NOTES ON DESIRABILITY AND CONDITIONAL LOWER PREVISIONS

ENRIQUE MIRANDA AND MARCO ZAFFALON

Abstract. We detail the relationship between sets of desirable gambles and conditional lower previsions. The former is one the most general models of uncertainty. The latter corresponds to Walley's celebrated theory of imprecise probability. We consider two avenues: when a collection of conditional lower previsions is derived from a set of desirable gambles, and its converse. In either case, we relate the properties of the derived model with those of the originating one. Our results constitute basic tools to move from one formalism to the other, and thus to take advantage of work done in the two fronts.

1. Introduction

Background and motivation. Uncertainty is at the very heart of much of artificial intelligence (AI), and so are the many theories and models proposed to deal with it. Among these, a central role is played by probabilistic theories, and in particular by Bayesian theory. On the other hand, the very general kind of uncertainty handling needed by AI has favored the emergence of theories able to deal with uncertainty more flexibly than traditional probability. In fact, recent years have seen more and more work being devoted to theories of so-called *imprecise probability*.[1] The common ground among them is the attempt to represent and deal with probabilities that can be imprecisely specified, for example by using sets of probabilities. Introducing imprecision has enabled probability to cope effectively with qualitative uncertainty statements, with incomplete (or missing) information, and to naturally embed robustness in its inferences.

A prominent theory of imprecise probability is Walley's behavioral theory of *coherent lower previsions* [19]. The distinguishing feature of this theory is its being founded on a rationality axiom: *coherence*. Coherence ensures that probabilistic inferences made under such a theory are self-consistent. Walley's theory shares this property with the Bayesian theory, of which it can actually be regarded as a generalisation: in fact, a coherent lower prevision is a lower expectation functional, which is in one-to-one correspondence with a closed convex set of probability distributions (that is, a so-called *credal set*). The approach by Walley includes also as particular cases most of the other imprecise probability models appearing in the literature (when these are interpreted as sets of probabilities and used coherently), such as possibility measures, belief functions, Choquet capacities and coherent lower and upper probabilities and expectations. Coherence is also at the basis of the inferential procedure called the *natural extension*, which allows one to derive, in a very general sense, probabilistic conclusions from probabilistic premises. A few

[1] See `http://www.sipta.org`.

special cases of the natural extension are logical deduction, de Finetti's fundamental theorem of probability, Lebesgue integration of a probability measure, Choquet integration of 2-monotone lower probabilities, Bayes' rule for probability measures and robust Bayesian analysis.

As Walley himself has repeatedly remarked (see [19, Appendix F], [21]), there is another theory which is even more general than coherent lower previsions: the theory of (coherent) *sets of desirable gambles*. In the theory of desirable gambles, a subject expresses his uncertainty about the outcome of an experiment through a set of gambles that he would accept. As with coherent lower previsions, a rationality axiom of coherence is imposed on a set of desirable gambles, and a related procedure of natural extension is defined as well.

Despite the unusual form as models of uncertainty, sets of desirable gambles have a number of remarkable properties. One of the most appealing is their inherent conceptual simplicity: for example, the founding notion of coherence, which in the case of coherent lower previsions is quite technical, becomes very clear and intuitive; updating a set of desirable gambles in the light of new evidence simply corresponds to focus on a subset of the original gambles. Also, we can derive coherent lower previsions easily from sets of desirable gambles, and moreover these are indeed more expressive than coherent lower previsions. This is obvious when we come to updating: coherent lower previsions, as well as traditional probability models, are not suited to update beliefs conditional on an event of probability zero.[2] The reason is that by its very definition Bayes' rule cannot be applied (or, stated alternatively, it leads to uninformative conclusions). On the other hand, the extra expressivity of desirable gambles permits to obtain meaningful conclusions also when the conditioning event has lower, and even upper, probability equal to zero.

The idea of using desirable gambles as models for uncertainty dates back to the early sixties from Smith's work [17], but it was elaborated and formalised with all the main modern ideas only in 1975 in the important work of Williams [24]. Then it was reconsidered by Walley, who used desirable gambles as a building block of his theory of coherent lower previsions. Walley also explored to a large extent (see [19, Section 3.7]) the relationship between desirability and unconditional lower previsions, showing in particular that a special type of desirable gambles, called *almost-desirable*, are models equivalent to unconditional lower previsions. He also introduced the related notions of *real* and *strict desirability*. More recent work has been done by Moral [15], who studied notions of irrelevance for desirable gambles; by de Cooman and Miranda [4], who made a general study of transformational symmetry assessments for desirable gambles; by Couso and Moral [1], who discussed the relationship with credal sets, computer representation, and maximal sets of desirable gambles; and finally by de Cooman and Quaeghebeur [5], who studied exchangeability in the framework of desirable gambles, and who also introduced the new notion of *weak desirability*.

Problems and contributions. Although the theory of desirable gambles has recently experienced a boost in research, some of its basic features are still relatively unexplored. This is the case of the relationship between sets of desirable gambles and *conditional* lower previsions, which is particularly important to relate, and take

[2]Some reasons why it is important to allow for conditioning on such events, in particular with imprecise probability models, are discussed in the Introduction of another paper [14].

advantage of, work done in the two fronts. This paper is a technical set of notes that detail such a relationship.

Consider a space of possibilities Ω that lists all the possible outcomes ω of an experiment. Consider a further experiment whose possible outcomes are the elements of a partition $\mathcal{B}$ of Ω. By yielding a set $B \in \mathcal{B}$, the latter experiment tells us something about where the outcome of the first experiment is going to be in Ω. Let $\mathcal{L}(\Omega)$ be the set of all *gambles* (bounded random variables) $f : \Omega \to \mathbb{R}$. A *conditional* coherent lower prevision $\underline{P}(\cdot|B)$, defined on a subset $\mathcal{K}$ of $\mathcal{L}(\Omega)$, is a conditional lower expectation functional equivalent to a (closed and convex) set of distributions conditional on some $B \in \mathcal{B}$. We summarise the conditional information through the gamble $\underline{P}(\cdot|\mathcal{B})$ which takes the value $\underline{P}(\cdot|B)$ on the elements of B; and we call $\underline{P}(\cdot|\mathcal{B})$, too, a conditional coherent lower prevision. The above reasoning can be repeated for different partitions $\mathcal{B}_1, \dots, \mathcal{B}_m$ of Ω, therefore it is common to take a collection of conditional coherent lower previsions $\underline{P}_1(\cdot|\mathcal{B}_1), \dots, \underline{P}_m(\cdot|\mathcal{B}_m)$ as the basic modelling unit in Walley's theory.

We can consider two situations at this point:

(1) In the first, we start from a set of desirable gambles $\mathcal{R}$, and given partitions $\mathcal{B}_1, \dots, \mathcal{B}_m$ of Ω, we derive from $\mathcal{R}$ a collection of conditional lower previsions $\underline{P}_1(\cdot|\mathcal{B}_1), \dots, \underline{P}_m(\cdot|\mathcal{B}_m)$.
(2) In the second, starting from a collection of conditional lower previsions $\underline{P}_1(\cdot|\mathcal{B}_1), \dots, \underline{P}_m(\cdot|\mathcal{B}_m)$ we derive a set of desirable gambles $\mathcal{R}$.

In this paper we study, in both cases (1) and (2), the relation between the properties of the derived model and those of the originating one.

As we have said, a thorough study of this type has already been done by Walley in the unconditional case [19, Section 3.7], which corresponds to focusing on the single trivial partition $\mathcal{B} = \{\Omega\}$. In the much more involved case where we focus on multiple partitions $\mathcal{B}_1, \dots, \mathcal{B}_m$, the contributions so far are basically confined within the work of Williams [24]. Williams has, in particular, focused mostly on case (1) above. Case (2) is nearly unexplored to date (although part of Walley's work is somewhat implicitly relying on it).

We start our work in Section 2, where we give background notions on conditional coherent lower previsions. We define a number of consistency conditions, already introduced by Walley: *separate coherence*, *avoiding uniform sure loss*, *avoiding partial loss*, *weak coherence*, and finally (*joint* or *strong*) *coherence*. We define also the natural extension of a collection of conditional coherent lower previsions, and prove some of its properties. Then we introduce the special case of coherent lower previsions called *linear previsions*, which are expectation functionals, that is, precise-probability models. Moreover, we discuss the updating of an unconditional coherent lower prevision in the form of a generalised Bayes' rule.

In doing all this, we take a very general stance: we do not place any restriction of the cardinality of the set Ω nor on the domain of the conditional coherent lower previsions. This is in contradistinction with Walley's (and Williams') work, which in the conditional case is restricted to domains being linear spaces, and is instead in the same spirit of more recent work [13, 16]. On the other hand, throughout the paper we do place the restriction that each conditioning partition of Ω be finite. We do this to avoid entering the controversy concerning how to deal with infinite partitions: in fact, Walley's approach relies on an axiom of so-called *conglomerability* [19, Section 6.8 and Appendix F], which is an important point of disagreement with Williams' and de Finetti's work (see also [24] and [16]).

In Section 3 we introduce *sets of really desirable gambles*, which is our main model through the paper, and define for this model the conditions of *avoiding partial loss*, *coherence*, and the procedure of *natural extension*. We provide some basic results, and introduce the related notions of *almost* and *strictly desirability*.

The main corpus of our work starts in Section 4, where we address case (1) above. We detail the conditions of the set $\mathcal{R}$ that enable the derived conditional lower previsions to be, in turn, well-defined (i.e., bounded), separately coherent, avoiding partial loss, and (jointly) coherent. We also highlight here the natural connections of these results with almost-desirability. Moreover, we discuss which properties of $\mathcal{R}$ can affect the domain of the derived lower previsions, and show under which conditions $\mathcal{R}$ yields linear previsions. This part of the paper is the closest to Williams' work, which is partly generalised here.

The work continues in Section 5, where we address point (2). There we define the set $\mathcal{R}$ of really desirable gambles that is derived from a collection of separately coherent conditional lower previsions $\underline{P}_1(\cdot|\mathcal{B}_1), \dots, \underline{P}_m(\cdot|\mathcal{B}_m)$, and provide two equivalent formulations for its natural extension. On this basis, we show that $\underline{P}_1(\cdot|\mathcal{B}_1), \dots, \underline{P}_m(\cdot|\mathcal{B}_m)$ avoid partial (or uniform sure) loss if and only if $\mathcal{R}$ does. Moreover, we show in Theorem 11 a somewhat unexpected result: that while a coherent collection $\underline{P}_1(\cdot|\mathcal{B}_1), \dots, \underline{P}_m(\cdot|\mathcal{B}_m)$ yields a coherent set $\mathcal{R}$, the coherence of $\mathcal{R}$ alone does not imply that the originating collection is coherent. Further results detail the link between the natural extension of the collection and that of the derived set $\mathcal{R}$.

These results are used in Section 6. We first discuss commutativity, which is something that Williams had already explored in his framework: what happens if we start from a collection $\underline{P}_1(\cdot|\mathcal{B}_1), \dots, \underline{P}_m(\cdot|\mathcal{B}_m)$, derive a set of desirable gambles $\mathcal{R}$, and from this we derive a new collection $\underline{P}'_1(\cdot|\mathcal{B}_1), \dots, \underline{P}'_m(\cdot|\mathcal{B}_m)$? What happens if we go the other way around? We show that in the first case we re-obtain the original collection, and moreover that the logical implications of set $\mathcal{R}$ coincide with those of the collection (so that it is possible to always work entirely in the framework of desirable gambles), while in the second case there is a loss of information. We then move on to deepen the properties of sets of desirable gambles that make them more expressive than collections of conditional coherent lower previsions: in this case, we describe the special sets of desirable gambles that are as expressive as collections of conditional lower previsions. When Ω is finite, this result is particularly revealing: it shows that there is a kind of information carried by sets of desirable gambles that cannot be disclosed through any conditional probabilistic statement. As we shall see, such a kind of information has to do with modelling preferences.

After drawing some conclusions in Section 7 on the problems studied and discussing open problems, we give some additional results in Appendix A. These results are related to the main discussion in the paper but are also somewhat less central to it. Thus, we have preferred to collect them separately in order to allow the reader to follow more easily the main discussion. The first result, relating to some recent work [1], shows which properties of $\mathcal{R}$ lead to conditional lower previsions that coincide with those obtained through the regular extension, which is a special updating procedure for lower previsions; the second investigates some properties of the concept of weak desirability, which has recently been introduced in [5], and its relationship with the results in this paper.

Finally, and again for clarity, in Appendix B we have collected all the proofs of the results in the paper.

2. Coherent lower previsions

Let us give a short introduction to the concepts and results from the behavioural theory of imprecise probabilities that we shall use in the rest of the paper. We refer to [19] for an in-depth study of these and other properties, and to [11] for a brief survey.

2.1. **The behavioural interpretation.** Given a possibility space Ω, a *gamble* f is a bounded real-valued function on Ω. This function represents a random reward $f(\omega)$, which depends on the a priori unknown value ω of Ω. We shall denote by $\mathcal{L}(\Omega)$ the set of all gambles on Ω, or by $\mathcal{L}$ when there is no confusion about the possibility space we are dealing with, and by $\mathcal{L}^+ := \{f \in \mathcal{L} : f \gneq 0\}$ the set of non-negative gambles different from zero.[3] A *lower prevision* $\underline{P}$ is a real functional defined on some set of gambles $\mathcal{K} \subseteq \mathcal{L}(\Omega)$. It is used to represent a subject's supremum acceptable buying prices for these gambles, in the sense that for all $\varepsilon > 0$ and all f in $\mathcal{K}$ the subject is disposed to accept the uncertain reward $f - \underline{P}(f) + \varepsilon$.

From any lower prevision $\underline{P}$ we can define an upper prevision $\overline{P}$ using conjugacy: $\overline{P}(f) := -\underline{P}(-f)$ for any gamble f in $-\mathcal{K} := \{f : -f \in \mathcal{K}\}$. $\overline{P}(f)$ can be interpreted as the infimum acceptable selling price for the gamble f. Because of this relationship, it will suffice for the purposes of this paper to concentrate on lower previsions for the most part.

We can also consider the supremum buying prices for a gamble, *conditional* on a non-empty subset of Ω. Given such a set B and a gamble f on Ω, the lower prevision $\underline{P}(f|B)$ represents the subject's supremum acceptable buying price for the gamble f, provided he later comes to know that the unknown value ω belongs to B, and nothing else. Equivalently, it can also be seen as the supremum value of δ for which our subject is disposed to accept the transaction given by the gamble $B(f - \delta)$,[4] where to simplify the notation we use B to denote also the indicator function $\mathbb{I}_B$ of the set B. If we consider a partition $\mathcal{B}$ of Ω (for instance a set of categories), then we shall represent by $\underline{P}(f|\mathcal{B})$ the gamble on Ω that takes the value $\underline{P}(f|B)$ if and only if ω belongs to the element B of the partition $\mathcal{B}$. The functional $\underline{P}(\cdot|\mathcal{B})$ that maps any gamble f on its domain into the gamble $\underline{P}(f|\mathcal{B})$ is called a *conditional lower prevision.* To any conditional lower prevision $\underline{P}(\cdot|\mathcal{B})$ we can associate a *conditional upper prevision* $\overline{P}(\cdot|\mathcal{B})$ by $\overline{P}(f|\mathcal{B}) := -\underline{P}(-f|\mathcal{B})$. It will represent the infimum acceptable selling price for the gamble f, contingent on the element of the partition $\mathcal{B}$ that we observe.

A gamble f on Ω is called $\mathcal{B}$*-measurable* when it is constant over the elements of $\mathcal{B}$. This is for instance the case of the conditional lower prevision $\underline{P}(f|\mathcal{B})$.

[3] In this paper we shall use $f < g$ to denote that $f(\omega) < g(\omega)$ for all $\omega \in \Omega$, and $f \leq g$ when $f(\omega) \leq g(\omega)$ for all $\omega \in \Omega$. The notation $f \lneq g$ (often adopted when either $f = 0$ or $g = 0$) is used in the case $f \leq g, f \neq g$, and similarly $f \gneq g$ means that $f \geq g, f \neq g$.

[4] These are called the *updated* and *contingent* interpretations of the conditional prevision, and represent our subject's beliefs *at the present time*, even if they take into account future scenarios. See [19, Section 6.1] or [11, Section 3.1] for more details.

We shall also use the notations

$$G(f|B) := B(f - \underline{P}(f|B)),\ G(f|\mathcal{B}) := \sum_{B\in\mathcal{B}} G(f|B) = f - \underline{P}(f|\mathcal{B})$$

for all f in the domain of $\underline{P}(\cdot|\mathcal{B})$ and all $B \in \mathcal{B}$. By $G(f|B)$ we represent the transaction where the gamble f is bought at the price $\underline{P}(f|B)$ contingent on B, and which is called off otherwise. In the case of an unconditional lower prevision $\underline{P}$, we shall let $G(f) := f - \underline{P}(f)$ for any gamble f in its domain. In our notation this is equivalent to have a conditional lower prevision $\underline{P}(\cdot|\mathcal{B})$ with $\mathcal{B} = \{\Omega\}$. Moreover, in this case $G(f|\Omega) = G(f)$.

These assessments modelled by a conditional lower prevision $\underline{P}(\cdot|\mathcal{B})$ can be made for many different partitions of Ω, and therefore it is not uncommon to model a subject's beliefs using a finite number of different conditional lower previsions. We should verify then that all the assessments modelled by these conditional lower previsions are coherent with one another. In this section we review the different consistency criteria. We give the particular definitions of these notions for finite partitions, which will be the ones considered in this paper, and refer to [13, 19] for more general definitions of these notions.

2.2. **Separate coherence.** The first requirement we make is that for any partition $\mathcal{B}$, the conditional lower prevision $\underline{P}(\cdot|\mathcal{B})$ defined on a subset $\mathcal{H}$ of $\mathcal{L}$ should be separately coherent.

Definition 1 (**Separate coherence**). A conditional lower prevision $\underline{P}(\cdot|\mathcal{B})$ with domain $\mathcal{H}$ is *separately coherent* if for every $B \in \mathcal{B}$, the gamble B belongs to $\mathcal{H}$ and $\underline{P}(B|B) = 1$, and moreover

$$\sup_{\omega\in B}\left[\sum_{j=1}^{n}\lambda_j G(f_j|B) - G(f_0|B)\right](\omega) \geq 0$$

for every $n \geq 0, j = 1,\ldots,n, f_j \in \mathcal{H}, \lambda_j \geq 0, f_0 \in \mathcal{H}$.

Separate coherence means that contingent on B the supremum acceptable buying price for a gamble f_0 cannot be raised by taking into account other acceptable transactions, and also that we should be prepared to bet on B at all odds after having observed it.

When the domain $\mathcal{H}$ is a linear set of gambles (i.e., closed under addition and under multiplication by real numbers), there is a simpler characterisation of separate coherence:

Theorem 1. *[19, Theorem 6.2.7] If the domain $\mathcal{H}$ of $\underline{P}(\cdot|\mathcal{B})$ is a linear set of gambles that contains all gambles $B \in \mathcal{B}$, then $\underline{P}(\cdot|\mathcal{B})$ is separately coherent if and only if the following conditions are satisfied for all $B \in \mathcal{B}, f, g \in \mathcal{H}$, and $\lambda > 0$:*

$$\underline{P}(f|B) \geq \inf_{\omega\in B} f(\omega), \tag{SC1}$$

$$\underline{P}(\lambda f|B) = \lambda\underline{P}(f|B), \tag{SC2}$$

$$\underline{P}(f+g|B) \geq \underline{P}(f|B) + \underline{P}(g|B). \tag{SC3}$$

It is also useful for this paper to explicitly consider the particular case where $\mathcal{B} = \{\Omega\}$, that is, when we have unconditional information. We have then a(n *unconditional*) *lower prevision* $\underline{P}$ on a subset $\mathcal{K}$ of the set $\mathcal{L}$ of all gambles. Separate coherence is simply called then *coherence*:

Definition 2 (**Coherence**). An unconditional lower prevision $\underline{P}$ with domain $\mathcal{K}$ is *coherent* when

$$\sup_{\omega\in\Omega}\left[\sum_{j=1}^{n}\lambda_j G(f_j) - G(f_0)\right](\omega) \geq 0 \tag{1}$$

for every $n \geq 0, j = 1,\dots,n, f_j \in \mathcal{K}, \lambda_j \geq 0, f_0 \in \mathcal{K}$.

Its interpretation is similar to that of separate coherence, now with $B = \Omega$. Again, we can give a simpler characterisation in the case of linear domains:

Theorem 2. *[19, Section 2.3.3] Let $\underline{P}$ be a lower prevision defined on a linear set of gambles $\mathcal{K}$. It is coherent if and only if the following conditions hold for all $f, g \in \mathcal{K}$, and $\lambda > 0$:*

$$\underline{P}(f) \geq \inf f, \tag{C1}$$

$$\underline{P}(\lambda f) = \lambda \underline{P}(f), \tag{C2}$$

$$\underline{P}(f+g) \geq \underline{P}(f) + \underline{P}(g). \tag{C3}$$

Remark 1. It is possible to deduce from Definition 1 that given a separately coherent conditional lower prevision $\underline{P}(\cdot|\mathcal{B})$ with domain $\mathcal{H}$, we may assume without loss of generality that $\mathcal{H}$ contains all the gambles $\lambda f - \mu$, where $f \in \mathcal{H}$, $\lambda \geq 0$ and $\mu \in \mathbb{R}$, and moreover that for each $f_B \in \mathcal{H}$, $B \in \mathcal{B}$, also the gamble $\sum_{B\in\mathcal{B}} Bf_B$ belongs to $\mathcal{H}$ (see [19, Lemma 6.2.4 and Section 6.2.6]). The above assumptions imply that the $\mathcal{B}$-measurable gambles are in $\mathcal{H}$. Two other useful consequences that we shall use repeatedly in the rest of the paper are the following:

- for all $f \in \mathcal{H}$, both $G(f|B)$ and $G(f|\mathcal{B})$ belong to $\mathcal{H}$;
- for all $f \in \mathcal{H}, \lambda \geq 0$, $\lambda G(f|B) = G(\lambda f|B)$ and $\lambda G(f|\mathcal{B}) = G(\lambda f|\mathcal{B})$ (note that the gamble λf belongs to $\mathcal{H}$).

The second point, in particular, will allow us to simplify the notation by removing the λ-coefficients from many formulae. ♦

2.3. **Avoiding partial and uniform sure loss.** Let $\mathcal{B}_1,\dots,\mathcal{B}_m$ be finite partitions of Ω and let $\underline{P}_1(\cdot|\mathcal{B}_1),\dots,\underline{P}_m(\cdot|\mathcal{B}_m)$ be separately coherent conditional lower previsions whose respective domains are subsets $\mathcal{H}^1,\dots,\mathcal{H}^m$ of $\mathcal{L}$.

Definition 3 (**Avoiding uniform sure loss**). The conditional lower previsions $\underline{P}_1(\cdot|\mathcal{B}_1),\dots,\underline{P}_m(\cdot|\mathcal{B}_m)$ *avoid uniform sure loss* if for every $j = 1,\dots,m, n_j \geq 1, k = 1,\dots,n_j, g_j^k \in \mathcal{H}^j$,

$$\sup_{\omega\in\Omega}\left[\sum_{j=1}^{m}\sum_{k=1}^{n_j} G_j(g_j^k|\mathcal{B}_j)\right](\omega) \geq 0.$$

The intuition behind this notion is that there at least should exist a possibility for the subject to not lose any utiles from the transactions that the subject has accepted, so a combination of transactions which are acceptable to our subject should not make him lose utiles for all the outcomes of the experiment. It is based on the rationality requirement that a gamble $f < 0$ should not be desirable.

A related stronger notion that restricts the set where we take the supremum in the definition above is called avoiding partial loss. In order to introduce it, we need to give first the notion of support.

Definition 4 (**$\mathcal{B}$-support**). Define the $\mathcal{B}$-*support* $S(f)$ of a gamble f in $\mathcal{L}$ as

$$S(f) := \{B \in \mathcal{B} : Bf \neq 0\}, \tag{2}$$

i.e., it is the set of conditioning events for which the restriction of f is not identically equal to the zero gamble.

Definition 5 (**Avoiding partial loss**). We say that a number of conditional lower previsions $\underline{P}_1(\cdot|\mathcal{B}_1), \dots, \underline{P}_m(\cdot|\mathcal{B}_m)$ *avoid partial loss* if for every $j = 1, \dots, m, n_j \geq 1, k = 1, \dots, n_j, g_j^k \in \mathcal{H}^j$, such that not all the g_j^k are zero gambles,

$$\sup_{\omega \in \mathbb{S}(g_j^k)} \left[\sum_{j=1}^{m} \sum_{k=1}^{n_j} G_j(g_j^k|\mathcal{B}_j) \right](\omega) \geq 0,$$

where by $\mathbb{S}(g_j^k) := \bigcup \cup_{j=1}^m \cup_{k=1}^{n_j} S_j(g_j^k)$ we mean the set of elements that belong to some set in $S_j(g_j^k)$ for some $j \in \{1, \dots, m\}, k \in \{1, \dots, n_j\}$.

With this stronger notion, we also reject the possibility that a combination of acceptable transactions make our subject lose utiles except in the set where all the transactions are equal to zero.

Remark 2. One might wonder whether the support of a gamble f could rather be defined as $S(f) := \{B \in \mathcal{B} : Bf \text{ not constant}\}$, because it is in these conditioning events where $G(f|B)$ is non-zero. Actually, it can be checked that the resulting condition of avoiding partial loss, which at first sight might seem stronger than the one in Definition 5, is equivalent to it provided that the domains are rich enough, in the sense that the domain of $\underline{P}(\cdot|\mathcal{B})$ contains all $\mathcal{B}$-measurable gambles (and this is something we have assumed in Section 2.2, see Remark 1). Similar considerations can be made for the notion of coherence we shall introduce in Definition 7. ♦

In order to explore the connection between avoiding partial loss and the desirability of a set of gambles, we are going to use a number of properties. The first one is an adaptation of a result we established somewhere else [14, Proposition 4] for lower previsions conditional on variables:

Proposition 1. *Let $\underline{P}_1(\cdot|\mathcal{B}_1), \dots, \underline{P}_m(\cdot|\mathcal{B}_m)$ be conditional lower previsions with respective domains $\mathcal{H}^1, \dots, \mathcal{H}^m$. The following are equivalent:*

1. *$\underline{P}_1(\cdot|\mathcal{B}_1), \dots, \underline{P}_m(\cdot|\mathcal{B}_m)$ avoid partial loss.*
2. *For every $\varepsilon > 0, g_j^k \in \mathcal{H}^j, j = 1, \dots, m, n_j \geq 1, k = 1, \dots, n_j$, such that not all the g_j^k are zero gambles, it holds that*

$$\sup_{\omega \in \Omega} \left[\sum_{j=1}^{m} \sum_{k=1}^{n_j} G_j(g_j^k|\mathcal{B}_j) + \varepsilon S_j(g_j^k) \right](\omega) > 0. \tag{3}$$

3. *For every $\varepsilon > 0, g_j^k \in \mathcal{H}^j, j = 1, \dots, m, n_j \geq 1, k = 1, \dots, n_j$, such that not all the g_j^k are zero gambles, it holds that*

$$\sup_{\omega \in \mathbb{S}(g_j^k)} \left[\sum_{j=1}^{m} \sum_{k=1}^{n_j} G_j(g_j^k|\mathcal{B}_j) + \varepsilon S_j(g_j^k) \right](\omega) \geq 0. \tag{4}$$

Remark 3. In the particular case where the domains of the conditional lower previsions $\underline{P}_1(\cdot|\mathcal{B}_1),\dots,\underline{P}_m(\cdot|\mathcal{B}_m)$ are linear spaces, we can assume without loss of generality that $n_j = 1$ for $j = 1,\dots,m$ in Definitions 3, 5, and in Eq. (3): it suffices to take into account that, because of the super-additivity (SC3) of the conditional lower previsions, $\sum_{k=1}^{n_j} G_j(g_j^k|\mathcal{B}_j) \geq G_j(\sum_{k=1}^{n_j} g_j^k|\mathcal{B}_j)$, and this for every $j = 1,\dots,m$. Similar considerations hold for the conditions of weak and strong coherence we shall introduce next, and for the natural extension we shall define in Eq. (7). We shall use this to simplify some of the proofs in Appendix B. ♦

2.4. **Weak and strong coherence.** We next give two notions that generalise the concept of coherence in Eq. (1) from the unconditional to the conditional case:

Definition 6 (**Weak coherence**). Let $\underline{P}_1(\cdot|\mathcal{B}_1),\dots,\underline{P}_m(\cdot|\mathcal{B}_m)$ be separately coherent conditional lower previsions with respective domains $\mathcal{H}^1,\dots,\mathcal{H}^m$. We say that they are *weakly coherent* if for every $j = 1,\dots,m, n_j \geq 1, k = 1,\dots,n_j, g_j^k \in \mathcal{H}^j$, and for every $j_0 \in \{1,\dots,m\}, g_0 \in \mathcal{H}^{j_0}, B_0 \in \mathcal{B}_{j_0}$, it holds that

$$\sup_{\omega\in\Omega}\left[\sum_{j=1}^{m}\sum_{k=1}^{n_j} G_j(g_j^k|\mathcal{B}_j) - G_{j_0}(g_0|B_0)\right](\omega) \geq 0. \tag{5}$$

With this condition we require that our subject should not be able to raise his supremum acceptable buying price $\underline{P}_{j_0}(g_0|B_0)$ for a gamble g_0 contingent on B_0 by taking into account the implications of other conditional assessments: if Eq. (5) does not hold and the supremum is strictly negative then we can deduce that there is some $\varepsilon > 0$ such that $G_{j_0}(g_0|B_0) - \varepsilon$ is also a desirable gamble, which means that $\underline{P}_{j_0}(g_0|B_0) + \varepsilon$ is an acceptable buying price.

