mathcal{F}_1, \dots, \mathcal{F}_m$ and a coherent lower prevision $\underline{P}$ such that for all $i = 1, \dots, m$ $\mathcal{R} \cup \mathcal{F}_i$ is $\mathcal{B}_i$-conglomerable coherent and it induces $\underline{P}, \underline{P}(\cdot|\mathcal{B}_i)$.*

However, since the coherence of $\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$ is a stronger notion that their weak coherence, we have that conglomerable coherence implies weak coherence by Theorem 5(2) and that the converse is not true by Example 1.

Now, if the lower prevision $\underline{P}$ satisfies $\underline{P}(B) > 0$ for all $B \in \mathcal{B}_1 \cup \dots \cup \mathcal{B}_m$ (whence $\mathcal{B}_1, \dots, \mathcal{B}_m$ can at most be countable), we deduce that $\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$ are conglomerable coherent:

Theorem 7. *Let $\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$ be separately coherent conditional lower previsions which are weakly coherent with some coherent lower prevision $\underline{P}$ satisfying that $\underline{P}(B) > 0$ for all $B \in \mathcal{B}_1 \cup \dots \cup \mathcal{B}_m$. Then:*

1. *$\mathcal{F}_i$ is $\mathcal{B}_j$-conglomerable for $i, j = 1, \dots, m$.*
2. *$\mathcal{F}_1 \oplus \dots \oplus \mathcal{F}_m$ is $\mathcal{B}_{1:m}$-conglomerable.*
3. *$\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$ are conglomerable coherent.*

Another interesting case where coherence and conglomerable coherence are equivalent is when we condition on partitions that are nested. Let $\mathcal{B}_1, \dots, \mathcal{B}_m$ be partitions of Ω such that $\mathcal{B}_j$ is finer than $\mathcal{B}_{j-1}$ for all $j = 2, \dots, m$, and let $\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$ be separately coherent conditional lower previsions.

Theorem 8. *$\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$ are coherent if and only if for all gambles $f \in \mathcal{L}$, $B_{j-1} \in \mathcal{B}_{j-1}$, $B_j \in \mathcal{B}_j$, it holds that $\underline{P}(G(f|B_j)|B_{j-1}) = 0$ and $\underline{P}(G(f|\mathcal{B}_j)|B_{j-1}) \geq 0$. Moreover, if $\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$ are coherent, then $\mathcal{F}_i$ and $\mathcal{F}_1 \oplus \dots \oplus \mathcal{F}_m$ are $\mathcal{B}_{1:m}$-coherent for all $i = 1, \dots, m$, and $\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$ are conglomerable coherent.*

5 Conclusions

This paper allows us to say a few conclusive words about the quest for a general behavioural theory of coherent (conditional) lower previsions. Now we know that Walley's theory does not consider all the implications of conglomerability and should better be understood as an approximation to the theory of conglomerable coherent lower previsions we have proposed here.

On the other hand, we have shown that in some special cases we can use Walley's theory in order to obtain the same outcomes as with conglomerable coherence: when the conditioning events have positive lower probability, or when the conditioning partitions are nested. Both cases are important in the applications of probability.

In our view, the most important next step to do is to try to make the new theory of practical use in general, not only in the cases already addressed in this paper. To this end, there is a main obstacle to overcome: the computation of the conglomerable natural extension of a set of desirable gambles. We know from [2] that we can approximate it by a sequence of sets, but we neither know whether it is attained in the limit nor whether the sequence is finite. This is the main challenge that has to be faced in future work.

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