However, under the behavioural interpretation, a number of weakly coherent conditional lower previsions can still present some forms of inconsistency with one another. See [19, Chapter 7], [12] and [22] for some discussion. On the other hand, weak coherence neither implies nor is implied by the notion of avoiding partial loss. Because of these two facts, we consider another notion which is stronger than both, and which is called (*joint* or *strong*) *coherence*:[5]

Definition 7 (**Strong coherence**). Let $\underline{P}_1(\cdot|\mathcal{B}_1),\dots,\underline{P}_m(\cdot|\mathcal{B}_m)$ be separately coherent conditional lower previsions with respective domains $\mathcal{H}^1,\dots,\mathcal{H}^m$. We say that they are *coherent* if for every $j = 1,\dots,m, n_j \geq 1, k = 1,\dots,n_j, g_j^k \in \mathcal{H}^j$, and for every $j_0 \in \{1,\dots,m\}, g_0 \in \mathcal{H}^{j_0}, B_0 \in \mathcal{B}_{j_0}$, it holds that

$$\sup_{\omega\in\mathbb{S}(g_j^k)\cup B_0}\left[\sum_{j=1}^{m}\sum_{k=1}^{n_j} G_j(g_j^k|\mathcal{B}_j) - G_{j_0}(g_0|B_0)\right](\omega) \geq 0. \tag{6}$$

Remark 4. There is another approach to the notion of coherence for imprecise previsions which is earlier than Walley's, and that was developed by Peter Williams in [24]. It is based on the idea of deriving conditional previsions from sets of desirable gambles that satisfy a number of consistency axioms, as we shall do in Section 4.

Although there are some differences between the two approaches, in the context of this paper, where all the partitions are finite, the two of them are equivalent. This

[5]The distinction between this and the unconditional notion of coherence mentioned above will always be clear from the context.

means that we could have formulated the subsequent results concerning coherent conditional lower previsions using Williams' terminology. We have opted to use Walley's instead because his is the theory most widespread, and also because we shall use a number of concepts, such as weak coherence, which were established by him. Nevertheless, some of the results we shall prove in Sections 4 and 6.1 are already present in a similar form in Williams' work, and we shall remark it when it is the case. See [11, Section 5.2], [19, Appendix K] and [18] for a comparison between both approaches. ⧫

The coherence of a collection of conditional lower previsions implies their weak coherence; although the converse does not hold in general, it does in the particular case when we only have a conditional and an unconditional lower prevision $\underline{P}_1(\cdot|\mathcal{B}), \underline{P}_2$ with respective domains $\mathcal{H}, \mathcal{K}$. To see this, note that the union of the supports in Eq. (6) is Ω unless all the gambles from $\mathcal{K}$ considered in the equation are equal to the zero gamble, and then Eq. (6) would follow from the separate coherence of $\underline{P}_1(\cdot|\mathcal{B})$.

Similarly, when we have only one conditional and one unconditional lower prevision, the notions of avoiding partial loss and avoiding uniform sure loss become equivalent. In that case, they are referred to as *avoiding sure loss* in [19, Chapter 6].

2.5. **Linear previsions and envelope theorems.** Given a conditional lower prevision $\underline{P}(\cdot|\mathcal{B})$ with domain $\mathcal{H}$, we define its conjugate *conditional upper prevision* by $\overline{P}(f|B) := -\underline{P}(-f|B)$ for every $f \in -\mathcal{H} := \{-f : f \in \mathcal{H}\}$. As we said at the beginning of the section, the value $\overline{P}(f|B)$ can be interpreted as the infimum acceptable selling price for the gamble f contingent on B. When the supremum acceptable buying price for a gamble coincides with the infimum acceptable selling price, we obtain the so-called *conditional linear previsions*.

Definition 8 (**Linear conditional previsions**). We say that a conditional lower prevision $\underline{P}(\cdot|\mathcal{B})$ with domain[6] $\mathcal{L}$ is *linear* if and only if it is separately coherent and moreover $\underline{P}(f+g|B) = \underline{P}(f|B) + \underline{P}(g|B)$ for all $B \in \mathcal{B}$ and $f, g \in \mathcal{L}$.

Conditional linear previsions correspond to the case where a subject's supremum acceptable buying price (lower prevision) coincides with his infimum acceptable selling price (or upper prevision) for any gamble on the domain. When a separately coherent conditional lower prevision $\underline{P}(\cdot|\mathcal{B})$ is linear we shall denote it by $P(\cdot|\mathcal{B})$; in the unconditional case, we shall use the notation P. A number of conditional linear previsions are coherent if and only if they avoid partial loss; and they are weakly coherent if and only if they avoid uniform sure loss.

Conditional linear previsions correspond to conditional expectations with respect to a probability. In particular, an unconditional linear prevision P is the expectation with respect to the probability which is the restriction of P to events. They can be used to give a Bayesian sensitivity analysis interpretation of the notion of coherence:

Theorem 3. *[19, Section 3.3.3] Given an unconditional lower prevision $\underline{P}$ with domain $\mathcal{K}$, we shall denote the set of* dominating *linear previsions by*

$$\mathcal{M}(\underline{P}) := \{P : P(f) \geq \underline{P}(f)\ \forall f \in \mathcal{K}\}.$$

Then $\underline{P}$ is coherent if and only if it is the lower envelope of $\mathcal{M}(\underline{P})$.

[6]We shall always assume in this paper that the domain of a conditional linear prevision is the whole set $\mathcal{L}$.

The conjugate unconditional upper prevision $\overline{P}$ on $-\mathcal{K}$ is then the upper envelope of $\mathcal{M}(\underline{P})$, so $\overline{P}(f) = \sup\{P(f) : P \in \mathcal{M}(\underline{P})\}$.

Following [10], we shall call any closed[7] and convex[8] set of linear previsions a *credal set*; an instance is the set $\mathcal{M}(\underline{P})$. Similarly, for a conditional lower prevision $\underline{P}(\cdot|\mathcal{B})$ with domain $\mathcal{H}$, we define

$$\mathcal{M}(\underline{P}(\cdot|\mathcal{B})) := \{P(\cdot|\mathcal{B}) : P(f|B) \geq \underline{P}(f|B)\ \forall f \in \mathcal{H}, B \in \mathcal{B}\}.$$

Then a conditional lower prevision $\underline{P}(\cdot|\mathcal{B})$ is separately coherent if and only if it is the lower envelope of $\mathcal{M}(\underline{P}(\cdot|\mathcal{B}))$, meaning that

$$\underline{P}(f|B) = \min\{P(f|B) : P(\cdot|\mathcal{B}) \geq \underline{P}(\cdot|\mathcal{B})\}\ \forall f \in \mathcal{H}, B \in \mathcal{B}.$$

Its conjugate conditional upper prevision $\overline{P}(\cdot|\mathcal{B})$ will be then the upper envelope of $\mathcal{M}(\underline{P}(\cdot|\mathcal{B}))$.

The situation is more complicated when we have more than one conditional lower prevision. In [19, Section 8.1] Walley proved that when the partitions are finite and the domains are linear spaces, coherent $\underline{P}_1(\cdot|\mathcal{B}_1), \dots, \underline{P}_m(\cdot|\mathcal{B}_m)$ are always the envelope of a set $\{P_\gamma(\cdot|\mathcal{B}_1), \dots, P_\gamma(\cdot|\mathcal{B}_m) : \gamma \in \Gamma\}$ of dominating coherent conditional linear previsions. Here, Γ denotes simply a (possibly infinite) set of indexes, which serves to identify the conditional linear previsions from which we are taking the lower envelope. In [12], a similar property was established for weak coherence.

Another interesting particular case is the following:

Theorem 4. *[19, Theorem 6.5.4] Consider an unconditional lower prevision $\underline{P}$ on a linear set of gambles $\mathcal{K}$ and a separately coherent conditional lower prevision $\underline{P}(\cdot|\mathcal{B})$ on a linear set of gambles $\mathcal{H}$ satisfying the properties mentioned in Remark 1, at the end of Section 2.2, where $\mathcal{B}$ is a finite partition of Ω. If $\mathcal{K} \supseteq \mathcal{H}$ then $\underline{P}, \underline{P}(\cdot|\mathcal{B})$ are coherent if and only if, for all $f \in \mathcal{H}$ and all $B \in \mathcal{B}$,*

$$\underline{P}(G(f|B)) = 0. \tag{GBR}$$

Condition (GBR) is called the *Generalised Bayes Rule*. When $\underline{P}(B) > 0$, GBR can be used to determine the value $\underline{P}(f|B)$: it is then the *unique* value $\mu \in \mathbb{R}$ for which $\underline{P}(B(f - \mu)) = 0$ holds.

If $\underline{P}(B) = 0$ and $\overline{P}(B) > 0$, then any value of μ in the interval

$$\left[\inf_{\omega \in B} f(\omega), \inf_{P \geq \underline{P}, P(B) > 0} \frac{P(Bf)}{P(B)}\right]$$

satisfies that $\underline{P}(B(f-\mu)) = 0$, and can therefore be used to define $\underline{P}(f|B)$; the upper limit of the above interval is what we shall call in Appendix A.1 the *regular extension* of $\underline{P}$. To see this, denote $\mu_1 := \inf_{\omega \in B} f(\omega)$ and $\mu_2 := \inf_{P \geq \underline{P}, P(B) > 0} \frac{P(Bf)}{P(B)}$, and observe that for any linear prevision $P \geq \underline{P}$, $P(B(f - \mu_2)) = 0$ if $P(B) = 0$, and $P(B(f - \mu_2)) \geq P\left(B(f - \frac{P(Bf)}{P(B)})\right) = 0$ if $P(B) > 0$. As a consequence,

$$0 \leq \underline{P}(B(f - \mu_2)) \leq \underline{P}(B(f - \mu_1)) \leq (\sup_{\omega \in B} f(\omega) - \inf_{\omega \in B} f(\omega))\underline{P}(B) = 0.$$

[7]In the *weak* topology*, which is the smallest topology for which all the evaluation functionals given by $f(P) := P(f)$, where $f \in \mathcal{L}$, are continuous.

[8]That is, for all linear previsions P_1, P_2 in the set and all $\alpha \in (0, 1)$, the linear prevision $\alpha P_1 + (1 - \alpha)P_2$ also belongs to this set.

Hence, both μ_1 and μ_2, and as a consequence also any value between them, are possible values for $\underline{P}(f|B)$, since the resulting conditional prevision satisfies (GBR).

Finally, if $\overline{P}(B) = 0$ then following a similar reasoning we see that any real number μ satisfies $\underline{P}(B(f-\mu)) = 0$, and therefore we can take any value in $[\inf_{\omega\in B} f(\omega), \sup_{\omega\in B} f(\omega)]$ to define $\underline{P}(f|B)$ with a separately coherent $\underline{P}(\cdot|B)$.

If P and $P(\cdot|\mathcal{B})$ are linear previsions, they are coherent if and only if for all f, $P(f) = P(P(f|\mathcal{B}))$. This is equivalent to requiring that $P(f|B) = \frac{P(fB)}{P(B)}$ for all f and all $B \in \mathcal{B}$ with $P(B) > 0$ (see [19, Section 6.5.7]).

2.6. **Extension of conditional lower previsions.** We next show how to determine the behavioural consequences of the assessments modelled by some conditional lower previsions.

Definition 9 (**Natural extension**). Let $\underline{P}_1(\cdot|\mathcal{B}_1), \dots, \underline{P}_m(\cdot|\mathcal{B}_m)$ be separately coherent conditional lower previsions with domains $\mathcal{H}^i$ for $i = 1, \dots, m$. Their *natural extensions* to $\mathcal{L}$ are defined, for every gamble f and every $B_0 \in \mathcal{B}_{j_0}$, with $j_0 \in \{1, \dots, m\}$ by

$$\underline{E}_{j_0}(f|B_0) := \sup\Big\{\alpha : \exists n_j \geq 1, k = 1, \dots, n_j, g_j^k \in \mathcal{H}^j \text{ for } j = 1, \dots, m \text{ s.t.} \sup_{\omega \in \mathbb{S}(g_j^k)\cup B_0} \Big[\sum_{j=1}^{m}\sum_{k=1}^{n_j} G_j(g_j^k|\mathcal{B}_j) - B_0(f-\alpha)\Big](\omega) < 0\Big\}. \quad (7)$$

Although the previous definition only requires that $\underline{P}_1(\cdot|\mathcal{B}_1), \dots, \underline{P}_m(\cdot|\mathcal{B}_m)$ are separately coherent, in practice the procedure of natural extension is useful when these conditional lower previsions avoid partial loss. This is the reason why the assumption of avoiding partial loss is made in [19, Definition 8.1.1] and [13, Definition 6]. In that case, and in the context of this paper, where all the partitions are finite, the natural extensions are the smallest conditional lower previsions which are coherent and dominate $\underline{P}_1(\cdot|\mathcal{B}_1), \dots, \underline{P}_m(\cdot|\mathcal{B}_m)$. Moreover, they coincide with the initial assessments if and only if $\underline{P}_1(\cdot|\mathcal{B}_1), \dots, \underline{P}_m(\cdot|\mathcal{B}_m)$ are themselves coherent. Otherwise, they 'correct' the initial assessments taking into account the implications of the notions of coherence. This is made precise in Lemma 1 below.

Remark 5. The natural extension can actually be computed for any non-empty $B_0 \subseteq \Omega$, not only for the elements of the pre-existing partitions $\mathcal{B}_1, \dots, \mathcal{B}_m$. In other words, Eq. (7) can be employed to compute the logical implications of the assessments $\underline{P}_1(\cdot|\mathcal{B}_1), \dots, \underline{P}_m(\cdot|\mathcal{B}_m)$ on the beliefs about a gamble f conditional on *any* $B_0 \subseteq \Omega$, $B_0 \neq \emptyset$. To see this, consider that, irrespective of whether or not B_0 is an element of one of the pre-existing partitions, one can (i) consider any partition $\mathcal{B}_0$ that includes B_0, and (ii) define a new conditional lower prevision $\underline{P}_0(\cdot|\mathcal{B}_0)$, whose domain is a trivial one, such as the set of $\mathcal{B}_0$-measurable gambles. In this way, by the new conditional lower prevision, we are not adding any assessment that is not already implied by the former ones. In fact, it can be checked that $\underline{P}_0(\cdot|\mathcal{B}_0), \underline{P}_1(\cdot|\mathcal{B}_1), \dots, \underline{P}_m(\cdot|\mathcal{B}_m)$ avoid partial loss if and only if $\underline{P}_1(\cdot|\mathcal{B}_1), \dots, \underline{P}_m(\cdot|\mathcal{B}_m)$ do. Then Eq. (7) can be readily used to compute $\underline{E}_0(f|B_0)$. Moreover, $\underline{E}_0(f|B_0)$ does not depend on the specific partition $\mathcal{B}_0$ considered, provided that it contains B_0. For this reason, in the following we shall sometimes adopt such an extended view of the natural extension referring directly to $\underline{E}_0(f|B_0)$ irrespective of whether B_0 is in a pre-existing partition or not. ♦

Some useful properties of the natural extension are the following:[9]

Lemma 1. *Let $\underline{P}_1(\cdot|\mathcal{B}_1),\dots,\underline{P}_m(\cdot|\mathcal{B}_m)$ be separately coherent conditional lower previsions.*

1. $\underline{E}_j(f_j|B_j) \geq \underline{P}_j(f_j|B_j)$*, for every* $j=1,\dots,m, f_j \in \mathcal{H}^j, B_j \in \mathcal{B}_j$.
2. $\underline{P}_1(\cdot|\mathcal{B}_1),\dots,\underline{P}_m(\cdot|\mathcal{B}_m)$ *avoid partial loss if and only if for all* $f \in \mathcal{L}$*, and all non-empty* $B_0 \subseteq \Omega$*, it holds that* $\underline{E}_0(f|B_0) < +\infty$.
3. *If* $\underline{P}_1(\cdot|\mathcal{B}_1),\dots,\underline{P}_m(\cdot|\mathcal{B}_m)$ *avoid partial loss, then* $\underline{E}_1(\cdot|\mathcal{B}_1),\dots,\underline{E}_m(\cdot|\mathcal{B}_m)$ *are coherent.*
4. *If* $\underline{P}'_1(\cdot|\mathcal{B}_1),\dots,\underline{P}'_m(\cdot|\mathcal{B}_m)$ *are coherent lower previsions on* $\mathcal{L}$ *that dominate* $\underline{P}_1(\cdot|\mathcal{B}_1),\dots,\underline{P}_m(\cdot|\mathcal{B}_m)$ *on their domains, then* $\underline{E}_j(\cdot|\mathcal{B}_j) \leq \underline{P}'_j(\cdot|\mathcal{B}_j)$ *for* $j=1,\dots,m$.
5. $\underline{P}_1(\cdot|\mathcal{B}_1),\dots,\underline{P}_m(\cdot|\mathcal{B}_m)$ *are coherent if and only if* $\underline{E}_j(f_j|B_j) = \underline{P}_j(f_j|B_j)$*, for every* $j=1,\dots,m, f_j \in \mathcal{H}^j, B_j \in \mathcal{B}_j$.

A consequence of the proof of the second point of this lemma is the following:

Corollary 1. *Let $\underline{P}_1(\cdot|\mathcal{B}_1),\dots,\underline{P}_m(\cdot|\mathcal{B}_m)$ be separately coherent conditional lower previsions. If $\underline{E}_j(f_j|B_j) < +\infty$ for all $j \in \{1,\dots,m\}, B_j \in \mathcal{B}_j$, and some $f_j \in \mathcal{H}^j$, then $\underline{P}_1(\cdot|\mathcal{B}_1),\dots,\underline{P}_m(\cdot|\mathcal{B}_m)$ avoid partial loss.*

We can also use the notion of natural extension to define an unconditional lower prevision. If we apply Eq. (7) with $\mathcal{B} := \{\Omega\}$, we obtain the functional $\underline{E}$ given by

$$\underline{E}(f) := \sup\Big\{\alpha : \exists j=1,\dots,m, n_j \geq 1, k=1,\dots,n_j, g_j^k \in \mathcal{H}^j \text{ s.t.} \quad (f-\alpha) > \sum_{j=1}^{m}\sum_{k=1}^{n_j} G_j(g_j^k|\mathcal{B}_j)\Big\} \tag{8}$$

for every gamble $f \in \mathcal{L}$. It follows that

$$\underline{E}(f) = \sup\Big\{\alpha : \exists j=1,\dots,m, n_j \geq 1, k=1,\dots,n_j, g_j^k \in \mathcal{H}^j \text{ s.t.} \quad (f-\alpha) \geq \sum_{j=1}^{m}\sum_{k=1}^{n_j} G_j(g_j^k|\mathcal{B}_j)\Big\} \tag{9}$$

for every gamble $f \in \mathcal{L}$: any α satisfying Eq. (8) satisfies Eq. (9), and conversely if α satisfies Eq. (9) then $\alpha - \varepsilon$ satisfies Eq. (8) for every $\varepsilon > 0$.

It has been established in [13, Proposition 14] that when the conditional lower previsions $\underline{P}_1(\cdot|\mathcal{B}_1),\dots,\underline{P}_m(\cdot|\mathcal{B}_m)$ are coherent, $\underline{E}$ is the smallest coherent lower prevision that is coherent with them, and it is called their (unconditional) natural extension.

Another particular case of interest is when we make the natural extension of a separately coherent conditional lower prevision $\underline{P}(\cdot|\mathcal{B})$ from its domain $\mathcal{H}$ to the set of all gambles $\mathcal{L}$; then $\underline{E}(\cdot|\mathcal{B})$ is the smallest coherent extension of $\underline{P}(\cdot|\mathcal{B})$, and it is the lower envelope of the credal set $\mathcal{M}(\underline{P}(\cdot|\mathcal{B}))$; similarly, if we have an (unconditional) coherent lower prevision $\underline{P}$ with domain $\mathcal{K}$, its natural extension $\underline{E}$ to $\mathcal{L}$ is its smallest coherent extension, and it is the lower envelope of the credal set $\mathcal{M}(\underline{P})$. Hence, in those cases we can keep the sensitivity analysis interpretation we mentioned in Section 2.5. See [19, Sections 3.4 and 6.7] for more details.

---

[9]Some related, but less general versions of points 1, 3, 4 and 5 can also be found in [13, Proposition 11 and Theorem 15].

3. Sets of desirable gambles

As we have seen in Section 2, lower previsions can be given a behavioural interpretation in terms of acceptable buying and selling prices, and the different consistency notions we have introduced can be better understood in terms of the desirability of a number of gambles. In this section, we summarise the formal structure of the sets of desirable gambles that we shall use later in the paper.

There are a number of different consistency notions for sets of gambles, which give rise to the notions of sets *almost desirable, really desirable, strictly desirable* or *weakly desirable* gambles. The difference between all these notions is in the inclusion of the gambles which are in the topological boundary of the set. Here, and for reasons that will become clearer in Section 4, we shall work mostly with sets of *really* desirable gambles.

Suppose $\mathcal{R} \subseteq \mathcal{Q} \subseteq \mathcal{L}$, where $\mathcal{Q}$ is a set of gambles whose desirability has been evaluated and $\mathcal{R}$ the subset of gambles that have been deemed desirable. Hence, $\mathcal{Q}$ could be interpreted as the domain of a function which tells us if a gamble is considered desirable or not. The notion of natural extension allows us to extend this domain to the set of all gambles.

Definition 10 (**Natural extension for gambles**). The *natural extension* of $\mathcal{R}$ is the set

$$\mathcal{E} := \mathcal{L}^+ \cup \Big\{ g \in \mathcal{L} : g \geq \sum_{j=1}^{r} \lambda_j g_j \text{ for some } r \geq 1, g_j \in \mathcal{R}, \lambda_j > 0 \Big\}. \qquad (10)$$

Note that $\mathcal{R} \subseteq \mathcal{Q} \cap \mathcal{E}$ and that we can express $\mathcal{E}$ equivalently as

$$\mathcal{E} = \Big\{ g \in \mathcal{L} : g = \sum_{j=1}^{r} \lambda_j g_j \text{ for some } r \geq 1, g_j \in \mathcal{R} \cup \mathcal{L}^+, \lambda_j > 0 \Big\},$$

and it follows also that $\mathcal{E}$ is closed under dominance. The natural extension models the consequences of the behavioural assessments expressed by $\mathcal{R}$, and does so in a least-committal way, in the sense that it produces the minimal set of gambles that we should judge desirable taking into account the set $\mathcal{R}$.

We next introduce two consistency conditions for a set of acceptable transactions. The first, less restrictive one, is called *avoiding partial loss*:[10]

Definition 11 (**Avoiding partial loss for gambles**). We say that $\mathcal{R}$ *avoids partial loss* if $0 \notin \mathcal{E}$.

Taking into account the previous interpretation of the natural extension, the intuition behind Definition 11 is that it is not rational to judge $\mathcal{R}$ desirable if it logically implies that we should judge as desirable the zero gamble (see also Corollary 2 below for further implications of avoiding partial loss).

The second consistency condition is called *coherence*; it means that the gambles in $\mathcal{R}$ are the only ones from $\mathcal{Q}$ that we should judge desirable, taking into account the consequences modelled by $\mathcal{E}$.

Definition 12 (**Coherence for gambles**). Say that $\mathcal{R}$ is *coherent relative to* $\mathcal{Q}$ if $\mathcal{R}$ avoids partial loss and $\mathcal{Q} \cap \mathcal{E} \subseteq \mathcal{R}$ (and hence $\mathcal{Q} \cap \mathcal{E} = \mathcal{R}$). In case $\mathcal{Q}$ coincides with $\mathcal{L}$ then we simply say that $\mathcal{R}$ *is coherent*.

[10] This concept is related to the notion of 'avoiding non-positivity' from [5].

We have the following axiomatic characterisation of coherence. A similar result for the notion of almost-desirability we shall use later can be found in [19, Section 3.7].

Proposition 2. *Suppose $\mathcal{Q}$ is a linear space containing constant gambles. Then $\mathcal{R}$ is coherent if and only if the following axioms hold:*

(APL) $0 \notin \mathcal{R}$. *[Avoiding Partial Loss]*
(APG) *If $g \in \mathcal{Q} \cap \mathcal{L}^+$, then $g \in \mathcal{R}$. [Accepting Partial Gains]*
(PHM) *If $g \in \mathcal{R}$ and $\lambda > 0$, then $\lambda g \in \mathcal{R}$. [Positive Homogeneity]*
(ADD) *If $f, g \in \mathcal{R}$, then $f + g \in \mathcal{R}$. [Additivity]*

A consequence of this result and Eq. (10) is that when $\mathcal{Q}$ satisfies the assumptions of Proposition 2, the natural extension of a coherent set of desirable gambles $\mathcal{R}$ with respect to $\mathcal{Q}$ is given by

$$\mathcal{E} = \mathcal{L}^+ \cup \{g \in \mathcal{L} : g \geq f \text{ for some } f \in \mathcal{R}\}. \tag{11}$$

In particular, when $\mathcal{Q} = \mathcal{L}$, a coherent set of desirable gambles $\mathcal{R}$ satisfies $\mathcal{R} = \mathcal{E}$.

Some basic properties of the natural extension are collected in the following:

Proposition 3. *Suppose $\mathcal{R} \subseteq \mathcal{Q}$ is a set of desirable gambles, and let $\mathcal{E}$ denote its natural extension. The following properties hold:*

(a) *The natural extension of $\mathcal{E}$ is $\mathcal{E}$ itself.*
(b) *If $\mathcal{R}$ is contained in a coherent set $\mathcal{E}'$, then $\mathcal{E} \subseteq \mathcal{E}'$.*
(c) *$\mathcal{R}$ avoids partial loss if and only if $\mathcal{E}$ avoids partial loss.*
(d) *$\mathcal{R}$ avoids partial loss if and only if $\mathcal{E}$ is coherent.*
(e) *$\mathcal{R}$ avoids partial loss if and only if there is a coherent set $\mathcal{E}'$ that includes $\mathcal{R}$.*
(f) *$\mathcal{R}$ is coherent relative to $\mathcal{Q}$ if and only if there is a coherent set $\mathcal{E}'$ such that $\mathcal{Q} \cap \mathcal{E}' = \mathcal{R}$.*
(g) *$\mathcal{R}$ is included in a coherent set $\mathcal{E}'$ if and only if $\mathcal{E}$ is coherent.*
(h) *If $\mathcal{E}$ is coherent, then it is the intersection of all the coherent sets that include $\mathcal{R}$.*

We deduce a corollary that clarifies the meaning of avoiding partial loss:

Corollary 2. *Suppose $\mathcal{R} \subseteq \mathcal{Q}$ is a set of desirable gambles, and let $\mathcal{E}$ denote its natural extension. If $\mathcal{R}$ avoids partial loss, then*

$$g \lneq 0 \Rightarrow g \notin \mathcal{E}. \tag{APL'}$$

A consequence of this is that whenever $\mathcal{R}$ avoids partial loss, then it does not include any gamble $g \lneq 0$, because $\mathcal{R} \subseteq \mathcal{E}$.

To every set of desirable gambles $\mathcal{R} \subseteq \mathcal{Q}$ we associate a set of linear previsions:

$$\mathcal{M}(\mathcal{R}) := \{P : P(g) \geq 0 \text{ for all } g \in \mathcal{R}\}.$$

In order to characterise this set, we next introduce the notion of almost-desirable gambles:

Definition 13 (**Almost-desirability**). We say that $\mathcal{D}$ is a *coherent set of almost-desirable gambles* (relative to $\mathcal{L}$) when it satisfies axioms (PHM), (ADD) and:

(ASL) If $f \in \mathcal{D}$, then $\sup f \geq 0$. [Avoiding Sure Loss]
(ASG) If $\inf f > 0$, then $f \in \mathcal{D}$. [Accepting Sure Gains]
(CLS) If $f + \delta \in \mathcal{D}$ for all $\delta > 0$, then $f \in \mathcal{D}$. [Closure]

From axioms (ASG) and (CLS), we deduce that the gamble $f = 0$ belongs to $\mathcal{D}$, and as a consequence any gamble $f \geq 0$ also belongs to $\mathcal{D}$. As a consequence, if $\mathcal{D}$ is a coherent set of almost-desirable gambles, it also satisfies axiom (APG). However, it does not satisfy condition (APL), although it may satisfy (APL'). An example is given by $\mathcal{D} := \{f \geq 0\}$.

Conversely, let $\mathcal{R}$ be a coherent set of desirable gambles with respect to $\mathcal{L}$, which as a consequence coincides with its natural extension $\mathcal{E}$. It follows that (APL') implies (ASL) and (APG) implies (ASG). However, a coherent set of desirable gambles does not satisfy condition (CLS), because it includes the constant gamble on δ for every $\delta > 0$, but it does not include the zero gamble.

Coherent sets of almost-desirable gambles are related to coherent lower previsions, as discussed in [19, Section 3.8]: if $\underline{P}$ is a coherent lower prevision on $\mathcal{L}$, the set $\mathcal{D} := \{f : \underline{P}(f) \geq 0\}$ is a coherent set of almost-desirable gambles; and conversely, if $\mathcal{D}$ is a coherent set of almost-desirable gambles with respect to $\mathcal{L}$, the lower prevision $\underline{P}_{\mathcal{D}}$ given by

$$\underline{P}_{\mathcal{D}}(f) := \max\{\alpha : f - \alpha \in \mathcal{D}\} \tag{12}$$

is coherent, and moreover $\mathcal{D} = \{f : \underline{P}_{\mathcal{D}}(f) \geq 0\}$. This equivalence allows us to characterise the set $\mathcal{M}(\mathcal{R})$ defined above (this is an adaptation from [14, Proposition 7]):

Proposition 4. *Let $\mathcal{R}$ be a set of desirable gambles that avoids partial loss. Then $\{g \in \mathcal{L} : P(g) \geq 0\ \forall P \in \mathcal{M}(\mathcal{R})\} = \{g \in \mathcal{L} : g + \delta \in \mathcal{E}\ \forall \delta > 0\} = \overline{\mathcal{E}}$, where $\mathcal{E}$ is the natural extension of $\mathcal{R}$ and $\overline{\mathcal{E}}$ denotes the closure of $\mathcal{E}$ in the topology of uniform convergence.*

In the unconditional case, there is a one-to-one correspondence between coherent sets of almost-desirable gambles and coherent lower previsions: every coherent lower prevision uniquely determines a coherent set of almost-desirable gambles, and viceversa; however, there are many different coherent sets of (really) desirable gambles that may correspond to the same set of almost-desirable gambles, and as a consequence coherent sets of desirable gambles are more informative (i.e, model more behavioural assessments) than coherent lower previsions.

It is also interesting to remark that with any coherent set $\mathcal{D}$ of almost-desirable gambles, as well as of really desirable gambles, we can associate a set $\underline{\mathcal{D}}$ of *strictly desirable* gambles, which is given by

$$\underline{\mathcal{D}} := \mathcal{L}^+ \cup \{f : \underline{P}_{\mathcal{D}}(f) > 0\},$$

where $\underline{P}_{\mathcal{D}}$ is the coherent lower prevision associated to $\mathcal{D}$ by means of Eq. (12). Using Proposition 2, it is easy to see that $\underline{\mathcal{D}}$ is a set of really desirable gambles. Moreover, if $\mathcal{R}$ is a coherent set of really desirable gambles, $\overline{\mathcal{R}}$ is its associated set of almost-desirable gambles, and $\underline{\mathcal{R}}$ its related set of strictly desirable gambles, we have $\underline{\mathcal{R}} \subseteq \mathcal{R} \subseteq \overline{\mathcal{R}}$ [19, Appendix F].

4. Conditional lower previsions derived from sets of desirable gambles

In this section, we are going to define a number of conditional lower previsions from a set of gambles that we judge desirable and we are going to investigate under which assumptions the different consistency notions from Section 2 are satisfied. We assume that we give a judgement about the desirability of all the gambles in some set $\mathcal{Q}$, turning out that we consider the gambles in some domain $\mathcal{R} \subseteq \mathcal{Q}$ to be desirable.

For the time being, we are not imposing any conditions on the set of gambles $\mathcal{R}$: hence, we are allowing for contradictory statements, such as considering that a constant reward on 1 is desirable for us, but a constant reward on 3 is not. What we shall show in this section is that if we derive conditional lower previsions from the assessments represented by $\mathcal{R}$ and we want these conditional lower previsions to be coherent, a number of consistency notions on $\mathcal{R}$ arise naturally.

The idea of deriving conditional lower previsions from sets of desirable gambles is already present in [19, Appendix F] and [21]. One of its advantages is the ability to deal easily with the problem of conditioning on sets of zero lower or upper probability: in that case there is usually not a unique way of deriving the conditional lower previsions from the unconditional ones ([19, Section 6.10]), while we can determine the behavioural implications of our assessments by working with sets of desirable gambles.

Let $\mathcal{B}$ be a partition of Ω, and let us define the conditional lower prevision $\underline{P}(\cdot|\mathcal{B})$ on $\mathcal{L}$ by

$$\underline{P}(f|B) := \sup\{\mu : B(f - \mu) \in \mathcal{R}\} \tag{13}$$

for every $f \in \mathcal{L}$ and every $B \in \mathcal{B}$.

If in particular $\mathcal{B} = \{\Omega\}$, we obtain a lower prevision $\underline{P}$ on $\mathcal{L}$ given by:[11]

$$\underline{P}(f) := \sup\{\mu : (f - \mu) \in \mathcal{R}\}. \tag{14}$$

We shall discuss why $\mathcal{L}$ is the domain of definition of these lower previsions in Proposition 7 later on: we shall show that we can in general assume that $\mathcal{Q} = \mathcal{L}$ or take the natural extension $\mathcal{E}$ of $\mathcal{R}$. But before establishing this, we are first going to determine under which conditions $\underline{P}(\cdot|\mathcal{B})$ is well-defined. By this we mean what follows:

Definition 14. (**Well-definedness**) $\underline{P}(\cdot|\mathcal{B})$ is well-defined if for every gamble f in $\mathcal{L}$ and every $B \in \mathcal{B}$, it holds that

$$\inf_{\omega \in B} f(\omega) \leq \underline{P}(f|B) \leq \sup_{\omega \in B} f(\omega).$$

In order to study this, we introduce an additional consistency axiom:

(SDa) If $f \leq 0, \sup_B f < 0$ for some $B \in \mathcal{B}$, then $f \notin \mathcal{R}$.

(SDb) If $f \geq 0, \inf_B f > 0$ for some $B \in \mathcal{B}$, then $f \in \mathcal{R}$.

We shall refer to these two conditions together as (SD) (which stands for *Strict Dominance*) with respect to $\mathcal{B}$, or simply to (SD) when it is clear which partition we are working with. They respectively follow from (APL') and (APG) (when in the latter we have $\mathcal{Q} = \mathcal{L}$), so any coherent set of desirable gambles satisfies (SDa), (SDb). To see that (SDa) is actually weaker than (APL'), consider $\Omega := \{1, 2, 3\}, \mathcal{B} := \{\{1\}, \{2, 3\}\}$ and $\mathcal{R} := \{f : f(1) \geq 0, \max(f(2), f(3)) \geq 0\}$: then $\mathcal{R}$ satisfies (SDa) with respect to $\mathcal{B}$, but not (APL'), because it includes the gamble $f = (f(1), f(2), f(3)) := (0, -1, 0)$. On the other hand, (SDa) is stronger than (ASL); to see that they are not equivalent, note that given $\Omega := \{1, 2\}$ and $\mathcal{B} := \{\{1\}, \{2\}\}$ the set of gambles $\{f : f(1) \geq 0\}$ is a coherent set of almost-desirable gambles but does not satisfy (SDa) because it includes the gamble $f = (f(1), f(2)) := (0, -1)$.

[11] Conversely, we can see Eq. (13) also as a consequence of Eq. (14), once we *update* the set $\mathcal{R}$ by B, by considering $\mathcal{R}_B := \{f : Bf \in \mathcal{R}\}$; see [5, 21] for further comments on this idea.

Hence, axiom (SD) allows us to differentiate between sets of desirable and almost-desirable gambles.

Theorem 5. *(1) If $\mathcal{R}$ satisfies (SDb), then $\underline{P}(f|B) \geq \inf_{\omega\in B} f(\omega)$ for every $f \in \mathcal{L}$ and every $B \in \mathcal{B}$.*

(2) If $\mathcal{R}$ satisfies (SDa), then $\underline{P}(f|B) \leq \sup_{\omega\in B} f(\omega)$ for every $f \in \mathcal{L}$ and every $B \in \mathcal{B}$.

(3) If $\mathcal{R}$ satisfies (SDb) and (ADD), then $B(f-\mu)$ belongs to $\mathcal{R}$ for every $\mu < \underline{P}(f|B)$.

(4) If $\mathcal{R}$ is closed under dominance, i.e., such that $g \geq f \in \mathcal{R}$ implies that $g \in \mathcal{R}$, then $\underline{P}(\cdot|\mathcal{B})$ is well-defined if and only if $\mathcal{R}$ satisfies (SD).

We deduce that $\underline{P}(\cdot|\mathcal{B})$ is well-defined when the set $\mathcal{R}$ satisfies (SD),[12] and in particular when it satisfies (APG) with respect to $\mathcal{L}$ and (APL'). Note that if $\mathcal{R}$ does not satisfy the dominance property, i.e., if we do not have that $g \geq f \in \mathcal{R}$ implies that $g \in \mathcal{R}$, we may still end up with a bounded conditional lower prevision by means of Eq. (13) (even if not satisfying the dominance property is rather counter-intuitive):

Example 1. Let $\Omega := \{\omega_1, \omega_2\}$, and let $\mathcal{R} := \mathcal{L}^+ \cap \{f \in \mathcal{L} : f \leq 1\}$. Let $\mathcal{B} := \{\{\omega_1\}, \{\omega_2\}\}$. To see that $\mathcal{R}$ does not satisfy neither (ASG) nor (SD), it suffices to see that the gamble $f := 2$ does not belong to $\mathcal{R}$. However, the conditional lower prevision we can define by means of Eq. (13) is bounded (it is even separately coherent): for every gamble f, it is easy to see that $\underline{P}(f|\{\omega_1\}) = f(\omega_1), \underline{P}(f|\{\omega_2\}) = f(\omega_2)$. ♦

We see then that axiom (SD), which is one of the differences between coherent sets of almost-desirable and really desirable gambles, is one of the keys for deriving conditional lower previsions that avoid partial loss. In fact, if $\mathcal{D}$ is a coherent set of almost-desirable gambles and we define the conditional lower previsions $\underline{P}_1(\cdot|\mathcal{B}_1), \dots, \underline{P}_m(\cdot|\mathcal{B}_m)$ from $\mathcal{D}$ by means of Eq. (13), these conditional lower previsions may not be well-defined:

Example 2. Let $\Omega := \{\omega_1, \omega_2\}$, and let $\mathcal{D} := \{f : f(\omega_1) \geq 0\}$. Let $\mathcal{B} := \{\{\omega_1\}, \{\omega_2\}\}$. It is easy to see that $\mathcal{D}$ satisfies all the axioms in Definition 13, and is therefore a coherent set of almost-desirable gambles. However, it does not satisfy (SD): the gamble $f := (0, -1)$ belongs to $\mathcal{D}$, and therefore (SDa) is not satisfied. In fact, using Eq. (13) we obtain $\underline{P}(f|\{\omega_2\}) = +\infty$, because the gamble $f_\alpha := \mathbb{I}_{\{\omega_2\}}(f - \alpha)$ satisfies $f_\alpha(\omega_1) = 0$ for every $\alpha \in \mathbb{R}$. ♦

This shows that when we want to derive conditional lower previsions from a set of gambles, the notion of almost-desirability may be too weak to produce any meaningful assessments, and this is one of the reasons why we are focusing on the notion of real desirability in this paper. Interestingly, for almost-desirable gambles axiom (SD) is related to the necessity of conditioning on sets of upper probability zero, which is related also to the results we shall develop in Appendix A.1:

Proposition 5. *Let $\mathcal{D}$ be a coherent set of almost-desirable gambles and B be a non-empty subset of Ω. Let $\underline{P}$ be the coherent lower prevision derived from $\mathcal{D}$ by means of Eq.* (12)*, and $\overline{P}$ its conjugate upper prevision. Then*

$$\overline{P}(B) = 0 \text{ if and only if there is } f \in \mathcal{D} \text{ s.t. } f \leq 0, \sup_B f < 0. \tag{15}$$

[12] This result can also be found in [25, Section 3].

As a consequence, $\mathcal{D}$ satisfies (SDa) with respect to a partition $\mathcal{B}$ if and only if $\overline{P}(B) > 0$ for every $B \in \mathcal{B}$.

We next investigate which conditions on the set of desirable gambles $\mathcal{R}$ guarantee that the lower prevision $\underline{P}(\cdot|\mathcal{B})$ satisfies the consistency axioms from Section 2. We begin with the property of separate coherence.

Theorem 6. *Let $\mathcal{R}$ be a set of gambles, and define $\underline{P}(\cdot|\mathcal{B})$ on $\mathcal{L}$ by Eq.* (13)*. Then $\underline{P}(\cdot|\mathcal{B})$ is separately coherent if $\mathcal{R}$ satisfies axioms (SD), (PHM) and (ADD).*

In the particular case where $\mathcal{B} = \{\Omega\}$, and where we define a lower prevision $\underline{P}$ on $\mathcal{L}$ by Eq. (14), the notion of separate coherence becomes the coherence property from Definition 1. In that case condition (SD) is equivalent to (ASL) together with (ASG). As we have already remarked, the connection between sets of desirable gambles and lower previsions in the unconditional case has been established by Walley in [19, Chapter 3].

To see that the sufficient conditions in Theorem 6 are not necessary, note that the set of gambles $\mathcal{R}$ considered in Example 1 does not satisfy *any* of the axioms (SD), (PHM), (ADD), and still it produces a separately coherent conditional lower prevision.

One particular case of interest of separately coherent conditional lower previsions are the linear previsions considered in Section 2.5. In that case, we must require that $\mathcal{R}$ satisfies some additional properties besides the ones in Theorem 6:

Proposition 6. *Assume $\mathcal{R}$ satisfies (ADD), (PHM) and (SD) and let $\underline{P}(\cdot|\mathcal{B}), \underline{P}$ be given by Eqs.* (13)*,* (14) *respectively.*

(1) $\underline{P}(\cdot|\mathcal{B})$ is a linear conditional prevision if and only if

$$\forall B \in \mathcal{B}, f \in \mathcal{L}, \varepsilon > 0 \text{ either } Bf \in \mathcal{R} \text{ or } B(\varepsilon - f) \in \mathcal{R}. \tag{LC}$$

(2) $\underline{P}$ is a linear prevision if and only if for every gamble f and every $\varepsilon > 0$, either f or $\varepsilon - f$ belongs to $\mathcal{R}$.

The intuition of this result is clear, once we recall the behavioural interpretation of linear conditional previsions from Section 2.5: for them the supremum acceptable buying price coincides with the infimum acceptable selling price, which means that for almost every real number μ and for every conditional event B either $B(f - \mu)$ or $B(\mu - f)$ should be an acceptable transaction for our subject.

In particular, we obtain linear (conditional) previsions when the set $\mathcal{R}$ is *maximal*, in the sense that for every non-zero gamble $f \in \mathcal{L}$ either f or $-f$ belongs to $\mathcal{R}$. Maximal sets have been studied in [1], and have the property that we cannot add any new gamble to $\mathcal{R}$ without violating one of the properties of coherence. Moreover, they can be used to express the natural extension $\mathcal{E}$ of a coherent set of gambles $\mathcal{R}$ as an intersection of maximal coherent sets. See [1, Section 5] for more information and [23] for some related work.

In particular, we can apply Proposition 6 when $\mathcal{R}$ is either a coherent set of almost-desirable gambles that satisfies (SD) (see Definition 13) or a coherent set of desirable gambles (Proposition 2). The reason why we are introducing the $\varepsilon > 0$ in the condition of the above proposition is that it may be that neither f nor $-f$ belong to $\mathcal{R}$, even if this set of gambles is coherent, and still it may give rise to a linear prevision:

Example 3. Consider $\Omega := \{\omega_1, \omega_2\}$ and let $\mathcal{R} := \{f \in \mathcal{L} : f(\omega_1)+f(\omega_2) > 0\}$. Then $\mathcal{R}$ satisfies all the axioms in Proposition 2, and as a consequence it is a coherent set of desirable gambles with respect to $\mathcal{L}$. Moreover, it gives rise to the linear prevision P given by $P(f) = \frac{f(\omega_1)+f(\omega_2)}{2}$ for all $f \in \mathcal{L}$. However, given $f := (1, -1)$, neither f nor $-f$ belong to $\mathcal{R}$. ♦

On the other hand, we can assume that $\varepsilon = 0$ when we work with a coherent set of almost-desirable gambles:

Corollary 3. *If $\mathcal{D}$ is a coherent set of almost-desirable gambles that satisfies (SD), then $\underline{P}(\cdot|\mathcal{B})$ is a linear conditional prevision if and only if for every set $B \in \mathcal{B}$ and every gamble $f \in \mathcal{L}$, either Bf or $-Bf$ belongs to $\mathcal{D}$.*

If $\mathcal{R}$ is a coherent set of desirable gambles with respect to a linear set of gambles $\mathcal{Q}$ and we want to use it to define a conditional lower prevision by means of Eq. (13), we may wonder if the domain of this conditional lower prevision should be the set of all gambles $\mathcal{L}$ or only the subset $\mathcal{Q}$ given by those gambles whose desirability we have evaluated.

Indeed, the coherence of $\mathcal{R}$ with respect to $\mathcal{Q}$ is not sufficient to guarantee that the conditional lower prevision it originates is separately coherent, because it must also satisfy condition (SDb); this property follows from (APG) when $\mathcal{Q} = \mathcal{L}$, but not in general:

Example 4. Let $\Omega := \{\omega_1, \omega_2, \omega_3\}$ and take $\mathcal{Q} := \{f \in \mathcal{L} : f(\omega_1) = f(\omega_2)\}, \mathcal{R} := \{f \in \mathcal{Q} : f(\omega_1) + f(\omega_3) > 0\}$. Then $\mathcal{Q}$ is a linear set of gambles that includes all constant gambles, and Proposition 2 shows that $\mathcal{R}$ is coherent with respect to $\mathcal{Q}$. If we consider the partition $\mathcal{B} := \{\{\omega_1, \omega_3\}, \{\omega_2\}\}$ of Ω, then for the gamble $f := (2, 2, 0) \in \mathcal{Q}$ it follows that given $B := \{\omega_1, \omega_3\}$ there is no value of μ such that $g_\mu := B(f - \mu)$ belongs to $\mathcal{R}$: the only μ for which $g_\mu(\omega_1) = g_\mu(\omega_2)$ is $\mu = 2$, and the gamble $g_\mu = (0, 0, -2)$ does not belong to $\mathcal{R}$. Hence, the lower prevision $\underline{P}(\cdot|\mathcal{B})$ derived from the coherent set $\mathcal{R}$ by means of Eq. (13) is not well-defined. ♦

Our next result shows, essentially, that in order to obtain meaningful conditional lower previsions it is helpful to work with the natural extension $\mathcal{E}$ of $\mathcal{R}$.

Proposition 7. *Let $\mathcal{R}$ be a coherent set of desirable gambles with respect to a linear set $\mathcal{Q}$, and let $\mathcal{E}$ denote its natural extension. Let $\mathcal{B}$ be a partition of Ω, and let $\underline{P}_1(\cdot|\mathcal{B}), \underline{P}_2(\cdot|\mathcal{B}), \underline{P}_3(\cdot|\mathcal{B})$, be the conditional lower previsions with respective domains $\mathcal{Q}, \mathcal{L}, \mathcal{L}$ given by*

$$\underline{P}_1(f|B) := \sup\{\mu : B(f - \mu) \in \mathcal{R}\}\ \forall B \in \mathcal{B}, f \in \mathcal{Q},$$
$$\underline{P}_2(f|B) := \sup\{\mu : B(f - \mu) \in \mathcal{R}\}\ \forall B \in \mathcal{B}, f \in \mathcal{L},$$
$$\underline{P}_3(f|B) := \sup\{\mu : B(f - \mu) \in \mathcal{E}\}\ \forall B \in \mathcal{B}, f \in \mathcal{L}.$$

(1) $\underline{P}_3(\cdot|\mathcal{B})$ is a separately coherent conditional lower prevision.
(2) If $\mathcal{R}$ satisfies (SD), then $\underline{P}_1(\cdot|\mathcal{B})$ is separately coherent, and $\underline{P}_2(\cdot|\mathcal{B}) = \underline{P}_3(\cdot|\mathcal{B})$ is the natural extension of $\underline{P}_1(\cdot|\mathcal{B})$.

Taking this result into account, whenever we have a coherent set of desirable gambles $\mathcal{R}$ with respect to $\mathcal{Q}$ that satisfies (SD), we can always use it to define a conditional lower prevision $\underline{P}(\cdot|\mathcal{B})$ on $\mathcal{L}$.

We move next to the consistency properties of several conditional lower previsions. Consider thus a number of partitions $\mathcal{B}_1, \dots, \mathcal{B}_m$ of Ω, and let us define conditional lower previsions $\underline{P}_1(\cdot|\mathcal{B}_1), \dots, \underline{P}_m(\cdot|\mathcal{B}_m)$ on $\mathcal{L}$.

Theorem 7. *If $\mathcal{R}$ satisfies (SD) with respect to $\mathcal{B}_1, \dots, \mathcal{B}_m$, (PHM) and (ADD), then the conditional lower previsions $\underline{P}_1(\cdot|\mathcal{B}_1), \dots, \underline{P}_m(\cdot|\mathcal{B}_m)$ avoid partial loss.*

When all the conditional lower previsions $\underline{P}_1(\cdot|\mathcal{B}_1), \dots, \underline{P}_m(\cdot|\mathcal{B}_m)$ are linear, the notion of avoiding partial loss is equivalent to coherence. Hence, Theorem 7 shows that conditions (SD),(PHM),(ADD) and (LC) imply the coherence of the derived conditional linear previsions. We next establish a similar result for conditional lower previsions:

Theorem 8. *If $\mathcal{R}$ includes the non-negative gambles and satisfies (SD) with respect to $\mathcal{B}_1, \dots, \mathcal{B}_m$,(PHM) and (ADD), then $\underline{P}_1(\cdot|\mathcal{B}_1), \dots, \underline{P}_m(\cdot|\mathcal{B}_m)$ are coherent.*

In particular we deduce the following:

Corollary 4. *([25, Proposition 1];[19, Appendix F, Theorem F3]) Let $\mathcal{R}$ be a coherent set of really desirable gambles and define conditional lower previsions $\underline{P}_1(\cdot|\mathcal{B}_1), \dots, \underline{P}_m(\cdot|\mathcal{B}_m)$ on $\mathcal{L}$ by means of Eq.* (13). *Then these conditional lower previsions are coherent.*

Hence, the coherence of a set of desirable gambles implies the coherence of the derived conditional lower previsions, but as Theorem 8 shows this sufficient condition is not necessary. This solves the main problem in one direction.

5. Desirable gambles derived from conditional lower previsions

In Section 4, we have started from a set of desirable gambles $\mathcal{R}$ and defined a number of conditional lower previsions $\underline{P}_1(\cdot|\mathcal{B}_1), \dots, \underline{P}_m(\cdot|\mathcal{B}_m)$, and have studied which conditions on $\mathcal{R}$ guarantee that $\underline{P}_1(\cdot|\mathcal{B}_1), \dots, \underline{P}_m(\cdot|\mathcal{B}_m)$ are coherent. We turn now to the converse problem.

Let $\underline{P}_1(\cdot|\mathcal{B}_1), \dots, \underline{P}_m(\cdot|\mathcal{B}_m)$ be separately coherent conditional lower previsions with respective domains $\mathcal{H}^1, \dots, \mathcal{H}^m$. Assume these domains satisfy the assumptions imposed in Remark 1, at the end of Section 2.2.

From the point of view of desirable gambles, we are evaluating the desirability of the gambles in the set

$$\mathcal{Q} := \{g \in \mathcal{L} : g = G_j(f_j|B_j) + \varepsilon B_j \text{ for some } j \in \{1, \dots, m\}, f_j \in \mathcal{H}^j, B_j \in \mathcal{B}_j, \varepsilon \neq 0\}. \quad (16)$$

Note that this set is not linear in general.

The lower previsions are equivalent to statements of desirability for some of the gambles in $\mathcal{Q}$. Taking into account that $\underline{P}_j(f_j|B_j)$ is interpreted as the supremum acceptable buying price for the gamble f_j contingent on B_j, and that therefore $\underline{P}_j(f_j|B_j) - \varepsilon$ is an acceptable buying price for every $\varepsilon > 0$, we obtain that the gambles in the following set are judged as desirable:

$$\mathcal{R} := \{g \in \mathcal{L} : g = G_j(f_j|B_j) + \varepsilon B_j \text{ for some } j \in \{1, \dots, m\}, f_j \in \mathcal{H}^j, B_j \in \mathcal{B}_j, \varepsilon > 0\}. \quad (17)$$

We give an equivalent characterisation of $\mathcal{R}$ in the following proposition.

Proposition 8. *Consider the sets $\mathcal{Q}, \mathcal{R}$ derived from the separately coherent conditional lower previsions $\underline{P}_1(\cdot|\mathcal{B}_1), \dots, \underline{P}_m(\cdot|\mathcal{B}_m)$ by means of Eqs.* (16) *and* (17).

Then $\mathcal{R}$ can be re-written equivalently as follows:

$$\mathcal{R} = \{g \in \mathcal{L} : g \in \mathcal{H}^j, gB_j^c = 0, \underline{P}_j(g|B_j) > 0 \text{ for some } j \in \{1, \dots, m\}, B_j \in \mathcal{B}_j\}. \tag{18}$$

Let us now apply considerations of avoiding partial loss and coherence to the set $\mathcal{R}$. For this, we reconsider the natural extension $\mathcal{E}$ of $\mathcal{R}$, as given by Eq. (10).

Theorem 9. *Consider the sets $\mathcal{Q}, \mathcal{R}$ derived from the separately coherent conditional lower previsions $\underline{P}_1(\cdot|\mathcal{B}_1), \dots, \underline{P}_m(\cdot|\mathcal{B}_m)$ by means of Eqs. (16) and (17). Let $\mathcal{E}$ be the natural extension of $\mathcal{R}$, given by Eq. (10). Then a gamble g belongs to $\mathcal{E}$ if and only if any of the following equivalent conditions holds:*

(1) Either $g \in \mathcal{L}^+$ or there are $j = 1, \dots, m, n_j \geq 1, k = 1, \dots, n_j, g_j^k \in \mathcal{H}^j$, not all of them zero, $B_j \in \mathcal{B}_j, \varepsilon > 0$ such that

$$g \geq \sum_{j=1}^{m} \sum_{k=1}^{n_j} G_j(g_j^k|\mathcal{B}_j) + \varepsilon S_j(g_j^k). \tag{19}$$

(2) Either $g \in \mathcal{L}^+$ or there are $j = 1, \dots, m, n_j \geq 1, k = 1, \dots, n_j, g_j^k \in \mathcal{H}^j$ not all of them zero, such that

$$\inf_{\mathbb{S}(g_j^k)} \left[g - \sum_{j=1}^{m} \sum_{k=1}^{n_j} G_j(g_j^k|\mathcal{B}_j) \right] > 0 \text{ and } g \geq 0 \text{ in } \mathbb{S}(g_j^k)^c. \tag{20}$$

We can use the equivalent expressions of $\mathcal{E}$ from this theorem to characterise the notions of avoiding partial and uniform sure loss in terms of desirable gambles:

Theorem 10. *Consider the set $\mathcal{R}$ derived from the separately coherent conditional lower previsions $\underline{P}_1(\cdot|\mathcal{B}_1), \dots, \underline{P}_m(\cdot|\mathcal{B}_m)$ by means of Eq. (17) and let $\mathcal{E}$ be the natural extension of $\mathcal{R}$.*

(1) $\underline{P}_1(\cdot|\mathcal{B}_1), \dots, \underline{P}_m(\cdot|\mathcal{B}_m)$ avoid partial loss if and only if $\mathcal{R}$ avoids partial loss.
(2) $\underline{P}_1(\cdot|\mathcal{B}_1), \dots, \underline{P}_m(\cdot|\mathcal{B}_m)$ avoid uniform sure loss if and only if $\mathcal{E}$ satisfies (ASL).

We next come to the main result of this section, where we characterise the coherence of the conditional lower previsions in terms of desirable gambles. Interestingly, we show that the coherence of the conditional lower previsions implies the coherence of the set $\mathcal{R}$ with respect to $\mathcal{Q}$, but both conditions are not equivalent: we need an additional technical condition, which is related to the fact that the same conditioning set can belong to two different partitions, and in that case we should require that the conditional previsions are defined in the same way:

Theorem 11. *Let $\mathcal{Q}, \mathcal{R}$ be derived from the separately coherent conditional lower previsions $\underline{P}_1(\cdot|\mathcal{B}_1), \dots, \underline{P}_m(\cdot|\mathcal{B}_m)$ by means of Eqs. (16) and (17). The following statements are equivalent:*

(1) $\mathcal{R}$ is coherent relative to $\mathcal{Q}$ and $G_j(f_j|B_j) - \varepsilon B_j \notin \mathcal{R}$ for any $f_j \in \mathcal{H}^j, B_j \in \mathcal{B}_j, \varepsilon > 0$ and any $j = 1, \dots, m$.
(2) $\underline{P}_j(f_j|B_j) = \underline{E}_j(f_j|B_j)$ for all $j = 1, \dots, m, f_j \in \mathcal{H}^j, B_j \in \mathcal{B}_j$.
(3) The conditional lower previsions $\underline{P}_1(\cdot|\mathcal{B}_1), \dots, \underline{P}_m(\cdot|\mathcal{B}_m)$ are coherent.

In order to show that point (1) in this theorem cannot be simplified, in the sense that if $\mathcal{R}$ is coherent relative to $\mathcal{Q}$ we do not necessarily have that $G_j(f_j|B_j)-\varepsilon B_j \notin \mathcal{R}$ for any $f_j \in \mathcal{H}^j, B_j \in \mathcal{B}_j, \varepsilon > 0$ and any $j = 1,\dots,m$, consider the following example:

Example 5. Consider $\Omega := \{1,2,3,4,5,6\}$ and the following two partitions of Ω: $\mathcal{B}_1 := \{\{1,2\},\{3,4,5,6\}\}$ and $\mathcal{B}_2 := \{\{1,2\},\{3,4\},\{5,6\}\}$. Let $\underline{P}_1(\cdot|\mathcal{B}_1)$ be the vacuous lower prevision, given by

$$\underline{P}_1(f|B_1) = \min_{\omega\in B_1} f(\omega)$$

for every gamble f and every $B_1 \in \mathcal{B}_1$, and let $\underline{P}_2(\cdot|B_2)$ be vacuous when $B_2 = \{3,4\}$ or $B_2 = \{5,6\}$, and uniform when $B_2 = \{1,2\}$. Then $\underline{P}_1(\cdot|\mathcal{B}_1)$ and $\underline{P}_2(\cdot|\mathcal{B}_2)$ are not coherent: if we consider the gamble $f := \mathbb{I}_{\{1\}}$, it holds that

$$G_2(f|\mathcal{B}_2) - G_1(f|\{1,2\}) \leq -0.5$$

in $S_2(f)\cup\{1,2\} = \{1,2\}$. Note on the other hand that $\underline{P}_1(\cdot|\mathcal{B}_1)$ and $\underline{P}_2(\cdot|\mathcal{B}_2)$ avoid partial loss (it suffices to take into account that $\underline{P}_1(\cdot|\mathcal{B}_1)$ is vacuous).

Let us derive the set $\mathcal{R}$ from them. According to the definition in (17), the generic gamble in $\mathcal{R}$ is equal to $G_j(f|B_j)+\varepsilon B_j$ for some $j \in \{1,2\}, f \in \mathcal{H}^j, B_j \in \mathcal{B}_j, \varepsilon > 0$. Let us consider the vacuous case first, for instance when $j = 1$ and $B_1 = \{1,2\}$. This gives rises to the following subset of $\mathcal{R}$: $\mathcal{R}_j^{B_j} = \mathcal{R}_1^{\{1,2\}} := \{B_1(f - \min_{B_1} f) + \varepsilon B_1 : f \in \mathcal{H}^1, \varepsilon > 0\}$, which is more conveniently written as

$$\mathcal{R}_1^{\{1,2\}} = \{g : \min_{\{1,2\}} g > 0, g\mathbb{I}_{\{3,4,5,6\}} = 0\}.$$

In fact, that the latter set includes $\mathcal{R}_1^{\{1,2\}}$ is trivial; conversely, it is enough to choose $\varepsilon := \min_{\{1,2\}} g$. We can proceed in much the same way in order to find out the expressions for the sets $\mathcal{R}_1^{\{3,4,5,6\}}, \mathcal{R}_2^{\{3,4\}}, \mathcal{R}_2^{\{5,6\}}$, which we summarise below:

$$\begin{aligned}
\mathcal{R}_1^{\{3,4,5,6\}} &= \{g : \min_{\{3,4,5,6\}} g > 0, g\mathbb{I}_{\{1,2\}} = 0\},\\
\mathcal{R}_2^{\{3,4\}} &= \{g : \min_{\{3,4\}} g > 0, g\mathbb{I}_{\{1,2,5,6\}} = 0\},\\
\mathcal{R}_2^{\{5,6\}} &= \{g : \min_{\{5,6\}} g > 0, g\mathbb{I}_{\{1,2,3,4\}} = 0\}.
\end{aligned}$$

In the remaining case where $j = 2$ and $B_2 = \{1,2\}$, we obtain that $\mathcal{R}_2^{\{1,2\}} = \{B_2(f - \frac{1}{2}f(1) - \frac{1}{2}f(2)) + \varepsilon B_2 : f \in \mathcal{H}^2, \varepsilon > 0\}$. Let us show that

$$\mathcal{R}_2^{\{1,2\}} = \{g : g(1) + g(2) > 0, g\mathbb{I}_{\{3,4,5,6\}} = 0\}.$$

That the latter set includes $\mathcal{R}_2^{\{1,2\}}$ is trivial. Conversely, it is enough to set $f(1) := g(1) - g(2), f(2) := 0$, and $\varepsilon := \frac{g(1)+g(2)}{2}$.

Observing that $\mathcal{R}_1^{\{1,2\}} \subseteq \mathcal{R}_2^{\{1,2\}}$, we can finally write that

$$\mathcal{R} = \mathcal{R}_1^{\{3,4,5,6\}} \cup \mathcal{R}_2^{\{3,4\}} \cup \mathcal{R}_2^{\{5,6\}} \cup \mathcal{R}_2^{\{1,2\}}.$$

Let us focus on the set $\mathcal{Q}$ now. With arguments analogous to those used with $\mathcal{R}$, we obtain that

$$\mathcal{Q} = \mathcal{Q}_1^{\{1,2\}} \cup \mathcal{Q}_1^{\{3,4,5,6\}} \cup \mathcal{Q}_2^{\{3,4\}} \cup \mathcal{Q}_2^{\{5,6\}} \cup \mathcal{Q}_2^{\{1,2\}},$$

with

$$\begin{aligned}
\mathcal{Q}_1^{\{1,2\}} &:= \{g : \min_{\{1,2\}} g \neq 0, g\mathbb{I}_{\{3,4,5,6\}} = 0\},\\
\mathcal{Q}_1^{\{3,4,5,6\}} &:= \{g : \min_{\{3,4,5,6\}} g \neq 0, g\mathbb{I}_{\{1,2\}} = 0\},\\
\mathcal{Q}_2^{\{3,4\}} &:= \{g : \min_{\{3,4\}} g \neq 0, g\mathbb{I}_{\{1,2,5,6\}} = 0\},\\
\mathcal{Q}_2^{\{5,6\}} &:= \{g : \min_{\{5,6\}} g \neq 0, g\mathbb{I}_{\{1,2,3,4\}} = 0\},\\
\mathcal{Q}_2^{\{1,2\}} &:= \{g : g(1)+g(2) \neq 0, g\mathbb{I}_{\{3,4,5,6\}} = 0\}.
\end{aligned}$$

Let us focus now on the natural extension $\mathcal{E}$ of $\mathcal{R}$ (see Eq. (10)). We start by considering the related set

$$\mathcal{E}' := \{g \in \mathcal{L} : g = \sum_{j=1}^{r} \lambda_j g_j \text{ for some } r \geq 1, g_j \in \mathcal{R}, \lambda_j > 0\}.$$

Observe that for every set $\mathcal{R}_j^{B_j}$ and gamble $g_j \in \mathcal{R}_j^{B_j}$, the gamble $\lambda_j g_j$ belongs to $\mathcal{R}_j^{B_j}$, too, for all $\lambda_j > 0$. This allows λ_j to be dropped from the definition of $\mathcal{E}'$. For similar reasons, it is enough to consider at most one gamble from each set $\mathcal{R}_j^{B_j}$ in the sum that defines $\mathcal{E}'$. In other words, it holds that

$$\mathcal{E}' = \{g = g_1^{B_1} + \sum_{B_2 \in \mathcal{B}_2} g_2^{B_2} : B_1 = \{3,4,5,6\}, g_j^{B_j} \in \mathcal{R}_j^{B_j} \cup \{0\}, g \neq 0\}.$$

It can be checked that the sixteen elements making up $\mathcal{E}'$ can be recovered as follows:

$$\begin{aligned}
\mathcal{E}' = \quad & [(\{g : g(1)+g(2) > 0\} \cup \{g : g(1) = g(2) = 0\})\\
\cap \quad & (\{g : \min\{g(3), g(4)\} > 0\} \cup \{g : g(3) = g(4) = 0\})\\
\cap \quad & (\{g : \min\{g(5), g(6)\} > 0\} \cup \{g : g(5) = g(6) = 0\})] \setminus \{0\}.
\end{aligned}$$

The natural extension $\mathcal{E}$ is related to $\mathcal{E}'$ through the relation $\mathcal{E} = \mathcal{L}^+ \cup \{f : f \geq g \text{ for some } g \in \mathcal{E}'\}$. It follows that

$$\mathcal{E} = \left[(\{g : g(1)+g(2) > 0\} \cup \{g : g\mathbb{I}_{\{1,2\}} = 0\}) \cap \{g : g\mathbb{I}_{\{3,4,5,6\}} \geq 0\}\right] \setminus \{0\}.$$

Let us verify that $\mathcal{Q} \cap \mathcal{E} \subseteq \mathcal{R}$, and hence that $\mathcal{R}$ is coherent. Take $g \in \mathcal{Q}$. We have the following possibilities:

- If $g \in \mathcal{Q}_1^{\{1,2\}}$, then $g\mathbb{I}_{\{3,4,5,6\}} = 0$ and $\min\{g(1), g(2)\} \neq 0$; since $g \in \mathcal{E}$, we obtain that $g(1)+g(2) > 0$ and hence $g \in \mathcal{R}$.
- If $g \in \mathcal{Q}_1^{\{3,4,5,6\}}$, then $g\mathbb{I}_{\{1,2\}} = 0$ and $\min g\mathbb{I}_{\{3,4,5,6\}} \neq 0$; since $g \in \mathcal{E}$, we obtain that $\min g\mathbb{I}_{\{3,4,5,6\}} > 0$ and hence $g \in \mathcal{R}$.
- If $g \in \mathcal{Q}_2^{\{3,4\}}$, then $g\mathbb{I}_{\{1,2,5,6\}} = 0$ and $\min g\mathbb{I}_{\{3,4\}} \neq 0$; since $g \in \mathcal{E}$, we obtain that $\min g\mathbb{I}_{\{3,4\}} > 0$ and hence $g \in \mathcal{R}$.
- If $g \in \mathcal{Q}_2^{\{5,6\}}$, then $g\mathbb{I}_{\{1,2,3,4\}} = 0$ and $\min g\mathbb{I}_{\{5,6\}} \neq 0$; since $g \in \mathcal{E}$, we obtain that $\min g\mathbb{I}_{\{5,6\}} > 0$ and hence $g \in \mathcal{R}$.
- If $g \in \mathcal{Q}_2^{\{1,2\}}$ then $g\mathbb{I}_{\{3,4,5,6\}} = 0$ and $g(1)+g(2) \neq 0$; since $g \in \mathcal{E}$, we obtain that $g(1)+g(2) > 0$ and hence $g \in \mathcal{R}$.

Hence, the set $\mathcal{R}$ is coherent but the conditional lower previsions $\underline{P}_1(\cdot|\mathcal{B}_1), \underline{P}_2(\cdot|\mathcal{B}_2)$ are not. Taking into account Theorem 11, we deduce that the first statement in the theorem does not hold, and as consequence the second condition in that statement does not follow from the coherence of $\mathcal{R}$. ⧫

We can also relate the natural extension $\mathcal{E}$ of the set of gambles $\mathcal{R}$ derived by some conditional lower previsions $\underline{P}_1(\cdot|\mathcal{B}_1), \dots, \underline{P}_m(\cdot|\mathcal{B}_m)$ to their conditional natural extensions. Let us consider a gamble $f \in \mathcal{L}$ and a non-empty set $B_0 \subseteq \Omega$; taking into account Remark 5, we can calculate the natural extension $\underline{E}_0(f|B_0)$ of f conditional on B_0 as the supremum value of α for which there are $j = 1, \dots, m, n_j \geq 1, k = 1, \dots, n_j, g_j^k \in \mathcal{H}^j$ and $\delta > 0$, such that

$$-\delta > \sum_{j=1}^{m} \sum_{k=1}^{n_j} G_j(g_j^k|\mathcal{B}_j) - B_0(f - \alpha)$$

in $\mathbb{S}(g_j^k) \cup B_0$.

Definition 15 (**Support**). For every $f \in \mathcal{L}$, we shall denote by B_f the event $\{\omega \in \Omega : f(\omega) \neq 0\}$, and refer to it as the *support* of f. Similarly, given $\varepsilon > 0$ we shall denote by B_f^ε the event $\{\omega \in \Omega : |f(\omega)| \geq \varepsilon\}$, and we shall call it the *ε-support* of f.

Lemma 2. *Let $\underline{P}_1(\cdot|\mathcal{B}_1), \dots, \underline{P}_m(\cdot|\mathcal{B}_m)$ be separately coherent conditional lower previsions that avoid partial loss. Let $\mathcal{R}$ be the set they originate by means of Eq.* (17) *and let $\mathcal{E}$ be the natural extension of $\mathcal{R}$. Take $f \in \mathcal{E}$. Then there is some $\overline{\varepsilon} > 0$ such that $\underline{E}_0(f|B_f^\varepsilon) > 0$ for all $\varepsilon \in (0, \overline{\varepsilon})$. As a consequence, when Ω is finite $\underline{E}_0(f|B_f) > 0$.*

If we compute the natural extension for all non-empty $B_0 \subseteq \Omega$ and all gambles $f \in \mathcal{L}$, we can determine, in the usual way, a corresponding set of desirable gambles:

$$\mathcal{E}_{\underline{P}} := \{g \in \mathcal{L} : g = G_0(f|B_0) + \varepsilon B_0 \text{ for some } f \in \mathcal{L}, \emptyset \neq B_0 \subseteq \Omega, \varepsilon > 0\}, \quad (21)$$

where G_0 corresponds to $\underline{E}_0$. In our next theorem we give a number of properties of the set $\mathcal{E}_{\underline{P}}$ and establish its relationship with $\mathcal{E}$:

Theorem 12. *Assume $\underline{P}_1(\cdot|\mathcal{B}_1), \dots, \underline{P}_m(\cdot|\mathcal{B}_m)$ avoid partial loss, and consider the natural extension $\mathcal{E}$ of the set $\mathcal{R}$ given by Eq.* (17)*. Let $\mathcal{E}_{\underline{P}}$ be the set of gambles given by Eq.* (21)*.*

(1) $\mathcal{E}_{\underline{P}} = \{f : \underline{E}_0(f|B_f) > 0\}$.
(2) $\{f : \underline{E}_0(f) > 0\} \cup \mathcal{L}^+ \subseteq \mathcal{E} \subseteq \{f : \underline{E}_0(f) \geq 0\}$.
(3) $\mathcal{E}_{\underline{P}} \subseteq \mathcal{E} \subseteq \overline{\mathcal{E}}_{\underline{P}}$.
(4) In the particular case where Ω is finite, $\mathcal{E} = \mathcal{E}_{\underline{P}}$.
(5) $\mathcal{E}$ is the natural extension of $\mathcal{E}_{\underline{P}}$.

Let us show that the fourth statement of this theorem cannot be extended to the case where Ω is infinite, and that there may be gambles in $\mathcal{E}$ which may not belong to $\mathcal{E}_{\underline{P}}$. This is because in that case the set $\mathcal{E}_{\underline{P}}$ may not be coherent:

Example 6. Let $\Omega := \mathbb{N}$, and let P be a linear prevision satisfying $P(\{n\}) = 0$ for all n, and let $\mathcal{Q}, \mathcal{R}$ be the sets derived from P by means of Eqs. (16) and (17). It follows from Theorem 11 that $\mathcal{R}$ is coherent relative to $\mathcal{Q}$, and from Proposition 3(d) that its natural extension $\mathcal{E}$ is coherent. Consider the gamble h given by $h(n) := \frac{1}{2^n}$. Then $h \in \mathcal{L}^+$ and as a consequence it belongs to $\mathcal{E}$. On the other hand, the support of h is $B_h = \mathbb{N}$, whence $\underline{E}_0(h|B_f) = \underline{E}_0(h) = P(h) = 0$, taking into account that the unconditional natural extension of P is P itself, and that $P(h) \leq P(\{1,\dots,n\}) + \frac{1}{2^n}P(\{n+1,\dots\}) = \frac{1}{2^n}$ for all n. Applying the first statement from Theorem 12, we deduce that h does not belong to $\mathcal{E}_P$. ⧫

Let us show also that both inclusions in the second statement can be strict:

Example 7. Consider first of all $\Omega := \{1,2,3,4\}, \mathcal{B} := \{\{1,2\},\{3,4\}\}$ and $P, P(\cdot|\mathcal{B})$ determined by $P(\{3,4\}) := 1, P(\{4\}|\{3,4\}) := 1 =: P(\{1\}|\{1,2\})$. It can be checked that these previsions are coherent. Given the gamble $f := \mathbb{I}_{\{1\}} - \mathbb{I}_{\{2\}}$, it holds that $\underline{E}_0(f|B_f) = P(f|\{1,2\}) > 0$, because of Lemma 1(5), whence $f \in \mathcal{E}_{\underline{P}} \subseteq \mathcal{E}$, using the third statement of Theorem 12. On the other hand, the natural extension $\underline{E}_0$ of $P, P(\cdot|\mathcal{B})$ is given by [19, Theorem 6.7.2] $\underline{E}_0 = P(P(\cdot|\mathcal{B}))$, and it satisfies $\underline{E}_0(f) = P(\{1,2\})P(f|\{1,2\}) = 0$.

For the second inclusion, take $\Omega := \{1,2\}$ and let P be the prevision associated to the uniform probability distribution on Ω. Let $\mathcal{R}$ be the set of desirable gambles derived from P through (17). Applying P to the expression for the natural extension of $\mathcal{R}$ in (19), we obtain that $\mathcal{E} \subseteq \{f : P(f) > 0\}$: to see this, note that for any non-zero gamble f it holds that $\underline{E}(G(f) + \varepsilon S(f)) = P(G(f) + \varepsilon) = \varepsilon > 0$, and since $P(1) = P(2) > 0$ we also have $\mathcal{L}^+ \subseteq \{f : P(f) > 0\}$. Applying the second statement in Theorem 12, we deduce that $\mathcal{E} = \{f : P(f) > 0\}$, and this is a strict subset of $\{f : P(f) \geq 0\}$, because the gamble $f := (1,-1)$ belongs to the latter set but not to the former. ⧫

6. Commutativity and equal expressivity

6.1. **Commutativity.** So far, we have introduced two ways of relating conditional lower previsions and sets of really desirable gambles: we can derive the conditional lower previsions from the desirable gambles (Section 4) or we can derive a set of desirable gambles from the conditional lower previsions (Section 5). We now proceed to investigate whether these two procedures commute.

First of all, we should like to show that one can take conditional lower previsions $\underline{P}_1(\cdot|\mathcal{B}_1),\dots,\underline{P}_m(\cdot|\mathcal{B}_m)$ that avoid partial loss, turn them into a (coherent) set of desirable gambles $\mathcal{R}$, and do inferences from it that are equal to those that can be obtained from the natural extensions of $\underline{P}_1(\cdot|\mathcal{B}_1),\dots,\underline{P}_m(\cdot|\mathcal{B}_m)$. This would allow us to always work in the domain of desirable gambles even if we start from conditional lower previsions. We proceed to show that this is indeed the case. A similar result was already established, in a slightly different context, by Peter Williams in [25, Propositions 2 and 3]. The differences are basically in his formulation of the coherence condition, which is nevertheless equivalent to the one we are using in this paper when we have finite partitions, and on the use of conditional *upper* previsions. Another difference is that, unlike us, he assumes the zero gamble to be desirable. For the sake of completeness, we also establish the result within our framework:

Theorem 13. *Consider the sets $\mathcal{Q}, \mathcal{R}$ derived from the jointly coherent conditional lower previsions $\underline{P}_1(\cdot|\mathcal{B}_1),\dots,\underline{P}_m(\cdot|\mathcal{B}_m)$ by means of Eqs.* (16) *and* (17). *Let $\underline{P}'_1(\cdot|\mathcal{B}_1),\dots,\underline{P}'_m(\cdot|\mathcal{B}_m)$ be the conditional lower previsions obtained from $\mathcal{R}$ through Eq.* (13). *Then $\underline{P}_j(f_j|B_j) = \underline{P}'_j(f_j|B_j)$ for all $j = 1,\dots,m, f_j \in \mathcal{H}^j, B_j \in \mathcal{B}_j$.*

Let us give an example of illustration of this theorem:

Example 8. Consider $\Omega := \{1,2,3,4\}, \mathcal{B}_1 := \{\{1,2\},\{3,4\}\}, \mathcal{B}_2 := \{\{1,3\},\{2,4\}\}$, and $\underline{P}_1(\cdot|\mathcal{B}_1), \underline{P}_2(\cdot|\mathcal{B}_2)$ given by

$$\underline{P}_1(f|\{1,2\}) := \min\{f(1), f(2)\},$$
$$\underline{P}_1(f|\{3,4\}) := \frac{f(3)+f(4)}{2},$$
$$\underline{P}_2(f|\{1,3\}) := \min\{f(1), f(3)\},$$
$$\underline{P}_2(f|\{2,4\}) := \frac{f(2)+f(4)}{2},$$

for every gamble $f \in \mathcal{L}$. Then $\underline{P}_1(\cdot|\mathcal{B}_1), \underline{P}_2(\cdot|\mathcal{B}_2)$ are coherent: this follows applying [12, Theorem 6] to the lower prevision $\underline{P}$ given by $\underline{P}(f) := \min\{f(1), \frac{f(2)+f(3)+f(4)}{3}\}$.

The set $\mathcal{R}$ of desirable gambles they originate is given by $\mathcal{R} := \mathcal{R}_1^{\{1,2\}} \cup \mathcal{R}_1^{\{3,4\}} \cup \mathcal{R}_2^{\{1,3\}} \cup \mathcal{R}_2^{\{2,4\}}$, where

$$\mathcal{R}_1^{\{1,2\}} := \{f \in \mathcal{L} : \min\{f(1), f(2)\} > 0, f(3) = f(4) = 0\},$$
$$\mathcal{R}_1^{\{3,4\}} := \{f \in \mathcal{L} : f(1) = f(2) = 0, f(3) + f(4) > 0\},$$
$$\mathcal{R}_2^{\{1,3\}} := \{f \in \mathcal{L} : \min\{f(1), f(3)\} > 0, f(2) = f(4) = 0\},$$
$$\mathcal{R}_2^{\{2,4\}} := \{f \in \mathcal{L} : f(1) = f(3) = 0, f(2) + f(4) > 0\}.$$

The definition of set $\mathcal{R}_i^B$, $i = 1,2$, immediately follows from the application of Eq. (18) to the corresponding conditional $\underline{P}_i(\cdot|B)$.

Now, if we consider the conditional lower previsions $\underline{P}'_1(\cdot|\mathcal{B}_1)$ and $\underline{P}'_2(\cdot|\mathcal{B}_2)$ derived from Eq. (13) we recover $\underline{P}_1(\cdot|\mathcal{B}_1), \underline{P}_2(\cdot|\mathcal{B}_2)$. For instance, given $f \in \mathcal{L}$,

$$\begin{aligned}\underline{P}'_1(f|\{1,2\}) &= \sup\{\mu : \mathbb{I}_{\{1,2\}}(f-\mu) \in \mathcal{R}\} \\ &= \sup\{\mu : \mathbb{I}_{\{1,2\}}(f-\mu) \in \mathcal{R}_1^{\{1,2\}}\} = \min\{f(1), f(2)\}.\end{aligned}$$

Here, the second equality follows because if $g := \mathbb{I}_{\{1,2\}}(f-\mu) \in \mathcal{R}_1^{\{3,4\}}$ we contradict that $g(3)+g(4) > 0$, if it belongs to $\mathcal{R}_2^{\{1,3\}}$ we contradict $g(3) > 0$, and if it belongs to $\mathcal{R}_2^{\{2,4\}}$ we should have $g(2) + g(4) = g(2) > 0$ and $g(1) = 0$, which means that $\mu = f(1) < f(2)$ and then again $\mu \leq \min\{f(1), f(2)\}$.

The other cases can be established similarly. ⧫

In addition, Williams also showed in [25, Theorem 1] that we can use the set of gambles $\mathcal{E}$ to derive the natural extensions of the initial assessments. We next establish this result in our context:

Theorem 14. *Consider conditional lower previsions* $\underline{P}_1(\cdot|\mathcal{B}_1), \dots, \underline{P}_m(\cdot|\mathcal{B}_m)$ *that avoid partial loss, and let* $\underline{E}_0(f|B_0)$ *be their generic natural extension to a gamble* $f \in \mathcal{L}$*, conditional on a non-empty subset* B_0 *of* Ω*. Let* $\mathcal{Q}, \mathcal{R}$ *be the sets derived from* $\underline{P}_1(\cdot|\mathcal{B}_1), \dots, \underline{P}_m(\cdot|\mathcal{B}_m)$ *by means of Eqs.* (16) *and* (17)*, and let* $\mathcal{E}$ *be the natural extension of* $\mathcal{R}$*. Let* $\underline{E}'_0(f|B_0)$ *be the conditional lower prevision obtained from* $\mathcal{E}$ *through Eq.* (13)*. Then* $\underline{E}_0(f|B_0) = \underline{E}'_0(f|B_0)$*.*

Conversely, we can also start from a coherent set of desirable gambles $\mathcal{E}$ with respect to $\mathcal{L}$, and define conditional lower previsions $\underline{P}_1(\cdot|\mathcal{B}_1), \dots, \underline{P}_m(\cdot|\mathcal{B}_m)$ by means of Eq. (13). We can use these conditional previsions to derive another set $\mathcal{R}'$ of desirable gambles by means of Eq. (17). Let $\mathcal{E}'$ be the natural extension of $\mathcal{R}'$, defined by Eq. (10).

Proposition 9. *The set of gambles $\mathcal{E}'$ is included in $\mathcal{E}$.*

To see that we may not have the equality in general, consider the following example:

Example 9. Consider $\Omega := \{\omega_1, \omega_2\}$, and let $\mathcal{E} := \{f : f(\omega_1) + f(\omega_2) > 0\}$. Take $\mathcal{B} := \{\{\omega_1\}, \{\omega_2\}\}$. Then applying Eq. (13) given any gamble f on $\mathcal{L}$, $\underline{P}(f|\{\omega_1\}) = f(\omega_1)$ and $\underline{P}(f|\{\omega_2\}) = f(\omega_2)$. If we now consider the set $\mathcal{R}'$ derived from $\underline{P}(\cdot|\mathcal{B})$ through Eq. (18), we obtain

$$\mathcal{R}' = \{f : f(\omega_1) > 0, f(\omega_2) = 0\} \cup \{f : f(\omega_1) = 0, f(\omega_2) > 0\},$$

whose natural extension is (Eq. (10)) $\mathcal{E}' = \mathcal{L}^+$, which is a strict subset of $\mathcal{E}$. ♦

What these two results provide is further evidence that sets of desirable gambles are more informative than coherent lower previsions: although for a number of coherent conditional lower previsions there is always a coherent set of desirable gambles with the same behavioural implications, not every coherent set of desirable gambles can be recovered from a set of coherent conditional lower previsions.

6.2. **Equal expressivity.** Taking the previous results into account, we shall investigate next if there is some particular subclass of coherent sets of desirable gambles which is as expressive as coherent conditional lower previsions. In the next definition we make precise the idea of equal expressivity.

Definition 16 (**Equal expressivity**). Let $\underline{P}_1(\cdot|\mathcal{B}_1), \dots, \underline{P}_m(\cdot|\mathcal{B}_m)$ be a set of coherent conditional lower previsions, $\mathcal{Q}, \mathcal{R}$ be the sets derived from them by means of Eqs. (16) and (17), and let $\mathcal{E}$ be the natural extension of $\mathcal{R}$. Let $\mathcal{E}'$ be a coherent set of desirable gambles. We say that $\underline{P}_1(\cdot|\mathcal{B}_1), \dots, \underline{P}_m(\cdot|\mathcal{B}_m)$ and $\mathcal{E}'$ are *equally expressive* if $\mathcal{E} = \mathcal{E}'$.

This definition is well posed as $\mathcal{E}$ is coherent thanks to Theorem 11 and Proposition 3(d). More generally speaking, the definition hinges on the consideration that desirability is a more primitive concept than that of lower prevision. Therefore, when the desirability statements implied by the lower previsions coincide with those in $\mathcal{E}'$, then all the conclusions we may draw from either of them will coincide with those obtained from the other (see also Theorem 14).

Now, let us focus on some special sets of desirable gambles.

Definition 17 (**Conditional strict desirability**). Let $\mathcal{E}$ be a coherent set of desirable gambles. Consider $f \in \mathcal{E}$, and its ε-support B_f^ε for all $\varepsilon > 0$. We say that $\mathcal{E}$ is a coherent set of *conditionally strictly desirable gambles* if it satisfies the following condition:

$$f \in \mathcal{E} \Rightarrow \exists \bar{\varepsilon} > 0 : \forall \varepsilon \in (0, \bar{\varepsilon}) \text{ there is } \delta_\varepsilon > 0 \text{ s.t. } B_f^\varepsilon (f - \delta_\varepsilon) \in \mathcal{E}. \tag{22}$$

The next proposition shows that the desirability-counterpart of sets of conditional lower previsions are sets of conditionally strictly desirable gambles. This shows in a definite sense that sets of conditional lower previsions are at most as expressive as a special class of desirable gambles.

Proposition 10. *For each finite set of coherent conditional lower previsions there is a coherent set of conditionally strictly desirable gambles that is equally expressive.*

When Ω is finite, it is possible to provide a tighter link between sets of conditional lower previsions and sets of conditionally strictly desirable gambles. In that case, the condition of conditional strict desirability in (22) is equivalent to

$$f \in \mathcal{E} \Rightarrow \exists \delta > 0 : B_f(f-\delta) \in \mathcal{E}. \tag{23}$$

To see that (22) implies (23), note that when Ω is finite there is some $\varepsilon' > 0$ for which $B_f^\varepsilon = B_f$ for every $\varepsilon \in (0, \varepsilon')$. Then it suffices to consider the δ_ε associated to any $\varepsilon \in (0, \min\{\varepsilon', \bar{\varepsilon}\})$ and define $\delta := \delta_\varepsilon$. Conversely, consider again $\varepsilon' > 0$ for which $B_f^\varepsilon = B_f$ for every $\varepsilon \in (0, \varepsilon')$. Define $\bar{\varepsilon} := \varepsilon'$, and let $\delta_\varepsilon := \delta$ for all $\varepsilon \in (0, \bar{\varepsilon})$.

On this basis, we obtain the next result.

Theorem 15. *Let Ω be a finite set.*

(1) For each coherent set of conditionally strictly desirable gambles there is a finite set of coherent conditional lower previsions that is equally expressive.
(2) For each finite set of coherent conditional lower previsions there is a coherent set of conditionally strictly desirable gambles that is equally expressive.

What this theorem says is that in the finite case there is a correspondence between sets of conditionally strictly desirable gambles and sets of conditional lower previsions. This result may look surprising at first. In fact, Walley motivated the introduction of real desirability (see [19, Appendix F]), among other things, by stressing in particular that there may be different sets of desirable gambles that yield different conditional lower previsions while yielding the same unconditional ones.[13] This shows that really desirable gambles are more expressive than unconditional lower previsions. Our result goes a step further by showing that this holds also when we consider sets of conditional lower previsions. And the question is that, while in the cases discussed by Walley, the differences in the sets of desirable gambles were possible to reveal by looking at the conditional lower previsions they originate, in our case this is not possible: what we show, in fact, is that there is some extra expressivity of real desirable gambles that is not revealed by any conditional probabilistic statement.

To see what this extra expressivity is for, we have to briefly consider the notion of preference. In fact, Walley has pointed out long ago [19, Section 3.7] that there is one-to-one correspondence between sets of desirable gambles $\mathcal{R}$ and partial preference orderings among gambles: given a set $\mathcal{R}$, say that f is *preferred* to g (or $f \succ g$, in symbols) whenever $f-g$ belongs to $\mathcal{R}$;[14] conversely, a partial preference ordering $\succ$ originates a set of desirable gambles through the definition $\mathcal{R} := \{f-g : f \succ g\}$. The interpretation at the basis of these transformations is straightforward: f is preferred to g if and only if it is desirable to give away g in order to have f. We can also consider a weaker notion of preference: say that f is *weakly preferred* to g (in symbols, $f \succeq g$) if and only if gamble $f-g+\varepsilon$ is desirable for all $\varepsilon > 0$. In this case $f+\varepsilon$ is preferred to g for all $\varepsilon > 0$, but f itself may not. At this point we are ready to show how the extra expressivity of desirable gambles comes about:

Example 10. Consider $\Omega := \{\omega_1, \omega_2\}$, and let $\mathcal{R}_1 := \{f \in \mathcal{L} : f(\omega_1) + f(\omega_2) > 0\}, \mathcal{R}_2 := \mathcal{R}_1 \cup \{f \in \mathcal{L} : f(\omega_1) = -f(\omega_2) < 0\}$. $\mathcal{R}_1$ and $\mathcal{R}_2$ are coherent sets of desirable gambles with respect to $\mathcal{L}$ (use Proposition 2). Moreover, they originate

[13] This is true even when the conditional event has probability zero.

[14] Such a relation is in fact a partial order in general, as it can be the case that neither $f-g$ nor $g-f$ belong to $\mathcal{R}$.

the same conditional and unconditional lower previsions on $\mathcal{L}$ through (13): in the unconditional case we obtain $\underline{P}(f) = \frac{f(\omega_1)+f(\omega_2)}{2}$, and in the conditional case $\underline{P}(f|\{\omega_1\}) = f(\omega_1), \underline{P}(f|\{\omega_2\}) = f(\omega_2)$. Therefore $\mathcal{R}_1$ and $\mathcal{R}_2$ are indistinguishable as far as probabilistic statements are concerned. This is not the case of preferences. Consider $f := (2, -1)$ and $g := (1, 0)$: under both $\mathcal{R}_1$ and $\mathcal{R}_2$ it holds that $f \succeq g$; but under $\mathcal{R}_2$ we obtain the additional, and perhaps unexpected, information that $f \prec g$. ♦

In other words, desirable gambles give us the opportunity to distinguish preference from weak preference, which is something that probabilities (that is, lower previsions) do not allow us to do. This point has already been made in particular in a paper also authored by Walley [21, Example 7(f), and Section 4(f)].

7. Conclusions

The behavioural theory of imprecise probabilities can be formulated by means of lower and upper previsions, credal sets of linear previsions, or sets of desirable gambles. In the unconditional case, there is a well-known correspondence between the three representations, which allows us to move from one to another. In this paper we have investigated the more involved situation when we consider beliefs conditional on some evidence.

We have focused on two problems: how to derive conditional lower previsions from a set of desirable gambles, and viceversa. For the first problem, we have established sufficient conditions for the conditional lower previsions to satisfy the different consistency axioms in [19] (separate coherence, avoiding partial and uniform sure loss, weak and strong coherence). The most important result in this section is Theorem 8, where we give sufficient conditions on the set of desirable gambles so that the derived conditional lower previsions are coherent, thus extending some results from the literature. In this section, we also detail the connections with the important notion of almost-desirability.

With respect to the second problem, we have derived sets of desirable gambles from conditional lower previsions, and determined under which conditions the consistency properties of the previsions hold onto the gambles. Moreover, we have showed that these gambles can be used effectively to compute the conditional natural extensions of our assessments, which represent their behavioural implications. Specifically, in Theorem 11 we give conditions for the equivalence between the coherence of the conditional lower previsions and their derived sets of desirable gambles.

These results have highlighted a well-known fact within the behavioural theory of imprecise probabilities: that sets of really desirable gambles are more informative than coherent lower previsions. This is made clearer in Theorem 13 and Proposition 9 in Section 6.1. However, and in parallel to the situation in the unconditional case, we prove in Theorem 15 that when the referential space is finite we can consider a subset of the class of sets of really desirable gambles which is as expressive, in the sense that it allows us to produce the same inferences, as conditional lower previsions: those which satisfy conditional strict desirability. These are an analog of the sets of strictly desirable gambles from the unconditional case.

An important remark throughout is that there is an alternative definition of coherence for sets of desirable gambles which assumes that the zero gamble is desirable, and as a consequence it includes it in the natural extension of a set of desirable gambles that avoids partial loss; see, for example, [3, 19, 25] for some papers that follow this approach. The results we have established in this paper can be adapted to this alternative definition, by making some minor modifications: for instance the expression of the natural extension in Definition 10 will be different now, and this means that the equivalent expressions in Theorem 9 should be slightly modified.

With respect to the open problems deriving from this paper, one of the most important is the generalisation of the results in this paper to conditional lower previsions on infinite partitions. In fact, despite the formulation of desirability in this paper is very general (among other things, we allow for any cardinality of the possibility space), we have restricted the attention to finite conditioning partitions. In order to relax this requirement, one has to deal carefully with the issue of *conglomerability*, which is discussed in some detail in [19, Chapter 6] (see also [24] and [16]) and which is one of the points of disagreement between the approaches to coherence of Walley and Williams, as well as de Finetti's. On our view, this would mean (i) to deepen the discussion concerning whether or not conglomerability can be justified as a rationality requirement (which is the major source of disagreement between the above-mentioned authors); and (ii) to verify how severe are the technical implications of adding a conglomerability axiom to our notion of coherence for sets of desirable gambles, as it is already quite clear that this is going to complicate the computation of the natural extension. Another open problem would be the extension of the results on equal expressivity to infinite spaces.

Finally, it is useful also to point out that the results of this paper have a direct bearing on a topic in artificial intelligence that is currently subject of much attention: modelling preferences (and in particular partial preferences, see [9] for some recent work). Preferences are increasingly important for AI sectors as diverse as agents, machine learning, and argumentation, to say a few. The relationship between desirability and preferences is very tight, as we have described at the end of Section 6.2: there is actually one-to-one correspondence between sets of desirable gambles and partial preference orderings, so that the material in this paper can immediately be re-phrased as a study on the relationship between preference modelling and imprecise probability. Exploring this link further appears to be a promising research avenue.

Acknowledgements

The research in this paper has been partially supported by the projects TIN2008-06796-C04-01, MTM2010-17844, by the Swiss NSF grants n. 200020_134759 / 1, 200020-121785/1, and by the Hasler foundation grant n. 10030.

Appendix A. Additional results

This appendix collects some additional results that are related to the main discussion in the paper but are not necessary to follow it. We report them here to ease the accessibility of the paper to the reader more interested in the main development.

A.1. **Regular conditioning.** In this section we investigate when the conditional lower previsions obtained from a coherent set of desirable gambles $\mathcal{R}$ through Eq. (13) match those that would be obtained by applying regular extension on the associated unconditional lower prevision. Taking into account Proposition 7, we shall assume $\mathcal{Q} = \mathcal{L}$ here.

Definition 18 (**Regular extension**). Given a set $\mathcal{M}$ of linear previsions and a partition $\mathcal{B}$ of Ω, the *regular extension* $\underline{R}(\cdot|\mathcal{B})$ is given by

$$\underline{R}(f|B) := \inf \left\{ \frac{P(Bf)}{P(B)} : P \in \mathcal{M}, P(B) > 0 \right\}$$

for every $B \in \mathcal{B}, f \in \mathcal{L}$ whenever there is some $P \in \mathcal{M}$ such that $P(B) > 0$, and is given by $\underline{R}(f|B) := \inf_{\omega \in B} f(\omega)$ otherwise. This amounts to applying Bayes' rule to the dominating linear previsions whenever possible (i.e., disregarding the linear previsions that assign zero probability to the conditioning event).

If we have an unconditional coherent lower prevision $\underline{P}$, the regular extension $\underline{R}(\cdot|\mathcal{B})$ is derived from $\underline{P}$ by applying the above definition to the credal set $\mathcal{M}(\underline{P})$. When the partition $\mathcal{B}$ of Ω is finite, as it is the case in this paper, it follows from [19, Appendix (J3)] that $\underline{P}, \underline{R}(\cdot|\mathcal{B})$ are coherent. The regular extension has been proposed and used a number of times in the literature as an updating rule [2, 6, 7, 8, 19, 20]. For the case of finite Ω, a comparison with natural extension has been made in [12, 14].

Now, denote by $\underline{P}(f|B)$ the lower prevision of $f \in \mathcal{L}$ conditional on $B \in \mathcal{B}$ obtained from $\mathcal{R}$ through (13). Let $\overline{\mathcal{R}}$ denote the closure of $\mathcal{R}$ in the topology of uniform convergence. $\overline{\mathcal{R}}$ is a set of almost-desirable gambles according to Proposition 4. Let $\mathcal{M}$ be the corresponding set of linear previsions, $\underline{P}$ be its lower envelope, and $\underline{R}(f|B)$ the associated regular extension. The following proposition gives a necessary and sufficient condition for the procedure of Eq. (13) to provide us with the regular extension.

Theorem 16. *Consider $B \in \mathcal{B}$ and $f \in \mathcal{L}$. Then $\overline{P}(B) > 0$ and $\underline{P}(f|B) = \underline{R}(f|B)$ if and only if*

$$\mu \in \mathbb{R}, B(f - \mu) \in \overline{\mathcal{R}} \Rightarrow B(f - (\mu - \varepsilon)) \in \mathcal{R} \ \forall \varepsilon > 0. \tag{24}$$

An immediate corollary that we establish without proof is the following:

Corollary 5. *Consider $B \in \mathcal{B}$. Then $\overline{P}(B) > 0$ and $\underline{P}(f|B) = \underline{R}(f|B)$ for all $f \in \mathcal{L}$ if and only if*

$$Bf \in \overline{\mathcal{R}} \Rightarrow B(f + \varepsilon) \in \mathcal{R} \ \forall \varepsilon > 0. \tag{25}$$

This corollary generalises the result given by Theorem 3 in [1]. Such a theorem shows that the condition

$$f \in \overline{\mathcal{R}} \Rightarrow f + \varepsilon B \in \mathcal{R} \ \forall \varepsilon > 0 \tag{26}$$

implies $\overline{P}(B) > 0$ and $\underline{P}(f|B) = \underline{R}(f|B)$. The above corollary shows that this condition is unnecessarily strong as we do not need to take into account gambles f such that $Bf \neq f$.

Remark 6. The same reference [1] makes two additional claims, which we briefly discuss in our language. The first (in Lemma 1 from that paper) is that $\underline{P}(B) > 0$ is sufficient for (26) to hold. An analogous claim is easy to obtain in our context: in fact, when $\underline{P}(B) > 0$, GBR determines uniquely the lower prevision conditional on B, and therefore the regular extension $\underline{R}(f|B)$ coincides with $\underline{P}(f|B)$, so that (25) follows.

The second claim (Lemma 2 in [1]) can be reformulated in our context as follows: if $\overline{P}(B) > 0$, $B(f-\mu) \in \overline{\mathcal{R}}$ and $B((\mu-\varepsilon)-f) \notin \overline{\mathcal{R}}\ \forall \varepsilon > 0 \Rightarrow B(f-(\mu-\varepsilon)) \in \mathcal{R}\ \forall\varepsilon > 0$, then (24) holds. But since $\overline{P}(B) > 0$, we know, thanks to Proposition 5, that there cannot be $\varepsilon > 0$ such that $B((\mu-\varepsilon)-f) \in \overline{\mathcal{R}}$, as otherwise $-\varepsilon B = B((f-\mu)+((\mu-\varepsilon)-f)) \in \overline{\mathcal{R}}$, thus violating (15). In other words, under the condition $\overline{P}(B) > 0$, the assumption

$$B(f-\mu) \in \overline{\mathcal{R}} \text{ and } B((\mu-\varepsilon)-f) \notin \overline{\mathcal{R}}\ \forall\varepsilon > 0 \Rightarrow B(f-(\mu-\varepsilon)) \in \mathcal{R}\ \forall\varepsilon > 0$$

reduces to $B(f-\mu) \in \overline{\mathcal{R}} \Rightarrow B(f-(\mu-\varepsilon)) \in \mathcal{R}\ \forall\varepsilon > 0$, which is Eq. (24), and which is equivalent to the equality $\underline{P}(f|B) = \underline{R}(f|B)$ because of Theorem 16. ♦

A.2. **On weak desirability.** In this section, we briefly discuss another approach to sets of desirable gambles which was recently introduced by de Cooman and Quaeghebeur in [5]: that of a set of *weakly desirable* gambles. This is an intermediate notion between those of desirability and almost-desirability, and it was introduced in the context of exchangeable imprecise models.

Given a coherent set of really desirable gambles $\mathcal{R}$, the set of weakly desirable gambles associated to $\mathcal{R}$ is defined by

$$\mathcal{D}_{\mathcal{R}} := \{f : f+g \in \mathcal{R}\ \forall g \in \mathcal{R}\}. \tag{27}$$

It follows immediately from this definition that $\mathcal{R} \cup \{0\} \subseteq \mathcal{D}_{\mathcal{R}}$.

We can study the properties of a set of weakly desirable gambles by means of the following axioms:

(WD1) $f \lneq 0 \Rightarrow f \notin \mathcal{D}_{\mathcal{R}}$.
(WD2) $f \geq 0 \Rightarrow f \in \mathcal{D}_{\mathcal{R}}$.
(WD3) $f \in \mathcal{D}_{\mathcal{R}}, \lambda > 0 \Rightarrow \lambda f \in \mathcal{D}_{\mathcal{R}}$.
(WD4) $f, g \in \mathcal{D}_{\mathcal{R}} \Rightarrow f+g \in \mathcal{D}_{\mathcal{R}}$.
(WD5) If $f+\delta \in \mathcal{D}_{\mathcal{R}}\ \forall\delta > 0$ and it does not hold that $f \lneq 0$ then $f \in \mathcal{D}_{\mathcal{R}}$.

From [5, Proposition 5], the set $\mathcal{D}_{\mathcal{R}}$ originated by a coherent set of really desirable gambles satisfies (WD1)–(WD4). We next give a sufficient condition for a set of gambles to be a set of weakly desirable gambles.

Proposition 11. *For every set of gambles $\mathcal{D}$ satisfying (WD1)–(WD5) there is a coherent set of really desirable gambles $\mathcal{R}$ such that $\mathcal{D}_{\mathcal{R}} = \mathcal{D}$.*

However, not every set of weakly desirable gambles associated to a set $\mathcal{R}$ of desirable gambles satisfies (WD5), as the following example shows:

Example 11. Let $\Omega := \{\omega_1, \omega_2\}$ and let $\mathcal{R} := \{f : f(\omega_1)+f(\omega_2) > 0\} \cup \{f : f(\omega_1) > 0, f(\omega_2) = -f(\omega_1)\}$. Then $\mathcal{R}$ is a coherent set of really desirable gambles. Its associated set of weakly desirable gambles is $\mathcal{D}_{\mathcal{R}} = \mathcal{R} \cup \{0\}$: given a gamble f such that $f(\omega_1)+f(\omega_2) = -\varepsilon < 0$, the constant gamble equal to $\frac{\varepsilon}{2}$ belongs to $\mathcal{R}$ and $f+\frac{\varepsilon}{2}$ does not belong to $\mathcal{R}$; on the other hand, if $f(\omega_1)+f(\omega_2) = 0$ and $f(\omega_1) < 0$, then $-\frac{f}{2}$ belongs to $\mathcal{R}$ but $f-\frac{f}{2} = \frac{f}{2}$ does not.

To see that $\mathcal{R}$ does not satisfy (WD5), note that the gamble f given by $f(\omega_1) = -1, f(\omega_2) = 1$ satisfies that $f+\delta \in \mathcal{R} \subseteq \mathcal{D}_{\mathcal{R}}$ for every $\delta > 0$, but $f \notin \mathcal{D}_{\mathcal{R}}$. ♦

On the other hand, a coherent set of really desirable gambles $\mathcal{R}$ lies between the associated sets of strictly desirable $\underline{\mathcal{R}}$ and almost-desirable $\overline{\mathcal{R}}$ gambles. It is not difficult to see that the associated set of weakly desirable gambles lies between the coherent set of really desirable gambles and the coherent set of almost-desirable gambles: remember that $\overline{\mathcal{R}}$ is the topological closure of set $\mathcal{R}$ (see Proposition 4); then it is clear that $\overline{\mathcal{R}}$ undergoes the same condition as in Eq. (27) but restricting ourselves to gambles g which are constant on some $\varepsilon > 0$.

We next show that for the purposes of our work in this paper, really desirable gambles and weakly desirable gambles are going to provide the same information:

Proposition 12. *Let $\mathcal{R}$ be a coherent set of really desirable gambles, and let $\mathcal{D}_{\mathcal{R}}$ be its associated set of weakly desirable gambles. Then for every subset B of Ω and for every gamble f,*

$$\sup\{\mu : B(f-\mu) \in \mathcal{R}\} = \sup\{\mu : B(f-\mu) \in \mathcal{D}_{\mathcal{R}}\}. \tag{28}$$

Appendix B. Proofs

This appendix gathers the proofs of all the results in the paper.

Proof of Proposition 1. We make a circular proof. Let us show that the first statement implies the second. Assume Eq. (3) fails. Then there are $\varepsilon > 0$, $g_j^k \in \mathcal{H}^j$, $j = 1, \dots, m$, $n_j \geq 1$, $k = 1, \dots, n_j$, such that for all $\omega \in \Omega$,

$$\left[\sum_{j=1}^{m}\sum_{k=1}^{n_j} G_j(g_j^k|\mathcal{B}_j) + \varepsilon S_j(g_j^k)\right](\omega) \leq 0,$$

and hence

$$\left[\sum_{j=1}^{m}\sum_{k=1}^{n_j} G_j(g_j^k|\mathcal{B}_j)\right](\omega) \leq -\varepsilon \left[\sum_{j=1}^{m}\sum_{k=1}^{n_j} S_j(g_j^k)\right](\omega) < 0$$

for every $\omega \in \mathbb{S}(g_j^k)$, since not all the g_j^k are zero gambles. This implies that the conditional lower previsions $\underline{P}_1(\cdot|\mathcal{B}_1), \dots, \underline{P}_m(\cdot|\mathcal{B}_m)$ incur partial loss.

That the second statement implies the third follows by taking into account that the sum in Eq. (4) is zero outside $\mathbb{S}(g_j^k)$.

Finally, assume that Eq. (4) holds. If our conditional lower previsions incur partial loss, then there are $\delta > 0$, $n_j \geq 1, g_j^k \in \mathcal{H}^j$, $j = 1, \dots, m$, $k = 1, \dots, n_j$, such that not all the g_j^k are zero gambles, which lead to

$$\sum_{j=1}^{m}\sum_{k=1}^{n_j} G_j(g_j^k|\mathcal{B}_j)(\omega) \leq -\delta$$

for all $\omega \in \mathbb{S}(g_j^k)$. Therefore, we can define $\varepsilon := \frac{\delta}{1+\sum_{j=1}^{m} n_j}$, and obtain that for all $\omega \in \mathbb{S}(g_j^k)$,

$$\left[\sum_{j=1}^{m}\sum_{k=1}^{n_j} G_j(g_j^k|\mathcal{B}_j) + \varepsilon S_j(g_j^k)\right](\omega) =$$

$$\left[\sum_{j=1}^{m}\sum_{k=1}^{n_j} G_j(g_j^k|\mathcal{B}_j)\right](\omega) + \varepsilon \left[\sum_{j=1}^{m}\sum_{k=1}^{n_j} S_j(g_j^k)\right](\omega) \leq$$

$$-\delta + \delta \cdot \frac{\sum_{j=1}^{m} n_j}{1+\sum_{j=1}^{m} n_j} < 0.$$

This implies that expression (4) fails, a contradiction. $\square$

Proof of Lemma 1. 1. Consider $j \in \{1, \dots, m\}, f_j \in \mathcal{H}^j, B_j \in \mathcal{B}_j$, and let us consider the gamble $B_j f_j \in \mathcal{H}^j$. Then for all $\alpha < \underline{P}_j(f_j|B_j)$ it holds that

$$\sup_{S(B_j f_j) \cup B_j} [G_j(B_j f_j|\mathcal{B}_j) - B_j(f_j - \alpha)]$$
$$= \sup_{B_j} [G_j(f_j|B_j) - B_j(f_j - \alpha)] = \alpha - \underline{P}_j(f_j|B_j) < 0,$$

whence Eq. (7) implies that $\underline{E}_j(f_j|B_j) \geq \alpha$. As a consequence, $\underline{E}_j(f_j|B_j) \geq \underline{P}_j(f_j|B_j)$.

2. Let us consider the direct implication. Take $f \in \mathcal{L}$ and $B_0 \subseteq \Omega$, $B_0 \neq \emptyset$. By definition of natural extension, we know that for all $\alpha < \underline{E}_0(f|B_0)$, there are $j = 1, \dots, m, n_j \geq 1, k = 1, \dots, n_j, g_j^k \in \mathcal{H}^j$, $\delta > 0$ such that

$$B_0(f - \alpha) - \delta > \sum_{j=1}^{m} \sum_{k=1}^{n_j} G_j(g_j^k|\mathcal{B}_j) \tag{29}$$

in $\mathbb{S}(g_j^k) \cup B_0$. Let us show that any such α is smaller than $\sup_{\omega \in B_0} f(\omega)$, from which we deduce that $\underline{E}_0(f|B_0) \leq \sup_{\omega \in B_0} f(\omega)$. In the case where all the g_j^k are zero gambles, that follows immediately from (29) as $\alpha < \inf_{\omega \in B_0} f(\omega) \leq \sup_{\omega \in B_0} f(\omega)$.

Let us assume that not all the g_j^k are zero gambles. It follows from (29) that (i) the gamble $\sum_{j=1}^{m} \sum_{k=1}^{n_j} G_j(g_j^k|\mathcal{B}_j)$ is bounded by $-\delta$ in $\mathbb{S}(g_j^k) \setminus B_0$, and (ii) that $\sup_{\omega \in B_0} [B_0(f - \alpha)](\omega) > \sup_{\omega \in B_0} [\sum_{j=1}^{m} \sum_{k=1}^{n_j} G_j(g_j^k|\mathcal{B}_j)](\omega)$. On the other hand, knowing that $\underline{P}_1(\cdot|\mathcal{B}_1), \dots, \underline{P}_m(\cdot|\mathcal{B}_m)$ avoid partial loss tells us (iii) that $\sum_{j=1}^{m} \sum_{k=1}^{n_j} G_j(g_j^k|\mathcal{B}_j)$ has non-negative supremum over $\mathbb{S}(g_j^k)$, and hence also over $\mathbb{S}(g_j^k) \cup B_0$. By using (i) and (iii), we obtain that $\sup_{\omega \in B_0} [\sum_{j=1}^{m} \sum_{k=1}^{n_j} G_j(g_j^k|\mathcal{B}_j)](\omega) \geq 0$; this, together with (ii) allows us to deduce that $\sup_{\omega \in B_0} [B_0(f - \alpha)](\omega) > 0$, and hence that $\alpha < \sup_{\omega \in B_0} f(\omega)$.

For the converse implication, let us assume that $\underline{P}_1(\cdot|\mathcal{B}_1), \dots, \underline{P}_m(\cdot|\mathcal{B}_m)$ incur partial loss, and show that this leads to an infinite natural extension. From Definition 5, there are $j = 1, \dots, m, n_j \geq 1, k = 1, \dots, n_j, g_j^k \in \mathcal{H}^j$, not all the g_j^k equal to the zero gamble, such that

$$\sup_{\omega \in \mathbb{S}(g_j^k)} \left[\sum_{j=1}^{m} \sum_{k=1}^{n_j} G_j(g_j^k|\mathcal{B}_j) \right](\omega) = -\delta < 0.$$

Multiplying both sides of the equality by $\lambda > 0$, and taking into account that $\lambda G_j(g_j^k|\mathcal{B}_j) = G_j(\lambda g_j^k|\mathcal{B}_j)$, $S_j(g_j^k) = S_j(\lambda g_j^k)$ when $\lambda > 0$, and that $g_{j,\lambda}^k := \lambda g_j^k$ belongs to $\mathcal{H}^j$, we see that for every $\lambda > 0$ there are $j = 1, \dots, m, n_j \geq 1, k = 1, \dots, n_j, g_{j,\lambda}^k \in \mathcal{H}^j$, where not all the $g_{j,\lambda}^k$ are equal to the zero gamble, such that

$$\sup_{\omega \in \mathbb{S}(g_j^k)} \left[\sum_{j=1}^{m} \sum_{k=1}^{n_j} G_j(g_{j,\lambda}^k|\mathcal{B}_j) \right](\omega) = -\lambda\delta < 0.$$

The key point here is that $\mathbb{S}(g_j^k)$ does not depend on λ. This means, in other words, that we can find gambles that make the double sum as small as we wish on each element of the *fixed* set $\mathbb{S}(g_j^k)$.

Now, take any $f \in \mathcal{L}$, and $j_0 \in \{1,\dots,m\}$, $B_0 \in \mathcal{B}_{j_0}$ s.t. $B_0 \in \mathbb{S}(g_j^k)$ (we can do so because $\mathbb{S}(g_j^k)$ is not empty as not all the g_j^k are zero gambles). Choose also $\alpha > 0$, and let $\mu := \inf_{\mathbb{S}(g_j^k)} B_0(f-\alpha)$. Then it is enough to choose $\lambda > 0$ such that $-\lambda\delta < \mu$, in order to know that there are $j = 1,\dots,m, n_j \geq 1, k = 1,\dots,n_j, g_{j,\lambda}^k \in \mathcal{H}^j$, such that $\sum_{j=1}^m \sum_{k=1}^{n_j} G_j(g_{j,\lambda}^k|\mathcal{B}_j) \leq -\lambda\delta < \mu \leq B_0(f-\alpha)$ in $\mathbb{S}(g_j^k) \cup B_0 = \mathbb{S}(g_j^k)$, whence

$$\sum_{j=1}^{m}\sum_{k=1}^{n_j} G_j(g_{j,\lambda}^k|\mathcal{B}_j) - B_0(f-\alpha) \leq -\lambda\delta - \mu < 0$$

on $\mathbb{S}(g_j^k) \cup B_0 = \mathbb{S}(g_j^k)$, taking into account that $-B_0(f-\alpha) \leq -\mu$ and that $-\lambda\delta - \mu < \mu - \mu = 0$. Since we can do this for any $\alpha > 0$, it follows that $\underline{E}_{j_0}(f|B_0) = +\infty$.

3. Consider $f_j \in \mathcal{L}$ for $j = 1,\dots,m$, $j_0 \in \{1,\dots,m\}, B_0 \in \mathcal{B}_{j_0}, f_0 \in \mathcal{L}$, and let us show that

$$\sup_{\omega \in \mathbb{S}(f_j) \cup B_0} \Big[\sum_{j=1}^{m} f_j - \underline{E}_j(f_j|\mathcal{B}_j) - B_0(f_0 - \underline{E}_{j_0}(f_0|B_0))\Big](\omega) \geq 0. \qquad (30)$$

Assume ex-absurdo that the above supremum in Eq. (30) is smaller than $-\delta$ for some $\delta > 0$. Fix $\varepsilon := \frac{\delta}{2m} > 0$. Then, since from the second statement $\underline{E}_i(f_i|B_i)$ is finite for every $i = 1,\dots,m$ and every $B_i \in S_i(f_i)$, there is an integer $n_j \geq 1$ and gambles $g_{B_i,j}^k \in \mathcal{H}^j, j = 1,\dots,m, k = 1,\dots,n_j$ such that

$$\sup_{\omega \in \mathbb{S}(g_{B_i,j}^k) \cup B_i} \Big[\sum_{j=1}^{m}\sum_{k=1}^{n_j} G_j(g_{B_i,j}^k|\mathcal{B}_j) - B_i(f_i - \underline{E}_i(f_i|B_i) + \varepsilon)\Big](\omega) < -\delta_{i,B_i} < 0$$

for some positive real δ_{i,B_i}. By making the sum over all $B_i \in S_i(f_i)$, we deduce that

$$\Big[\sum_{B_i \in S_i(f_i)} \Big(\sum_{j=1}^{m}\sum_{k=1}^{n_j} G_j(g_{B_i,j}^k|\mathcal{B}_j) - B_i(f_i - \underline{E}_i(f_i|B_i) + \varepsilon)\Big)\Big](\omega) < -\min_{B_i \in S_i(f_i)} \delta_{i,B_i} < 0$$

for all $\omega \in S_i(f_i) \cup \mathbb{S}(g_{B_i,j}^k)$. If we now make the sum over all the partitions, we deduce that

$$\Big[\sum_{i=1}^{m}\sum_{B_i \in S_i(f_i)} \Big(\sum_{j=1}^{m}\sum_{k=1}^{n_j} G_j(g_{B_i,j}^k|\mathcal{B}_j) - B_i(f_i - \underline{E}_i(f_i|B_i) + \varepsilon)\Big)\Big](\omega) < -\min_{i, B_i \in S_i(f_i)} \delta_{i,B_i}$$

on $\mathbb{S}(f_i)\cup\mathbb{S}(g^k_{B_i,j})$. Consider now $\gamma\in(0,\frac{\delta}{4})$. Given $\omega\in(\mathbb{S}(g^k_{B_i,j})\cap\mathbb{S}(f_i))\cup B_0$, it follows that

$$\begin{aligned}&\Big[\sum_{i=1}^m\sum_{B_i\in S_i(f_i)}\sum_{j=1}^m\sum_{k=1}^{n_j}G_j(g^k_{B_i,j}|\mathcal{B}_j)-B_0(f_0-\underline{E}_{j_0}(f_0|B_0)-\gamma)\Big](\omega)\\&=\Big[\sum_{i=1}^m\sum_{B_i\in S_i(f_i)}\Big(\sum_{j=1}^m\sum_{k=1}^{n_j}G_j(g^k_{B_i,j}|\mathcal{B}_j)-B_i(f_i-\underline{E}_i(f_i|\mathcal{B}_i)+\varepsilon)\Big)\Big](\omega)\\&+\Big[\sum_{i=1}^m\sum_{B_i\in S_i(f_i)}B_i(f_i-\underline{E}_i(f_i|\mathcal{B}_i)+\varepsilon)-B_0(f_0-\underline{E}_{j_0}(f_0|B_0)-\gamma)\Big](\omega)\\&\leq 0-\delta+\gamma+\frac{\delta}{2}<0.\end{aligned}$$

On the other hand, if $\omega\in\mathbb{S}(g^k_{B_i,j})\setminus(\mathbb{S}(f_i)\cup B_0)$, then

$$\begin{aligned}&\Big[\sum_{i=1}^m\sum_{B_i\in S_i(f_i)}\sum_{j=1}^m\sum_{k=1}^{n_j}G_j(g^k_{B_i,j}|\mathcal{B}_j)-B_0(f_0-\underline{E}_{j_0}(f_0|B_0)-\gamma)\Big](\omega)\\&\qquad=\Big[\sum_{i=1}^m\sum_{B_i\in S_i(f_i)}\sum_{j=1}^m\sum_{k=1}^{n_j}G_j(g^k_{B_i,j}|\mathcal{B}_j)\Big](\omega)<-\min_{i,B_i\in S_i(f_i)}\delta_{i,B_i}<0;\end{aligned}$$

we conclude that we can increase $\underline{E}_{j_0}(f_0|B_0)$ in γ, a contradiction with the definition of the natural extension.

4. Let $\underline{P}'_1(\cdot|\mathcal{B}_1),\dots,\underline{P}'_m(\cdot|\mathcal{B}_m)$ be coherent conditional lower previsions on $\mathcal{L}$ that dominate $\underline{P}_1(\cdot|\mathcal{B}_1),\dots,\underline{P}_m(\cdot|\mathcal{B}_m)$ on their domains. Assume there are some $j_0\in\{1,\dots,m\}$, $B_0\in\mathcal{B}_{j_0}, f_0\in\mathcal{L}$, such that $\underline{P}'_{j_0}(f_0|B_0)<\underline{E}_{j_0}(f_0|B_0)$. Then it follows from the definition of $\underline{E}_{j_0}(f_0|B_0)$ that there are $g^k_j\in\mathcal{H}^j, n_j\geq 1, j=1,\dots,m,k=1,\dots,n_j$ such that

$$\sup_{\omega\in\mathbb{S}(g^k_j)\cup B_0}\Big[\sum_{j=1}^m G_j(g^k_j|\mathcal{B}_j)-B_0(f_0-\underline{P}'_{j_0}(f_0|B_0))\Big](\omega)<0,$$

and since $\underline{P}'_j(g^k_j|\mathcal{B}_j)\geq\underline{P}_j(g^k_j|\mathcal{B}_j)$ for all $j=1,\dots,m,k=1,\dots,n_j$,

$$\sup_{\omega\in\mathbb{S}(g^k_j)\cup B_0}\Big[\sum_{j=1}^m G'_j(g^k_j|\mathcal{B}_j)-B_0(f_0-\underline{P}'_{j_0}(f_0|B_0))\Big](\omega)<0.$$

This contradicts the coherence of $\underline{P}'_1(\cdot|\mathcal{B}_1),\dots,\underline{P}'_m(\cdot|\mathcal{B}_m)$.

5. Assume first of all that $\underline{P}_j(\cdot|\mathcal{B}_j)=\underline{E}_j(\cdot|\mathcal{B}_j)$ for $j=1,\dots,m$. Since $\underline{P}_1(\cdot|\mathcal{B}_1),\dots,\underline{P}_m(\cdot|\mathcal{B}_m)$ are separately coherent, they are in particular finite. Applying the second statement, $\underline{P}_1(\cdot|\mathcal{B}_1),\dots,\underline{P}_m(\cdot|\mathcal{B}_m)$ avoid partial loss, and using now the third statement we deduce that the natural extensions $\underline{E}_1(\cdot|\mathcal{B}_1),\dots,\underline{E}_m(\cdot|\mathcal{B}_m)$, and therefore also $\underline{P}_1(\cdot|\mathcal{B}_1),\dots,\underline{P}_m(\cdot|\mathcal{B}_m)$, are coherent.

   Conversely, if the lower previsions $\underline{P}_1(\cdot|\mathcal{B}_1),\dots,\underline{P}_m(\cdot|\mathcal{B}_m)$ are coherent but there is some $j_0\in\{1,\dots,m\}$ and $g_0\in\mathcal{H}^{j_0},B_0\in\mathcal{B}_{j_0}$ such that $\underline{P}_{j_0}(g_0|B_0)<\underline{E}_{j_0}(g_0|B_0)$, then Eq. (7) implies that there are $j=$

$1,\dots,m, n_j \geq 1, k=1,\dots,n_j, g_j^k \in \mathcal{H}^j$ such that

$$\sup_{\mathbb{S}(g_j^k)\cup B_0}\left[\sum_{j=1}^{m}\sum_{k=1}^{n_j} G_j(g_j^k|\mathcal{B}_j) - G_{j_0}(g_0|B_0)\right] < 0,$$

thus contradicting the coherence of $\underline{P}_1(\cdot|\mathcal{B}_1),\dots,\underline{P}_m(\cdot|\mathcal{B}_m)$. □

Proof of Corollary 1. The statement is a direct consequence of the second part of the proof of statement 2 in Lemma 1, which shows that if $\underline{P}_1(\cdot|\mathcal{B}_1),\dots,\underline{P}_m(\cdot|\mathcal{B}_m)$ incur partial loss, there must be $j \in \{1,\dots,m\}, B_j \in \mathcal{B}_j$, such that $\underline{E}_j(f|B_j) = +\infty$ for all $f \in \mathcal{L}$. □

Proof of Proposition 2. Let us assume that the axioms hold and prove that $\mathcal{Q}\cap\mathcal{E} \subseteq \mathcal{R}$. We use Eq. (10). Given $g \in \mathcal{Q}\cap\mathcal{E}$, if $g \in \mathcal{L}^+$ then it follows from (APG) that $g \in \mathcal{R}$. Otherwise, if $g \notin \mathcal{L}^+$ then $g \geq \sum_{j=1}^r \lambda_j g_j$ for some $r \geq 1, j \in \{1,\dots,r\}, g_j \in \mathcal{R}, \lambda_j > 0$. By using (PHM) and (ADD), we obtain that $\sum_{j=1}^r \lambda_j g_j \in \mathcal{R}$. Since $\mathcal{Q}$ is linear, it holds that $g_0 := g - \sum_{j=1}^r \lambda_j g_j$ belongs to $\mathcal{Q}$, too. If $g_0 = 0$ then $g = \sum_{j=1}^r \lambda_j g_j \in \mathcal{R}$. Otherwise, condition (APG) implies that $g_0 \in \mathcal{R}$ and then applying (ADD) we obtain that g belongs to $\mathcal{R}$.

Let us show now that $\mathcal{R}$ avoids partial loss as in Definition 11. Assume, by contradiction, that $0 \in \mathcal{E}$. Then $0 \geq \sum_{j=1}^r \lambda_j g_j$ for some $r \geq 1, j \in \{1,\dots,r\}, g_j \in \mathcal{R}, \lambda_j > 0$. Let $g_0 := 0 - \sum_{j=1}^r \lambda_j g_j$. The case $g_0 = 0$ is not possible because the previous part of the proof has shown that $\sum_{j=1}^r \lambda_j g_j \in \mathcal{R}$, and $0 \notin \mathcal{R}$ by (APL). Therefore it holds that $g_0 \in \mathcal{L}^+$. But g_0 belongs also to $\mathcal{Q}$, because $\mathcal{Q}$ is linear, and by (APG) we see that $g_0 \in \mathcal{R}$. By (ADD) we obtain that $0 = \sum_{j=1}^r \lambda_j g_j - \sum_{j=1}^r \lambda_j g_j \in \mathcal{R}$. This is a contradiction with (APL).

Assume conversely that $\mathcal{R}$ avoids partial loss and $\mathcal{R} = \mathcal{Q}\cap\mathcal{E}$, and let us prove that the axioms hold. That (APL) holds follows trivially from $\mathcal{R} \subseteq \mathcal{E}$ and the fact that $\mathcal{R}$ avoids partial loss. Concerning (APG), if $g \in \mathcal{Q}$ is such that $g \in \mathcal{L}^+$, then $g \in \mathcal{E}$ and hence $g \in \mathcal{Q}\cap\mathcal{E} = \mathcal{R}$. As for (PHM), if $g \in \mathcal{R}$ and $\lambda > 0$, then $\lambda g \in \mathcal{Q}$, because $\mathcal{Q}$ is a linear set, and $\lambda g \in \mathcal{E}$, by definition: whence, $\lambda g \in \mathcal{Q}\cap\mathcal{E} = \mathcal{R}$. Finally, and analogously, if $f, g \in \mathcal{R}$, then $f+g \in \mathcal{Q}$, because $\mathcal{Q}$ is a linear set, and $f+g \in \mathcal{E}$, by definition: whence, $f+g \in \mathcal{Q}\cap\mathcal{E} = \mathcal{R}$, and (ADD) holds. □

Proof of Proposition 3. Regarding point (a), if a gamble g in the natural extension of $\mathcal{E}$ belongs to $\mathcal{L}^+$, then it belongs to $\mathcal{E}$ too; otherwise g is such that $g \geq \sum_{j=1}^r \lambda_j g_j$ for some $r \geq 1, j \in \{1,\dots,r\}, g_j \in \mathcal{E}, \lambda_j > 0$, where we can assume without loss of generality that $g_j \notin \mathcal{L}^+$ for all $j \in \{1,\dots,r\}$. Moreover, each g_j in the sum belongs to $\mathcal{E}$ and hence it is such that $g_j \geq \sum_{k_j=1}^{r_j} \lambda_{k_j} g_{k_j}$, for some $r_j \geq 1, g_{k_j} \in \mathcal{R}, \lambda_{k_j} > 0$. It follows that $g \geq \sum_{j=1}^r \sum_{k_j=1}^{r_j} \lambda_j \lambda_{k_j} g_{k_j}$ for some $r \geq 1, r_j \geq 1, g_{k_j} \in \mathcal{R}, \lambda_j > 0, \lambda_{k_j} > 0$. This shows that g belongs to $\mathcal{E}$, and hence that the natural extension of $\mathcal{E}$ is included in $\mathcal{E}$. The opposite inclusion holds trivially. Concerning point (b), if $\mathcal{R}$ is included in a coherent set $\mathcal{E}'$ then $\mathcal{E}$ is included in the natural extension of $\mathcal{E}'$, which from the first point is again $\mathcal{E}'$. Point (c) follows trivially from point (a). Point (d) follows trivially from points (a) and (c). The direct implication in point (e) is trivial, given the definition of $\mathcal{E}$ and point (d). For the converse implication, consider that: $\mathcal{E}'$ must include the natural extension $\mathcal{E}$, because of point (b); this implies that $\mathcal{E}$ avoids partial loss, considered that its natural extension is $\mathcal{E}$ itself, as in point (a); then $\mathcal{R}$ avoids partial loss because of point (c). The direct implication in point (f) is trivial. For the converse implication, consider that $\mathcal{R}$ avoids partial loss since $\mathcal{E}'$ contains $\mathcal{R}$ and because of point (e). Now, take $g \in \mathcal{Q} \cap \mathcal{E}$. Since $\mathcal{E} \subseteq \mathcal{E}'$, thanks to point (b), then $g \in \mathcal{Q} \cap \mathcal{E}' = \mathcal{R}$. For the direct implication in (g), note that if $\mathcal{R}$ is included in a coherent set $\mathcal{E}'$, then $\mathcal{E}'$ avoids partial loss. Applying (b), we deduce that so does $\mathcal{E}$, and then points (c) and (d) imply that $\mathcal{E}$ is coherent. The converse implication in (g) is trivial. Finally, (h) is a consequence of points (b) and (g). □

Proof of Corollary 2. Consider $g \lneq 0$, and assume by contradiction that $g \in \mathcal{E}$. $\mathcal{E}$ is coherent because of Proposition 3(d); hence it includes $-g \in \mathcal{L}^+$. Moreover, by (ADD) in Proposition 2, we obtain that $0 = g - g \in \mathcal{E}$, a contradiction. □

Proof of Proposition 4. Remember that $\mathcal{E}$ is coherent given that $\mathcal{R}$ avoids partial loss, because of Proposition 3(d). Let us define $\mathcal{E}' := \{g \in \mathcal{L} : g + \delta \in \mathcal{E}\ \forall \delta > 0\}$, and let us prove that this is a coherent set of almost-desirable gambles. For this, we are going to prove that it satisfies the axioms in Definition 13:

(PHM) Let $g \in \mathcal{E}'$, $\lambda > 0$. Then $g + \delta$ belongs to $\mathcal{E}$ for every $\delta > 0$, and $\lambda(g + \delta) = \lambda g + \lambda\delta$ also belongs to $\mathcal{E}$ for every $\delta > 0$, thanks to (PHM) in Proposition 2. As a consequence, $\lambda g \in \mathcal{E}'$.

(ADD) Given $f, g \in \mathcal{E}'$ and $\delta > 0$, $f + g + \delta = (f + \frac{\delta}{2}) + (g + \frac{\delta}{2}) \in \mathcal{E}$, thanks to (ADD) and (APG) in Proposition 2. Hence, $f + g \in \mathcal{E}'$.

(ASL) Let $g \in \mathcal{L}$ satisfy $\sup g < 0$. Then there is a $\delta > 0$ such that $g + \delta < 0$. Then $g + \delta \notin \mathcal{E}$ because of (APL'). This implies that $g \notin \mathcal{E}'$.

(ASG) Let $g \in \mathcal{L}$ satisfy $\inf g > 0$. Then $g + \delta \in \mathcal{E}$ for all $\delta > 0$ because $\mathcal{E}$ satisfies (APG) in Proposition 2, and as a consequence $g \in \mathcal{E}'$.

(CLS) Finally, if $g + \delta \in \mathcal{E}'$ for every $\delta > 0$, we deduce that $g + \delta' \in \mathcal{E}$ for all $\delta' > 0$, and as a consequence $g \in \mathcal{E}'$.

Applying [19, Theorem 3.8.5], $\mathcal{E}'$ is equal to

$$\{g \in \mathcal{L} : P(g) \geq 0\ \forall P \in \mathcal{M}(\mathcal{E}')\},$$

where

$$\mathcal{M}(\mathcal{E}') := \{P : P(g) \geq 0\ \forall g \in \mathcal{E}'\} = \{P : P(g) \geq 0\ \forall g \in \mathcal{E}\} =: \mathcal{M}(\mathcal{E}) = \mathcal{M}(\mathcal{R}).$$

To see that $\mathcal{M}(\mathcal{E}') = \mathcal{M}(\mathcal{E})$, consider first that by definition $g \in \mathcal{E}'$ implies that $g + \delta \in \mathcal{E}$ for all $\delta > 0$. For the inclusion $\mathcal{M}(\mathcal{E}') \supseteq \mathcal{M}(\mathcal{E})$, take P non-negative on all the elements of $\mathcal{E}$ and $g \in \mathcal{E}'$; then $P(g) + \delta = P(g + \delta) \geq 0$ for all $\delta > 0$, and this implies $P(g) \geq 0$. As a consequence P is non-negative on all the elements of $\mathcal{E}'$. The inclusion $\mathcal{M}(\mathcal{E}') \subseteq \mathcal{M}(\mathcal{E})$ is trivial. To see that $\mathcal{M}(\mathcal{E}) = \mathcal{M}(\mathcal{R})$, it suffices to prove that $\mathcal{M}(\mathcal{E}) \supseteq \mathcal{M}(\mathcal{R})$, since the inclusion $\mathcal{M}(\mathcal{E}) \subseteq \mathcal{M}(\mathcal{R})$ is trivial as $\mathcal{R} \subseteq \mathcal{E}$. Take P non-negative on all the elements of $\mathcal{R}$. By definition $g \in \mathcal{E}$ implies that either $g \in \mathcal{L}^+$, and then trivially $P(g) \geq 0$, or $g \geq \sum_{j=1}^r \lambda_j g_j$, for some $r \geq 1$, $g_j \in \mathcal{R}$, $\lambda_j > 0$. In this second case, the linearity of P implies that $P(\sum_{j=1}^r \lambda_j g_j) \geq 0$ and its monotonicity that $P(g) \geq 0$. Hence,

$$\{g \in \mathcal{L} : P(g) \geq 0\ \forall P \in \mathcal{M}(\mathcal{R})\} = \mathcal{E}' = \{g \in \mathcal{L} : g + \delta \in \mathcal{E}\ \forall \delta > 0\}.$$

It remains to prove that $\mathcal{E}' = \overline{\mathcal{E}}$, where the closure is taken in the topology of uniform convergence. To see that $\mathcal{E}' \subseteq \overline{\mathcal{E}}$, note that for any gamble g in $\mathcal{E}'$, g is the uniform limit of the sequence $\{g + \frac{1}{n} : n \in \mathbb{N}\}$, and that each element of the sequence belongs to $\mathcal{E}$; as a consequence, g belongs to $\overline{\mathcal{E}}$. Conversely, to see that

$\mathcal{E}' \supseteq \overline{\mathcal{E}}$, let $(g_n)_n$ be a sequence of elements in $\mathcal{E}$ that converges uniformly to g. Then for every $\delta > 0$, there is some $n_\delta \in \mathbb{N}$ such that $\|g_n - g\| < \delta \ \forall n \geq n_\delta$, whence $g + \delta \geq g_n \ \forall n \geq n_\delta$ and therefore $g + \delta \in \mathcal{E}$, because $\mathcal{E}$ is closed under dominance. This implies that $g \in \mathcal{E}'$. As a consequence, $\mathcal{E}' = \overline{\mathcal{E}}$. □

Proof of Theorem 5. Let f be a gamble on Ω, $B \in \mathcal{B}$.

(1) Given $\mu < \inf_{\omega \in B} f(\omega)$, it follows from (SDb) that the gamble $B(f - \mu)$ belongs to $\mathcal{R}$, and as a consequence $\underline{P}(f|B) \geq \inf_{\omega \in B} f(\omega)$.
(2) Similarly, if $\mathcal{R}$ satisfies (SDa) then for every $\mu > \sup_{\omega \in B} f(\omega)$ the gamble $B(f - \mu)$ does not belong to $\mathcal{R}$, whence $\underline{P}(f|B) \leq \sup_{\omega \in B} f(\omega)$.
(3) For every $\varepsilon > 0$, there is some $\mu \in \mathbb{R}$ such that $\underline{P}(f|B) - \varepsilon \leq \mu < \underline{P}(f|B)$, and such that $B(f-\mu) \in \mathcal{R}$. Then for every $\mu' < \mu$ it holds that $B(f-\mu') = B(f-\mu) + B(\mu-\mu')$. If $\mathcal{R}$ satisfies (SDb), it follows that $B(\mu-\mu')$ belongs to $\mathcal{R}$, and then applying condition (ADD) we deduce that $B(f - \mu')$ also belongs to $\mathcal{R}$.
(4) To conclude with the fourth statement, we have showed above that condition (SD) is sufficient for $\underline{P}(\cdot|\mathcal{B})$ to be well-defined. To see that it is also necessary, assume that $\mathcal{R}$ does not satisfy axiom (SDa), i.e., that there is a gamble $f \leq 0$ and some $B \in \mathcal{B}$ such that $\sup_{\omega \in B} f(\omega) < 0$ and $f \in \mathcal{R}$. Then it holds that the gamble $Bf \geq f$ also belongs to $\mathcal{R}$ because this set is closed under dominance, whence $\underline{P}(f|B) \geq 0 > \sup_{\omega \in B} f(\omega)$. Similarly, if it does not satisfy (SDb), there is a gamble $f \geq 0$ and some $B \in \mathcal{B}$ such that $\inf_B f > 0$ and $f \notin \mathcal{R}$. Since $f \geq Bf$, we deduce that $Bf \notin \mathcal{R}$ and therefore $\underline{P}(f|B) \leq 0 < \inf_{\omega \in B} f(\omega)$.

□

Proof of Proposition 5. We begin with the direct implication in (15). Assume that $\overline{P}(B) = 0$. Then given any gamble g and $\mu > \sup_B g$, it holds that

$$0 \geq \underline{P}(B(g - \mu)) = -\overline{P}(B(\mu - g)) \geq -(\mu - \inf_B g)\overline{P}(B) = 0$$

whence $\underline{P}(B(g - \mu)) = 0$. From [19, Theorem 3.8.1], we deduce that $B(g - \mu) =: f \in \mathcal{D}$.

Conversely, assume that there is a gamble $f \leq 0$ in $\mathcal{D}$ such that $\sup_B f < 0$. Since $f \in \mathcal{D}$ implies that $\underline{P}(f) \geq 0$, and since $Bf \geq f$ because f is non-positive, we deduce that $\underline{P}(Bf) \geq 0$, whence $-\overline{P}(-Bf) \geq 0$, or, equivalently, $\overline{P}(-Bf) \leq 0$. But since $-Bf \geq 0$ implies that $\overline{P}(-Bf) \geq 0$, we deduce from this that $0 = \overline{P}(-Bf) \geq \inf_B(-f)\overline{P}(B) \geq 0$, whence $\overline{P}(B) = 0$.

The proof of the remaining part of the proposition is trivial given (15). □

Proof of Theorem 6. Since the domain of $\underline{P}(\cdot|\mathcal{B})$ is the linear set of gambles $\mathcal{L}$, separate coherence is equivalent to conditions (SC1)–(SC3). Let us show that these conditions are satisfied when $\mathcal{R}$ satisfies the axioms (SD), (PHM) and (ADD):

(SC1) This follows from condition (SD) because of Theorem 5.
(SC2) Let $f \in \mathcal{L}, \lambda > 0$. From Eq. (13), for every $B \in \mathcal{B}$, it holds that $\underline{P}(\lambda f|B) = \sup\{\mu | B(\lambda f - \mu) \in \mathcal{R}\} = \sup\{\lambda\mu' | B(\lambda f - \lambda\mu') \in \mathcal{R}\} = \sup\{\lambda\mu' | \lambda B(f - \mu') \in \mathcal{R}\} = \lambda \sup\{\mu' | \lambda B(f - \mu') \in \mathcal{R}\} = \lambda \sup\{\mu' | B(f - \mu') \in \mathcal{R}\} = \lambda\underline{P}(f|B)$, where the one-but-last equality holds because, from (PHM), a gamble g belongs to $\mathcal{R}$ if and only if $\lambda g \in \mathcal{R}$ for every $\lambda > 0$.

(SC3) Consider gambles $f, g \in \mathcal{R}$, and $\varepsilon > 0$. Then there are $\mu_1 \in [\underline{P}(f|B) - \frac{\varepsilon}{2}, \underline{P}(f|B)), \mu_2 \in [\underline{P}(g|B) - \frac{\varepsilon}{2}, \underline{P}(g|B))$ such that the gambles $B(f - \mu_1)$ and $B(g - \mu_2)$ belong to $\mathcal{R}$. Applying (ADD), we deduce that $B(f + g - \mu_1 - \mu_2)$ belongs to $\mathcal{R}$ and therefore $\underline{P}(f + g|B) \geq \underline{P}(f|B) + \underline{P}(g|B) - \varepsilon$. Since we can do this for every $\varepsilon > 0$, we deduce that $\underline{P}(f + g|B) \geq \underline{P}(f|B) + \underline{P}(g|B)$. □

Proof of Proposition 6. It suffices to prove the first point, since the second follows as a particular case. Let us define the conditional upper prevision $\overline{P}(\cdot|\mathcal{B})$ by

$$\overline{P}(f|B) := -\underline{P}(-f|B) = \inf\{\mu : B(\mu - f) \in \mathcal{R}\}$$

for every $f \in \mathcal{L}$ and every $B \in \mathcal{B}$. It follows from Theorem 6 that $\underline{P}(\cdot|\mathcal{B})$ is separately coherent, and from [19, Theorem 6.2.6] that $\underline{P}(f|B) \leq \overline{P}(f|B)$ for every $f \in \mathcal{L}, B \in \mathcal{B}$.

Assume that $\underline{P}(\cdot|B)$ is a linear conditional prevision, i.e., $\overline{P}(f|B) = -\underline{P}(-f|B) = \underline{P}(f|B)$ for every gamble f. Given some gamble g and some $\varepsilon > 0$, there are two possibilities: either $\underline{P}(g|B) = \overline{P}(g|B) > 0$, and then since $B(g - \mu) \in \mathcal{R}$ for every $\mu < \underline{P}(g|B)$ because of Theorem 5(3), we deduce that Bg belongs to $\mathcal{R}$; or $\underline{P}(g|B) = \overline{P}(g|B) \leq 0$, whence the definition of $\overline{P}(g|B)$ and the same result implies that $B(\varepsilon - g) \in \mathcal{R}$ for every $\varepsilon > 0$.

Conversely, assume (LC) holds and let us show that $\underline{P}(f|B) = \overline{P}(f|B)$ for every gamble f and every $B \in \mathcal{B}$: assume ex-absurdo there is some gamble f for which $\underline{P}(f|B) < \overline{P}(f|B)$; take $0 < \delta < \overline{P}(f|B) - \underline{P}(f|B)$, and $\varepsilon := \overline{P}(f|B) - \underline{P}(f|B) - \delta$. Then the definition of $\underline{P}(f|B)$ implies that $B((f - \underline{P}(f|B)) - \frac{\varepsilon}{2})$ does not belong to $\mathcal{R}$ because $\varepsilon > 0$, and similarly the definition of $\overline{P}(f|B)$ implies that

$$B\left(\frac{\varepsilon}{2} - (f - \underline{P}(f|B) - \frac{\varepsilon}{2})\right) = B\left(\varepsilon - (f - \underline{P}(f|B))\right) = B(\overline{P}(f|B) - \delta - f) \notin \mathcal{R}.$$

This contradicts (LC). From the equality $\underline{P}(f|B) = \overline{P}(f|B)$ for all $f \in \mathcal{L}$ and all $B \in \mathcal{B}$ we deduce applying [19, Thm. 6.2.6(c)] that $\underline{P}(\cdot|\mathcal{B})$ is a linear conditional prevision. □

Proof of Corollary 3. If $\mathcal{D}$ is a coherent set of almost-desirable gambles that satisfies (SD), it satisfies the hypotheses of Proposition 6. As a consequence, if $\underline{P}(\cdot|\mathcal{B})$ is a linear conditional prevision, then given $B \in \mathcal{B}$ and a gamble $f \in \mathcal{L}$ either $Bf \in \mathcal{D}$ or $B(\varepsilon - f) \in \mathcal{D}$ for every $\varepsilon > 0$. But in this second case conditions (ASG) and (CLS) imply that $B^c\varepsilon$ belongs to $\mathcal{D}$, and using (ADD) we deduce that $B(\varepsilon - f) + B^c\varepsilon = \varepsilon - Bf$ belongs to $\mathcal{D}$; since this holds for every $\varepsilon > 0$, we can apply (CLS) and deduce that $-Bf \in \mathcal{D}$. Conversely, it follows from axioms (APG) and (ADD) that if $\mathcal{D}$ is a coherent set of almost-desirable gambles and $-Bf \in \mathcal{D}$, then also $B(\varepsilon - f) \in \mathcal{D}$ for every $\varepsilon > 0$, so condition (LC) holds and therefore $\mathcal{D}$ originates a linear conditional prevision. □

Proof of Proposition 7. If $\mathcal{R}$ is coherent it avoids in particular partial loss, and therefore $\mathcal{E}$ is coherent with respect to $\mathcal{L}$ thanks to Proposition 3(d). Since conditions (APG) (with respect to $\mathcal{L}$) and (APL') guarantee that (SD) holds, we deduce from Theorem 6 that $\underline{P}_3(\cdot|\mathcal{B})$ is separately coherent.

For the second statement, note first of all that if $\mathcal{R}$ satisfies (SD), Theorem 6 guarantees that the conditional lower prevision $\underline{P}_2(\cdot|\mathcal{B})$ is separately coherent, and as a consequence so is $\underline{P}_1(\cdot|\mathcal{B})$, which is its restriction to $\mathcal{Q}$.

Now, since $\mathcal{R}$ is a coherent set of really desirable gambles with respect to the linear set $\mathcal{Q}$, it follows that $\mathcal{E} = \mathcal{L}^+ \cup \{g \geq h \text{ for some } h \in \mathcal{R}\}$ (cf. (11)). As a consequence,

$$\begin{aligned}\underline{P}_3(f|B) &= \max\{\sup\{\alpha : B(f-\alpha) \geq h \text{ for some } h \in \mathcal{R}\}, \sup\{\alpha : B(f-\alpha) \in \mathcal{L}^+\}\} \\ &= \max\{\sup\{\alpha : B(f-\alpha) \geq h \text{ for some } h \in \mathcal{R}\}, \inf_B f\} \\ &= \sup\{\alpha : B(f-\alpha) \geq h \text{ for some } h \in \mathcal{R}\}\end{aligned}$$

for every $f \in \mathcal{L}$ and every $B \in \mathcal{B}$, where last equality follows because, using the separate coherence of $\underline{P}_2(\cdot|\mathcal{B})$,

$$\sup\{\alpha : B(f-\alpha) \geq h \text{ for some } h \in \mathcal{R}\} \geq \sup\{\alpha : B(f-\alpha) \in \mathcal{R}\} = \underline{P}_2(f|B) \geq \inf_B f. \tag{31}$$

Let us prove that in fact

$$\underline{P}_3(f|B) = \sup\{\alpha : B(f-\alpha) \in \mathcal{R}\} = \underline{P}_2(f|B). \tag{32}$$

Consider $\alpha < \underline{P}_3(f|B)$, and take $h \in \mathcal{R}$ such that $B(f-\alpha) \geq h$. Then for every $\varepsilon > 0$,

$$B(f-(\alpha-\varepsilon)) = h + \varepsilon B + B(f-\alpha) - h;$$

the gamble $B(f-\alpha) - h$ is non-negative, whence $B(f-\alpha) - h + \varepsilon B$ is a non-negative gamble which is strictly positive in B. Applying (SD), we deduce that $B(f-\alpha) - h + \varepsilon B$ belongs to $\mathcal{R}$, and since $\mathcal{R}$ also satisfies (ADD), also $h + B(f-\alpha) - h + \varepsilon B = B(f-\alpha) + \varepsilon B$ belongs to $\mathcal{R}$. As a consequence,

$$\sup\{\mu : B(f-\mu) \in \mathcal{R}\} \geq \alpha - \varepsilon \text{ for every } \alpha < \underline{P}_3(f|B), \varepsilon > 0,$$

whence $\sup\{\mu : B(f-\mu) \in \mathcal{R}\} \geq \underline{P}_3(f|B)$. Since the converse inequality is already shown in (31), we deduce that Eq. (32) holds. From this we also deduce that $B(f-\alpha)$ belongs to $\mathcal{R}$ for every $\alpha < \underline{P}_3(f|B)$.

To complete the proof, let us show that $\underline{P}_3(\cdot|\mathcal{B})$ is the natural extension of $\underline{P}_1(\cdot|\mathcal{B})$. Since from the above result we see that $\underline{P}_3(f|B) = \underline{P}_1(f|B)$ for every gamble $f \in \mathcal{Q}$ and every $B \in \mathcal{B}$, we see that $\underline{P}_3(\cdot|\mathcal{B})$ is a separately coherent extension of $\underline{P}_1(\cdot|\mathcal{B})$ to all gambles, which from the fourth statement in Lemma 1 dominates the natural extension $\underline{E}_1(\cdot|\mathcal{B})$ of $\underline{P}_1(\cdot|\mathcal{B})$. Conversely, given $B \in \mathcal{B}, f \in \mathcal{L}$ and $\alpha < \underline{P}_3(f|B)$, the gamble $g := B(f-\alpha)$ belongs to $\mathcal{R} \subseteq \mathcal{Q}$, whence $\underline{P}_1(g|B) \geq 0$, and therefore

$$G_1(g|\mathcal{B}) - B(f-\alpha) \leq g - B(f-\alpha) = 0.$$

As a consequence, given $\varepsilon > 0$,

$$G_1(g|\mathcal{B}) - B(f-\alpha) - \varepsilon B = G_1(g|\mathcal{B}) - B(f-(\alpha-\varepsilon)) < 0$$

on $S(g) \cup B = B$. This implies that $\underline{E}_1(f|B) \geq \alpha - \varepsilon$ for every $\alpha < \underline{P}_3(f|B)$ and every $\varepsilon > 0$, and therefore $\underline{E}_1(f|B) \geq \underline{P}_3(f|B)$. Hence, $\underline{P}_3(\cdot|\mathcal{B})$ coincides with the natural extension $\underline{E}_1(\cdot|\mathcal{B})$ of $\underline{P}_1(\cdot|\mathcal{B})$ on all gambles. □

Proof of Theorem 7. Assume they do not. Then taking into account Proposition 1, there are $\varepsilon > 0$ and gambles f_j for $j = 1, \ldots, m$, not all of them equal to zero, such that

$$\sup_{\omega \in \mathbb{S}(f_j)} \left[\sum_{j=1}^m G_j(f_j|\mathcal{B}_j) + \varepsilon S_j(f_j)\right](\omega) < 0.$$

For every $j = 1, \dots, m$ and every $B_j \in \mathcal{B}_j$, the gamble $G_j(f_j|B_j) + \varepsilon B_j$ belongs to $\mathcal{R}$. Applying (ADD), and taking into account that the partitions are finite, we deduce that $\sum_{j=1}^m G_j(f_j|\mathcal{B}_j) + \varepsilon S_j(f_j)$ also belongs to $\mathcal{R}$. But this is a non-positive gamble which is strictly negative on each $B_j \in S_j(f_j)$ for $j = 1, \dots, m$. This is a contradiction with (SD). □

Proof of Theorem 8. Consider $f_j \in \mathcal{L}$ for $j = 0, \dots, m$, $j_0 \in \{1, \dots, m\}$ and $B_0 \in \mathcal{B}_{j_0}$. Let us show that

$$\sup_{\omega \in \mathbb{S}(f_j) \cup B_0} \left[\sum_{j=1}^m G_j(f_j|\mathcal{B}_j) - G_{j_0}(f_0|B_0) \right](\omega) \geq 0.$$

If the above supremum is equal to $-\delta < 0$ then given $\delta' := \frac{\delta}{m+1} > 0$ it holds that

$$\sum_{j=1}^m (G_j(f_j|\mathcal{B}_j) + \delta' S_j(f_j)) - G_{j_0}(f_0|B_0) + \delta' B_0 \leq 0,$$

whence

$$G_{j_0}(f_0|B_0) - \delta' B_0 \geq \sum_{j=1}^m (G_j(f_j|\mathcal{B}_j) + \delta' S_j(f_j)). \tag{33}$$

From the third statement in Theorem 5, we see that for every $j = 1, \dots, m$ and every $B_j \in S_j(f_j)$, the gamble $B_j(f_j - \underline{P}(f_j|B_j) + \delta')$ belongs to $\mathcal{R}$. Applying (ADD), we deduce that the right-hand-side in Eq. (33) belongs to $\mathcal{R}$, and taking into account that $\mathcal{R}$ includes all non-negative gambles and satisfies axiom (ADD), we deduce that $G_{j_0}(f_0|B_0) - \delta' B_0$ also belongs to $\mathcal{R}$. But if $B_0(f_0 - \underline{P}_{j_0}(f_0|B_0) - \delta')$ belongs to $\mathcal{R}$ we can increase the value of $\underline{P}_{j_0}(f_0|B_0)$. This is a contradiction with Eq. (13). □

Proof of Proposition 8. Call $\mathcal{R}_1$ the set in the r.h.s. of the equality in the statement. Take $g := G_j(f_j|B_j) + \varepsilon B_j$ from $\mathcal{R}$. It is clear that $gB_j^c = 0$, and also that $g \in \mathcal{H}^j$, as it follows from the assumptions on the domains that have been done in Remark 1, Section 2.2. Moreover, $\underline{P}_j(g|B_j) = \underline{P}_j(G_j(f_j|B_j) + \varepsilon B_j|B_j) = \underline{P}_j(f_j|B_j) - \underline{P}_j(f_j|B_j) + \varepsilon > 0$, where the second passage holds because of separate coherence (see [19, Lemma 6.2.4 and Section 6.2.6]). Conversely, consider $g \in \mathcal{R}_1$. Then there are $j \in \{1, \dots, m\}, B_j \in \mathcal{B}_j$, such that $g \in \mathcal{H}^j, gB_j^c = 0, \underline{P}_j(g|B_j) > 0$. Let $\varepsilon := \underline{P}_j(g|B_j)$. Then $B_j(g - (\underline{P}_j(g|B_j) - \varepsilon)) = gB_j = g$, and hence $g \in \mathcal{R}$. □

Proof of Theorem 9. Let $\mathcal{E}_1$ denote the set of gambles satisfying condition (19), and let $\mathcal{E}_2$ be the set of gambles satisfying condition (20). We shall make a circular proof of the equalities $\mathcal{E} = \mathcal{E}_1 = \mathcal{E}_2$.

Let us show that $\mathcal{E} \subseteq \mathcal{E}_1$. Take $g \in \mathcal{E}$. By definition, if $g \in \mathcal{L}^+$ then $g \in \mathcal{E}_1$. Otherwise, g can be written as follows:

$$g \geq \sum_{j=1}^m \sum_{k=1}^{n_j} \lambda_j^k (G_j(f_j^k|B_j^k) + \varepsilon_j^k B_j^k) = \sum_{j=1}^m \sum_{k=1}^{n_j} G_j(\tilde{f}_j^k|B_j^k) + \tilde{\varepsilon}_j^k B_j^k,$$

for some $j = 1, \dots, m, n_j \geq 1, k = 1, \dots, n_j, \lambda_j^k > 0, f_j^k \in \mathcal{H}^j, B_j^k \in \mathcal{B}_j, \varepsilon_j^k > 0$, and with $\tilde{f}_j^k := \lambda_j^k f_j^k, \tilde{\varepsilon}_j^k := \lambda_j^k \varepsilon_j^k$, where the second passage is possible thanks to the assumptions about the domains $\mathcal{H}^j$ made in Remark 1.

For the generic term $G_j(\tilde{f}_j^k|B_j^k)+\tilde{\varepsilon}_j^k B_j^k$ in the sum there are two possibilities: (i) if $B_j^k \in S_j(\tilde{f}_j^k)$, then we consider the gamble $g_j^k := \tilde{f}_j^k B_j^k$, so that $G_j(\tilde{f}_j^k|B_j^k)+\tilde{\varepsilon}_j^k B_j^k = G_j(g_j^k|\mathcal{B}_j) + \tilde{\varepsilon}_j^k S_j(g_j^k)$; (ii) If $B_j^k \notin S_j(\tilde{f}_j^k)$, then we consider the gamble $g_j^k := \tilde{\varepsilon}_j^k B_j^k$ so that, again, $G_j(\tilde{f}_j^k|B_j^k) + \tilde{\varepsilon}_j^k B_j^k = \tilde{\varepsilon}_j^k B_j^k = G_j(g_j^k|\mathcal{B}_j) + \tilde{\varepsilon}_j^k S_j(g_j^k)$.

As a consequence, if we take $\varepsilon := \min_{j,k} \tilde{\varepsilon}_j^k > 0$, we can write

$$g \geq \sum_{j=1}^{m} \sum_{k=1}^{n_j} G_j(g_j^k|\mathcal{B}_j) + \varepsilon S_j(g_j^k),$$

and this shows that $\mathcal{E} \subseteq \mathcal{E}_1$.

We next prove that $\mathcal{E}_1 \subseteq \mathcal{E}_2$. Let us consider $g \in \mathcal{E}_1$. Again, we can assume that $g \notin \mathcal{L}^+$, since for that case the inclusion is trivial. Then there are $j = 1,\dots,m, n_j \geq 1, k = 1,\dots,n_j, g_j^k \in \mathcal{H}^j, B_j \in \mathcal{B}_j, \varepsilon > 0$, such that

$$g \geq \sum_{j=1}^{m} \sum_{k=1}^{n_j} G_j(g_j^k|\mathcal{B}_j) + \varepsilon S_j(g_j^k). \tag{34}$$

Let us show that the above choice of gambles g_j^k make g belong to $\mathcal{E}_2$. We reason ex-absurdo, assuming that $g \notin \mathcal{E}_2$. There are two possibilities:

(i) If $g \not\geq 0$ in $\mathbb{S}(g_j^k)^c$, then there is some $\omega \in \mathbb{S}(g_j^k)^c$ such that $g(\omega) < 0$. But then it cannot hold that $g(\omega) \geq [\sum_{j=1}^m \sum_{k=1}^{n_j} G_j(g_j^k|\mathcal{B}_j) + \varepsilon S_j(g_j^k)](\omega) = 0$, a contradiction with (34).

(ii) If $\inf_{\omega \in \mathbb{S}(g_j^k)}[g - \sum_{j=1}^m \sum_{k=1}^{n_j} G_j(g_j^k|\mathcal{B}_j)] \leq 0$, then given the same $\varepsilon > 0$ there is some $\omega \in \mathbb{S}(g_j^k)$ such that

$$g(\omega) \leq \left[\sum_{j=1}^{m} \sum_{k=1}^{n_j} G_j(g_j^k|\mathcal{B}_j)\right](\omega) < \left[\sum_{j=1}^{m} \sum_{k=1}^{n_j} G_j(g_j^k|\mathcal{B}_j) + \varepsilon S_j(g_j^k)\right](\omega),$$

again a contradiction with (34).

We finally show that $\mathcal{E}_2 \subseteq \mathcal{E}$. Take $g \in \mathcal{E}_2$. We skip the trivial case $g \in \mathcal{L}^+$. Then there are $j = 1,\dots,m, n_j \geq 1, k = 1,\dots,n_j, g_j^k \in \mathcal{H}^j, g_j^k \neq 0$, such that $g \geq \sum_{j=1}^m \sum_{k=1}^{n_j} G_j(g_j^k|\mathcal{B}_j) + \delta$ in $\mathbb{S}(g_j^k)$ for some $\delta > 0$, and $g \geq 0$ in $\mathbb{S}(g_j^k)^c$. Let $\varepsilon := \frac{\delta}{m \max_{j,k} n_j}$. Then in $\mathbb{S}(g_j^k)$ we have that

$$\begin{aligned}\sum_{j=1}^{m} \sum_{k=1}^{n_j} G_j(g_j^k|\mathcal{B}_j) + \varepsilon S_j(g_j^k) &\leq \sum_{j=1}^{m} \sum_{k=1}^{n_j} G_j(g_j^k|\mathcal{B}_j) + \varepsilon m \max_{j,k} n_j \\ &= \sum_{j=1}^{m} \sum_{k=1}^{n_j} G_j(g_j^k|\mathcal{B}_j) + \delta \leq g,\end{aligned}$$

and since, whenever $\mathbb{S}(g_j^k)^c$ is not empty, it holds that in $\mathbb{S}(g_j^k)^c$

$$g \geq 0 = \sum_{j=1}^{m} \sum_{k=1}^{n_j} G_j(g_j^k|\mathcal{B}_j) + \varepsilon S_j(g_j^k);$$

it follows that $g \geq \sum_{j=1}^m \sum_{k=1}^{n_j} G_j(g_j^k|\mathcal{B}_j) + \varepsilon S_j(g_j^k)$. But taking into account that

$$G_j(g_j^k|\mathcal{B}_j) + \varepsilon S_j(g_j^k) = \sum_{B_j \in S_j(g_j^k)} G_j(g_j^k|B_j) + \varepsilon B_j,$$

it follows that $G_j(g_j^k|\mathcal{B}_j)+\varepsilon S_j(g_j^k)$ belongs to $\mathcal{E}$. Since $\mathcal{E}$ is closed under finite addition of gambles, and under dominance, we deduce that also g belongs to $\mathcal{E}$. This completes the proof. □

Proof of Theorem 10. Let us begin with the first statement. Assume $\mathcal{R}$ avoids partial loss. From Definition 11 and Corollary 2, this means that there is no $g\leq 0$ in $\mathcal{E}$, or, equivalently, that $\sup g>0$ for every $g\in\mathcal{E}$. Consider $j=1,\dots,m, n_j\geq 1, k=1,\dots,n_j, g_j^k\in\mathcal{H}^j$ not all of them zero gambles, $\varepsilon>0$. Then the previous comment implies that

$$\sup_{\omega\in\Omega}\left[\sum_{j=1}^{m}\sum_{k=1}^{n_j}G_j(g_j^k|\mathcal{B}_j)+\varepsilon S_j(g_j^k)\right](\omega)>0, \tag{35}$$

because the above gamble belongs to $\mathcal{E}$. Applying Proposition 1 we deduce that $\underline{P}_1(\cdot|\mathcal{B}_1),\dots,\underline{P}_m(\cdot|\mathcal{B}_m)$ avoid partial loss.

Conversely, if $\underline{P}_1(\cdot|\mathcal{B}_1),\dots,\underline{P}_m(\cdot|\mathcal{B}_m)$ avoid partial loss then Proposition 1 implies that Eq. (35) holds and as a consequence $\sup g>0$ for every $g\in\mathcal{E}$. Hence, $\mathcal{R}$ avoids partial loss.

Let us turn now to the second statement. We begin with the direct implication. Assume $\underline{P}_1(\cdot|\mathcal{B}_1),\dots,\underline{P}_m(\cdot|\mathcal{B}_m)$ avoid uniform sure loss, and consider a gamble $g\in\mathcal{E}$. The case $g\in\mathcal{L}^+$ is trivial. For other g, we can apply statement (1) of Theorem 9 to deduce that there are $g_j^k\in\mathcal{H}^j, n_j\geq 1, j=1,\dots,m, k=1,\dots,n_j$ and $\varepsilon>0$ such that $g\geq\sum_{j=1}^{m}\sum_{k=1}^{n_j}G_j(g_j^k|\mathcal{B}_j)+\varepsilon S_j(g_j^k)$. Then

$$\sup g\geq\sup\sum_{j=1}^{m}\sum_{k=1}^{n_j}G_j(g_j^k|\mathcal{B}_j)+\varepsilon S_j(g_j^k)\geq\sup\sum_{j=1}^{m}\sum_{k=1}^{n_j}G_j(g_j^k|\mathcal{B}_j)\geq 0,$$

using that $\underline{P}_1(\cdot|\mathcal{B}_1),\dots,\underline{P}_m(\cdot|\mathcal{B}_m)$ avoid sure loss.

Conversely, assume that $\sup g\geq 0$ for every $g\in\mathcal{E}$, and take $g_j^k\in\mathcal{H}^j, n_j\geq 1, j=1,\dots,m, k=1,\dots,n_j$. Then again Theorem 9, statement (1), implies that for every $\varepsilon>0$

$$\sup_{\omega\in\Omega}\left[\sum_{j=1}^{m}\sum_{k=1}^{n_j}G_j(g_j^k|\mathcal{B}_j)+\varepsilon S_j(g_j^k)\right](\omega)\geq 0,$$

whence

$$\sup_{\omega\in\Omega}\left[\sum_{j=1}^{m}\sum_{k=1}^{n_j}G_j(g_j^k|\mathcal{B}_j)\right](\omega)\geq -\varepsilon\sum_{j=1}^{m}n_j,$$

for every $\varepsilon>0$. As a consequence, $\underline{P}_1(\cdot|\mathcal{B}_1),\dots,\underline{P}_m(\cdot|\mathcal{B}_m)$ avoid uniform sure loss. □

Proof of Theorem 11. Let us show the equivalence between the first two conditions; the equivalence with (3) follows from Lemma 1, statement 5.

Assume first of all that $\underline{P}_j(f_j|B_j)=\underline{E}_j(f_j|B_j)$ for all $j=1,\dots,m, f_j\in\mathcal{H}^j, B_j\in\mathcal{B}_j$. From Corollary 1, recalling also that $\underline{P}_1(\cdot|\mathcal{B}_1),\dots,\underline{P}_m(\cdot|\mathcal{B}_m)$ are separately coherent, and hence bounded, we deduce that $\underline{P}_1(\cdot|\mathcal{B}_1),\dots,\underline{P}_m(\cdot|\mathcal{B}_m)$ avoid partial loss. Theorem 10 then implies that $\mathcal{R}$ (and therefore $\mathcal{E}$, through Proposition 3(c)) avoids partial loss.

Consider now $g\in\mathcal{Q}\cap\mathcal{E}$. Then there is some $f_j\in\mathcal{H}^j, B_j\in\mathcal{B}_j$ and $\varepsilon\neq 0$ such that $g=G_j(f_j|B_j)+\varepsilon B_j$. If $g\notin\mathcal{R}$, it follows from Eq. (17) that $\varepsilon<0$.

On the other hand, if $g \in \mathcal{E}$ we deduce from Theorem 9 that either $g \in \mathcal{L}^+$ or there are $j = 1, \dots, m, n_j \geq 1, k = 1, \dots, n_j, g_j^k \in \mathcal{H}^j, B_j \in \mathcal{B}_j, \delta > 0$, such that $g \geq \sum_{j=1}^m \sum_{k=1}^{n_j} G_j(g_j^k|\mathcal{B}_j) + \delta S_j(g_j^k)$. In the first case, we deduce that $\inf_{B_j} G_j(f_j|B_j) \geq -\varepsilon > 0$, a contradiction with the separate coherence of $\underline{P}_j(\cdot|\mathcal{B}_j)$. In the second, we deduce that

$$G_j(f_j|B_j) + \frac{\varepsilon}{2}B_j \geq \sum_{j=1}^m \sum_{k=1}^{n_j} G_j(g_j^k|\mathcal{B}_j) - \frac{\varepsilon}{2}B_j + \sum_{j=1}^m \sum_{k=1}^{n_j} \delta S_j(g_j^k),$$

or, equivalently,

$$\sum_{j=1}^m \sum_{k=1}^{n_j} G_j(g_j^k|\mathcal{B}_j) - B_j\left(f_j - (\underline{P}_j(f_j|B_j) - \frac{\varepsilon}{2})\right) \leq \frac{\varepsilon}{2}B_j - \sum_{j=1}^m \sum_{k=1}^{n_j} \delta S_j(g_j^k) < 0$$

on $\mathbb{S}(g_j^k) \cup B_j$. Using Eq. (7) we deduce that $\underline{E}_j(f_j|B_j) \geq \underline{P}_j(f_j|B_j) - \frac{\varepsilon}{2}$. This is a contradiction (remember that $\varepsilon < 0$). As a consequence, $\mathcal{Q} \cap \mathcal{E} \subseteq \mathcal{R}$ and therefore $\mathcal{R}$ is coherent.

Finally, if there is some $j \in \{1, \dots, m\}, B_j \in \mathcal{B}_j, f_j \in \mathcal{H}^j$ and $\varepsilon > 0$ s.t. $G_j(f_j|B_j) - \varepsilon B_j \in \mathcal{R}$, then there must be some $j' \in \{1, \dots, m\}, f'_j \in \mathcal{H}^{j'}, B'_j \in \mathcal{B}_{j'}$ and $\delta > 0$ s.t. $G_j(f_j|B_j) - \varepsilon B_j = G_{j'}(f_{j'}|B_{j'}) + \delta B_{j'}$, whence

$$G_{j'}(f_{j'}|B_{j'}) - G_j(f_j|B_j) = -\delta B_{j'} - \varepsilon B_j \leq -\min\{\delta, \varepsilon\} < 0$$

on $B_j \cup B_{j'}$; this is a contradiction with the coherence of $\underline{P}_j(\cdot|\mathcal{B}_j), \underline{P}_{j'}(\cdot|\mathcal{B}_{j'})$.

Conversely, let us show that the first statement implies the second. First of all, if $\mathcal{R}$ is coherent relative to $\mathcal{Q}$, it also avoids partial loss, which guarantees, via Theorem 10 and Lemma 1, statement 2, that the natural extensions are bounded, and through Proposition 3(d) that $\mathcal{E}$ is coherent. Furthermore Lemma 1, statement 1, establishes that $\underline{E}_j(f_j|B_j) \geq \underline{P}_j(f_j|B_j)$, for every $j = 1, \dots, m, f_j \in \mathcal{H}^j, B_j \in \mathcal{B}_j$. Assume ex-absurdo that there are $j_0 \in \{1, \dots, m\}, f_0 \in \mathcal{H}^{j_0}, B_0 \in \mathcal{B}_{j_0}$, such that $\underline{P}_{j_0}(f_0|B_0) < \underline{E}_{j_0}(f_0|B_0)$. Then given $0 < \delta < \underline{E}_{j_0}(f_0|B_0) - \underline{P}_{j_0}(f_0|B_0)$, the definition of the conditional natural extension (Eq. (7)) implies that there are $g_j^k \in \mathcal{H}^j, n_j \geq 1, j = 1, \dots, m, k = 1, \dots, n_j$, such that

$$\sup_{\mathbb{S}(g_j^k) \cup B_0} \Big[\sum_{j=1}^m \sum_{k=1}^{n_j} G_j(g_j^k|\mathcal{B}_j) - B_0(f_0 - (\underline{P}_{j_0}(f_0|B_0) + \delta))\Big] < 0,$$

and the second statement of Theorem 9 implies then that the gamble $B_0(f_0 - (\underline{P}_{j_0}(f_0|B_0) + \delta))$ belongs to $\mathcal{E}$. Since on the other hand Eq. (17) implies that the gamble $B_0(f_0 - \underline{P}_{j_0}(f_0|B_0)) + \frac{\delta}{2}B_0$ belongs to $\mathcal{R} \subseteq \mathcal{E}$, we deduce from the coherence of $\mathcal{E}$ (condition (ADD)) and the definition of $\mathcal{Q}$ that the gamble $B_0(f_0 - \underline{P}_{j_0}(f_0|B_0)) - \frac{\delta}{4}B_0$ belongs to $\mathcal{Q} \cap \mathcal{E}$, and as a consequence also to $\mathcal{R}$, because this set is coherent. But the first statement implies that this gamble cannot be in $\mathcal{R}$ because $\delta > 0$. This is a contradiction, from which we deduce that $\underline{P}_j(f_j|B_j) = \underline{E}_j(f_j|B_j)$ for all $f_j \in \mathcal{H}^j, B_j \in \mathcal{B}_j$ and $j = 1, \dots, m$. This completes the proof. □

Proof of Lemma 2. Since, $\underline{P}_1(\cdot|\mathcal{B}_1), \dots, \underline{P}_m(\cdot|\mathcal{B}_m)$ avoid partial loss, Lemma 1 implies that their natural extensions are coherent and using Theorem 10 and Proposition 3 we also deduce that $\mathcal{E}$ is coherent.

Let $f \in \mathcal{E}$. B_f is non-empty because $f \neq 0$, and moreover $f = B_f f$. If $f \in \mathcal{L}^+$ then for every $\varepsilon > 0$ s.t. $B_f^\varepsilon \neq \emptyset$ (and there is one such ε because $B_f = \cup_{\varepsilon>0} B_f^\varepsilon$) we have that $\underline{E}_0(f|B_f^\varepsilon) \geq \underline{E}_0(\varepsilon|B_f^\varepsilon) = \varepsilon > 0$, so the result holds.

If $f \notin \mathcal{L}^+$, we apply Eq. (20) to find $j = 1, \ldots, m, n_j \geq 1, k = 1, \ldots, n_j, g_j^k \in \mathcal{H}^j$, $\delta > 0$ such that

$$f = B_f f > \sum_{j=1}^{m} \sum_{k=1}^{n_j} G_j(g_j^k|\mathcal{B}_j) + \delta$$

in $\mathbb{S}(g_j^k)$, and such that $B_f f \geq 0$ in $\mathbb{S}(g_j^k)^c$. This second property implies that $f(\omega) > 0$ for every $\omega \in B_f \setminus \mathbb{S}(g_j^k)$, and $f(\omega) \geq \varepsilon$ for every $\omega \in B_f^\varepsilon \setminus \mathbb{S}(g_j^k)$ and for every $\varepsilon > 0$.

Fix $\varepsilon \in [\frac{\delta}{2}, \delta)$. Then $B_f^{\delta-\varepsilon} f \geq B_f f - (\delta - \varepsilon)$, and consequently

$$B_f^{\delta-\varepsilon} f - \sum_{j=1}^{m} \sum_{k=1}^{n_j} G_j(g_j^k|\mathcal{B}_j) \geq B_f f - \sum_{j=1}^{m} \sum_{k=1}^{n_j} G_j(g_j^k|\mathcal{B}_j) - (\delta-\varepsilon) > \delta - (\delta-\varepsilon) = \varepsilon > 0$$

in $\mathbb{S}(g_j^k)$.

Now, let $\alpha > 0$ be equal to $\inf_{\omega \in \mathbb{S}(g_j^k) \cup B_f^{\delta-\varepsilon}} [B_f^{\delta-\varepsilon} f - \sum_{j=1}^{m} \sum_{k=1}^{n_j} G_j(g_j^k|\mathcal{B}_j)](\omega)$. This infimum is positive because it dominates $\min\{\varepsilon, \delta - \varepsilon\} = \delta - \varepsilon$, which is positive. Then $B_f^{\delta-\varepsilon} f - \frac{\alpha}{2} B_f^{\delta-\varepsilon} - \sum_{j=1}^{m} \sum_{k=1}^{n_j} G_j(g_j^k|\mathcal{B}_j) \geq \frac{\alpha}{2} > 0$ in $\mathbb{S}(g_j^k) \cup B_f^{\delta-\varepsilon}$. As a consequence, $\underline{E}_0(f|B_f^{\delta-\varepsilon}) > 0$. Finally, note that we can do this for any $\varepsilon \in [\frac{\delta}{2}, \delta)$, so $\underline{E}_0(f|B_f^{\delta-\varepsilon}) > 0$ for every $\delta - \varepsilon \in (0, \frac{\delta}{2})$.

For the second part it suffices to note that when Ω is finite there is some $\varepsilon' > 0$ such that $B_f^\varepsilon = B_f$ for all positive $\varepsilon < \varepsilon'$, and apply the first part. □

Proof of Theorem 12. Since $\underline{P}_1(\cdot|\mathcal{B}_1), \ldots, \underline{P}_m(\cdot|\mathcal{B}_m)$ avoid partial loss, Lemma 1 implies that all the natural extensions are bounded. We use this fact throughout the proof.

(1) Given $g \in \mathcal{E}_{\underline{P}}, g = G_0(f|B_0) + \varepsilon B_0$; since $g = 0$ in B_0^c, we can assume without loss of generality that $B_f \subseteq B_0$; in fact, otherwise, it suffices to take $f' := B_0 f$, which also satisfies $G_0(f'|B_0) + \varepsilon B_0 = g$, thanks to separate coherence. Moreover, separate coherence also implies that $\underline{E}_0(g|B_0) = \underline{E}_0(G_0(f|B_0) + \varepsilon B_0|B_0) \geq 0 + \varepsilon > 0$. As a consequence, for some $\alpha > 0$ there are gambles $g_j^k \in \mathcal{H}^j$ such that

$$\sum_{j=1}^{m} \sum_{k=1}^{n_j} G_j(g_j^k|\mathcal{B}_j) - B_0(f - \alpha) \leq -\delta < 0$$

in $\mathbb{S}(g_j^k) \cup B_0$ for some $\delta > 0$. Since the support B_f is included in B_0, we deduce that

$$\sum_{j=1}^{m} \sum_{k=1}^{n_j} G_j(g_j^k|\mathcal{B}_j) - B_f(f - \alpha) \leq \sum_{j=1}^{m} \sum_{k=1}^{n_j} G_j(g_j^k|\mathcal{B}_j) - B_0(f - \alpha) \leq -\delta < 0$$

on $\mathbb{S}(g_j^k) \cup B_f$, and as a consequence $\underline{E}_0(f|B_f) > 0$. This shows that $\mathcal{E}_{\underline{P}}$ is included in $\{f : \underline{E}_0(f|B_f) > 0\}$. Conversely, if a gamble f satisfies $\underline{E}_0(f|B_f) = \alpha > 0$, then $f = B_f f = G_0(f|B_f) + \alpha B_f$ belongs to $\mathcal{E}_{\underline{P}}$.

(2) From the definition of the unconditional natural extension $\underline{E}_0$ (Eq. (9)), we have that $\underline{E}_0(f) > 0$ if and only if there is some $\alpha > 0$ and gambles $f_j^k \in \mathcal{H}^j, n_j \geq 1, j = 1, \ldots, m, k = 1, \ldots, n_j$, such that

$$f - \alpha \geq \sum_{j=1}^{m} \sum_{k=1}^{n_j} G(f_j^k | \mathcal{B}_j),$$

and this inequality implies that

$$f \geq \sum_{j=1}^{m} \sum_{k=1}^{n_j} G(f_j^k | \mathcal{B}_j) + \varepsilon S_j(f_j^k),$$

for $\varepsilon := \frac{\alpha}{\sum_{j=1}^{m} n_j} > 0$. Using Theorem 9 we deduce that if $\underline{E}_0(f) > 0$ then $f \in \mathcal{E}$. Since also $\mathcal{L}^+ \subseteq \mathcal{E}$ because this is a coherent set of really desirable gambles, we deduce that $\{f : \underline{E}_0(f) > 0\} \cup \mathcal{L}^+ \subseteq \mathcal{E}$.

On the other hand, if $f \in \mathcal{E}$, Theorem 9 implies that either $f \in \mathcal{L}^+$, and then the coherence of $\underline{E}_0$ guarantees that $\underline{E}_0(f) \geq 0$, or there are gambles $f_j^k \in \mathcal{H}^j, n_j \geq 1, j = 1, \ldots, m, k = 1, \ldots, n_j$, not all of them equal to zero, and $\varepsilon > 0$, such that

$$f \geq \sum_{j=1}^{m} \sum_{k=1}^{n_j} G(f_j^k | \mathcal{B}_j) + \varepsilon S_j(f_j^k).$$

We deduce that $f \geq \sum_{j=1}^{m} \sum_{k=1}^{n_j} G(f_j^k | \mathcal{B}_j)$, whence for every $\alpha > 0$

$$\sum_{j=1}^{m} \sum_{k=1}^{n_j} G(f_j^k | \mathcal{B}_j) - (f + \alpha) \leq -\alpha < 0;$$

as a consequence, $\underline{E}_0(f) \geq -\alpha$ for every $\alpha > 0$, and therefore $\underline{E}_0(f) \geq 0$. We conclude that $\mathcal{E} \subseteq \{f : \underline{E}_0(f) \geq 0\}$.

(3) Let us show that $\mathcal{E}_{\underline{P}} \subseteq \mathcal{E}$. Take $g \in \mathcal{E}_{\underline{P}}$, that is, $g = G_0(f|B_0) + \varepsilon B_0$. By definition of $\underline{E}_0(f|B_0)$, we know that for each $\varepsilon > 0$ there are $j = 1, \ldots, m, n_j \geq 1, k = 1, \ldots, n_j, g_j^k \in \mathcal{H}^j$, such that

$$\inf_{\omega \in \mathbb{S}(g_j^k) \cup B_0} \left[G_0(f|B_0) + \varepsilon B_0 - \sum_{j=1}^{m} \sum_{k=1}^{n_j} G_j(g_j^k | \mathcal{B}_j) \right] (\omega) > 0. \tag{36}$$

We distinguish two cases.

(i) If $g_j^k = 0$ for all j, k, from Eq. (36) we obtain that $G_0(f|B_0) + \varepsilon B_0 > 0$ in B_0. Since outside B_0 we have that $G_0(f|B_0) + \varepsilon B_0 = 0$, it follows that $G_0(f_0|B_0) + \varepsilon B_0 \in \mathcal{L}^+ \subseteq \mathcal{E}$.

(ii) If $g_j^k \neq 0$ for some j, k, we are going to prove that $g \in \mathcal{E}$. For this, we are going to use the equivalent expression (20) established in Theorem 9. From (36), the first requirement in (20) is satisfied, so it suffices to establish the second. Consider $\omega \in \mathbb{S}(g_j^k)^c$, provided $\mathbb{S}(g_j^k)^c$ is not empty. There are two possibilities: either $\omega \notin B_0$, whence trivially $g(\omega) = 0$; or $\omega \in B_0$, whence by applying Eq. (36) we conclude that $g(\omega) > 0$. This shows that $f \in \mathcal{E}$.

We establish next that $\mathcal{E} \subseteq \overline{\mathcal{E}}_{\underline{P}}$. Take $f \in \mathcal{E}$. We know by Lemma 2 that $\underline{E}_0(f|B_f^\varepsilon) > 0$ for every $\varepsilon \in (0, \overline{\varepsilon})$ for some $\overline{\varepsilon} > 0$. Setting $\alpha := \underline{E}_0(f|B_f^\varepsilon) >$

0 for one such ε, the gamble $G_0(f|B_f^\varepsilon)+\alpha B_f^\varepsilon$ belongs to $\mathcal{E}_{\underline{P}}$, and it is equal to $G_0(f|B_f^\varepsilon)+\alpha B_f^\varepsilon = B_f^\varepsilon(f-(\underline{E}_0(f|B_f^\varepsilon)-\alpha)) = B_f^\varepsilon f$. Now, since $f = B_f f$ is the uniform limit, as ε goes to zero, of the gambles $B_f^\varepsilon f$, we deduce that it belongs to $\overline{\mathcal{E}}_{\underline{P}}$.

(4) To see that $\mathcal{E}_{\underline{P}} = \mathcal{E}$ when Ω is finite, note that in that case we can deduce from Lemma 2 that also $\underline{E}_0(f|B_f) > 0$ for every $f \in \mathcal{E}$, and then the first statement implies that $\mathcal{E} \subseteq \mathcal{E}_{\underline{P}}$.

(5) Let us show that $\mathcal{E}$ is the natural extension of $\mathcal{E}_{\underline{P}}$: since $\mathcal{E}$ is a coherent set of desirable gambles that includes $\mathcal{E}_{\underline{P}}$, Proposition 3(b) implies that it must include also its natural extension; conversely, any gamble in $\mathcal{R}$ is of the type $B_j(f-\underline{P}_j(f|B_j))+\varepsilon B_j$ for some $B_j \in \mathcal{B}_j, f \in \mathcal{H}^j, \varepsilon > 0$; since $\underline{E}_0(f|B_j) \geq \underline{P}_j(f|B_j)$ from Lemma 1(1), it follows that $B_j(f-\underline{P}_j(f|B_j))+\varepsilon B_j$ dominates the gamble $B_j(f-\underline{E}_0(f|B_j))+\varepsilon B_j$ which belongs to $\mathcal{E}_{\underline{P}}$; as a consequence, $\mathcal{R}$ is included in the natural extension of $\mathcal{E}_{\underline{P}}$, and from Proposition 3(b) so is its natural extension $\mathcal{E}$.

□

Proof of Theorem 13. Consider an arbitrary $j \in \{1,\ldots,m\}$, $f_j \in \mathcal{H}^j$ and $B_j \in \mathcal{B}_j$. By definition $\mathcal{R}$ contains the gambles $G_j(f_j|B_j)+\varepsilon B_j = B_j(f_j-(\underline{P}_j(f_j|B_j)-\varepsilon))$ for all $\varepsilon > 0$, whence $\underline{P}'_j(f_j|B_j) \geq \underline{P}_j(f_j|B_j)$.

Now, let us assume ex-absurdo that $\underline{P}'_j(f_j|B_j) > \underline{P}_j(f_j|B_j)$, or, in other words, that $g := G_j(f_j|B_j) - \varepsilon' B_j$ belongs to $\mathcal{R}$ for some $\varepsilon' > 0$. Then g also belongs to the natural extension $\mathcal{E}$ of $\mathcal{R}$, given by Eq. (10).

If $g \in \mathcal{L}^+$, then we have that $\inf_{B_j} G_j(f_j|B_j) > 0$, a contradiction with the separate coherence of $\underline{P}_j(\cdot|\mathcal{B}_j)$. Then, if $g \in \mathcal{E}\setminus\mathcal{L}^+$, we can apply Theorem 9 to deduce that there are $g_j^k \in \mathcal{H}^j$, not all of them equal to zero, for $n_j \geq 1, j = 1,\ldots,m, k = 1,\ldots,n_j$ and $\varepsilon > 0$, such that

$$G_j(f_j|B_j) - \varepsilon' B_j \geq \sum_{j=1}^{m}\sum_{k=1}^{n_j} G_j(g_j^k|\mathcal{B}_j) + \varepsilon S_j(g_j^k).$$

As a consequence,

$$-\varepsilon' B_j - \varepsilon \sum_{j=1}^{m}\sum_{k=1}^{n_j} S_j(g_j^k) \geq \sum_{j=1}^{m}\sum_{k=1}^{n_j} G_j(g_j^k|\mathcal{B}_j) - G_j(f_j|B_j),$$

whence the supremum of the right-hand side in $\mathbb{S}(g_j^k) \cup B_j$ is smaller than zero. This is a contradiction with the coherence of the conditional lower previsions $\underline{P}_1(\cdot|\mathcal{B}_1),\ldots,\underline{P}_m(\cdot|\mathcal{B}_m)$. □

Proof of Theorem 14. It follows from Theorem 9 that $\underline{E}'_0(f|B_0)$ is the supremum value of α such that

$$B_0(f-\alpha) \geq \sum_{j=1}^{m}\sum_{k=1}^{n_j} G_j(g_j^k|\mathcal{B}_j) + \varepsilon S_j(g_j^k) \tag{37}$$

for some $j = 1,\ldots,m, n_j \geq 1, k = 1,\ldots,n_j, g_j^k \in \mathcal{H}^j, B_j \in \mathcal{B}_j, \varepsilon > 0$: to see this, note that we can restrict our attention to the gambles in $\mathcal{E}\setminus\mathcal{L}^+$, because with those in $\mathcal{L}^+$ we only attain $\inf_{B_0} f$, and this infimum can also be achieved in Eq. (37) by considering constant gambles.

On the other hand, from Eq. (7), $\underline{E}_0(f|B_0)$ is the supremum value of α such that there are $g_j^k \in \mathcal{H}^j, n_j \geq 1, j = 1, \dots, m, k = 1, \dots, n_j$, $\delta > 0$ such that

$$B_0(f-\alpha) - \sum_{j=1}^{m} \sum_{k=1}^{n_j} G_j(g_j^k|\mathcal{B}_j) \geq \delta > 0 \tag{38}$$

in $\mathbb{S}(g_j^k) \cup B_0$.

Hence, given $\alpha < \underline{E}'_0(f|B_0)$, there are $g_j^k \in \mathcal{H}^j, B_j \in \mathcal{B}_j, \varepsilon > 0$ satisfying Eq. (37), whence $B_0(f-\alpha) - \sum_{j=1}^{m} \sum_{k=1}^{n_j} G_j(g_j^k|\mathcal{B}_j) \geq \varepsilon$ in $\mathbb{S}(g_j^k)$, and $B_0(f-\alpha) \geq 0$ in $B_0 \backslash \mathbb{S}(g_j^k)$. Given μ in $(0, \varepsilon)$, we deduce that $B_0(f-\alpha+\mu) - \sum_{j=1}^{m} \sum_{k=1}^{n_j} G_j(g_j^k|\mathcal{B}_j) \geq \mu$ in $\mathbb{S}(g_j^k) \cup B_0$, and therefore $\underline{E}_0(f|B_0) \geq \alpha - \mu$, from which it follows that $\underline{E}_0(f|B_0) \geq \underline{E}'_0(f|B_0)$.

Conversely, given $\alpha < \underline{E}_0(f|B_0)$, there are $g_j^k \in \mathcal{H}^j, B_j \in \mathcal{B}_j, \delta > 0$ satisfying Eq. (38). Consider $\varepsilon := \frac{\delta}{m \max_{j,k} n_j}$. Then

$$B_0(f-\alpha) \geq \sum_{j=1}^{m} \sum_{k=1}^{n_j} G_j(g_j^k|\mathcal{B}_j) + \varepsilon S_j(g_j^k),$$

and as a consequence $\underline{E}'_0(f|B_0) \geq \underline{E}_0(f|B_0)$. This completes the proof. □

Proof of Proposition 9. Consider a gamble $f \in \mathcal{E}'$. If $f \in \mathcal{L}^+$ then trivially f belongs to the coherent set $\mathcal{E}$. If $f \notin \mathcal{L}^+$, Theorem 9 and the fact that $\mathcal{L}$ is a linear space implies that there are gambles $g_j \in \mathcal{L}$, $j = 1, \dots, m$ and $\varepsilon > 0$ such that $f \geq \sum_{j=1}^{m} G(g_j|\mathcal{B}_j) + \varepsilon S_j(g_j)$. From the definition of $\underline{P}_j(\cdot|\mathcal{B}_j)$ and (ADD), we deduce that the gamble $G(g_j|\mathcal{B}_j) + \varepsilon S_j(g_j) = \sum_{B_j \in S_j(g_j)} G(g_j|B_j) + \varepsilon B_j$ belongs to $\mathcal{E}$ for $j = 1, \dots, m$, and applying then (ADD) and (APG) we deduce that $f \in \mathcal{E}$. This implies that $\mathcal{E}' \subseteq \mathcal{E}$. □

Proof of Proposition 10. Let $\underline{P}_1(\cdot|\mathcal{B}_1), \dots, \underline{P}_m(\cdot|\mathcal{B}_m)$ be coherent conditional lower previsions, $\mathcal{Q}, \mathcal{R}$ be the sets derived from them by means of Eqs. (16) and (17), and let $\mathcal{E}$ be the natural extension of $\mathcal{R}$. We have to show that $\mathcal{E}$ satisfies condition (22).

Take $f \in \mathcal{E}$. From Lemma 2, there is some $\bar{\varepsilon} > 0$ such that for every $\varepsilon \in (0, \bar{\varepsilon})$, $\underline{E}_0(f|B_f^\varepsilon) > 0$; this means that for every $\varepsilon \in (0, \bar{\varepsilon})$ there is some $\delta_\varepsilon > 0$ such that $B_f^\varepsilon(f - \delta_\varepsilon) \in \mathcal{E}$, taking into account Theorem 14. This implies that condition (22) holds. □

Proof of Theorem 15. The second statement follows directly from Proposition 10, so we focus on the first. Consider a coherent set of conditionally strictly desirable gambles $\mathcal{E}$. Fix $\omega_0 \in \Omega$, and for every $\emptyset \neq A \subsetneq \Omega$ that contains ω_0 define the partition $\mathcal{B}_A := \{A, A^c\}$, and let $\mathcal{B}_\Omega := \{\Omega\}$. Then we deduce from $\mathcal{E}$ the following set of conditional lower previsions, each one defined on all $\mathcal{L}$:

$$\{\underline{P}(\cdot|\mathcal{B}_A)\}_{\omega_0 \in A \subseteq \Omega}, \tag{39}$$

where, as usual, $\underline{P}(f|B) := \sup\{\mu : B(f-\mu) \in \mathcal{E}\}$ for all $f \in \mathcal{L}$ and $\emptyset \neq B \subseteq \Omega$. It follows from Corollary 4 that the conditional lower previsions in (39) are coherent. Let $\mathcal{R}$ be the set of desirable gambles derived from those conditional lower previsions, and $\mathcal{E}_{\underline{P}}$ the set derived from the natural extensions of the lower previsions, as in (21). It is trivial that in the specific case of the lower previsions considered, it holds that $\mathcal{R} = \mathcal{E}_{\underline{P}}$, because the lower previsions $\{\underline{P}(\cdot|\mathcal{B}_A)\}_{\omega_0 \in A \subseteq \Omega}$ already encompass all their natural extensions. Therefore, since we know by Theorem 12 that $\mathcal{E}_{\underline{P}}$ is the natural extension of $\mathcal{R}$, what we are left to show is that $\mathcal{E}_{\underline{P}} = \mathcal{E}$.

That $\mathcal{E}_{\underline{P}} \subseteq \mathcal{E}$ follows directly from Proposition 9. Conversely, given $f \in \mathcal{E}$, it follows by Assumption (23) that $\underline{P}(f|B_f) > 0$, whence given $\varepsilon := \underline{P}(f|B_f)$, the gamble $G(f|B_f) + \varepsilon B_f = B_f(f - (\underline{P}(f|B_f) - \varepsilon)) = f$ belongs to $\mathcal{R} = \mathcal{E}_{\underline{P}}$. □

Proof of Theorem 16. Assume first of all that Eq. (24) holds, and let us show that then $\overline{P}(B) > 0$.

By contradiction, say that $\overline{P}(B) = 0$. From the correspondence between coherent sets of almost-desirable gambles and coherent lower previsions, $B(f-\mu) \in \overline{\mathcal{R}}$ if and only if $P(B(f-\mu)) \geq 0$ for all $P \in \mathcal{M}$, and therefore if $\overline{P}(B) = 0$ we get that $B(f-\mu) \in \overline{\mathcal{R}}$ for all $\mu \in \mathbb{R}$. This implies, through (24), that $B(f-\mu) \in \mathcal{R}$ for all $\mu \in \mathbb{R}$, and hence that $\underline{P}(f|B) = +\infty$. This means that $\underline{P}(f|B)$ is not well-defined according to Definition 14, and since $\mathcal{R}$ is closed under dominance, we deduce from Theorem 5 that it does not satisfy (SD), a contradiction.

To see that moreover we also have the equality between $\underline{R}(f|B)$ and $\underline{P}(f|B)$, we can start by exploiting some passages originally made with other purposes in [22, Eq. (12) in Section 3.7]:

$$\begin{aligned}
\underline{R}(f|B) &= \inf\{P(Bf)/P(B) : P \in \mathcal{M}, P(B) > 0\} \\
&= \sup\{\mu : P(Bf)/P(B) \geq \mu \ \forall P \in \mathcal{M}, P(B) > 0\} \\
&= \max\{\mu : P(Bf) \geq \mu P(B) \ \forall P \in \mathcal{M}\} \\
&= \max\{\mu : P(B(f-\mu)) \geq 0 \ \forall P \in \mathcal{M}\} \\
&= \max\{\mu : B(f-\mu) \in \overline{\mathcal{R}}\},
\end{aligned}$$

where the last passage is due to [19, Theorem 3.8.1]. As a consequence, $\underline{P}(f|B) = \underline{R}(f|B)$ if and only if $\sup\{\mu : B(f-\mu) \in \mathcal{R}\} \geq \max\{\mu : B(f-\mu) \in \overline{\mathcal{R}}\}$ (the converse inequality holds in general because $\mathcal{R} \subseteq \overline{\mathcal{R}}$).

Let $\mu^* := \max\{\mu : B(f-\mu) \in \overline{\mathcal{R}}\}$. From Eq. (24), we have that $B(f-(\mu^*-\varepsilon)) \in \mathcal{R}$ for all $\varepsilon > 0$. This implies that $\{\mu : B(f-\mu) \in \mathcal{R}\} \supseteq \{\mu^* - \varepsilon : \varepsilon > 0\}$, and hence that $\sup\{\mu : B(f-\mu) \in \mathcal{R}\} \geq \sup\{\mu^* - \varepsilon : \varepsilon > 0\} = \max\{\mu : B(f-\mu) \in \overline{\mathcal{R}}\}$. As a consequence, $\underline{R}(f|B) = \underline{P}(f|B)$.

Conversely, assume that $\overline{P}(B) > 0$ and $\underline{R}(f|B) = \underline{P}(f|B)$, or, equivalently, that $\sup\{\mu : B(f-\mu) \in \mathcal{R}\} \geq \max\{\mu : B(f-\mu) \in \overline{\mathcal{R}}\}$. Take $\mu \in \mathbb{R}$ such that $B(f-\mu) \in \overline{\mathcal{R}}$, and consider $\varepsilon > 0$. Then it follows from the assumption that $\mu - \varepsilon < \sup\{\mu' : B(f-\mu') \in \mathcal{R}\}$, whence $B(f-(\mu-\varepsilon)) \in \mathcal{R}$. As a consequence, Eq. (24) holds. □

Proof of Proposition 11. Consider a set of gambles $\mathcal{D}$ satisfying (WD1)–(WD5), and let us define $\mathcal{R}$ by

$$\mathcal{R} := \{f \in \mathcal{D} : -f \notin \mathcal{D}\}.$$

Let us show that this subset of $\mathcal{D}$ is a coherent set of really desirable gambles. We apply Proposition 2:

(APL) Trivially the gamble $f := 0$ coincides with $-f$, so we cannot have $f \in \mathcal{D}, -f \notin \mathcal{D}$. Hence, $0 \notin \mathcal{R}$.

(APG) Given a gamble $f \in \mathcal{L}^+$, it belongs to $\mathcal{D}$ because of (WD2). Since the gamble $-f \lneq 0$ does not belong to $\mathcal{D}$ because of (WD1), we deduce that $f \in \mathcal{R}$.

(PHM) Let $f \in \mathcal{R}, \lambda > 0$; then $f \in \mathcal{D}$, whence (WD3) implies that $\lambda f \in \mathcal{D}$. If $-\lambda f \in \mathcal{D}$, then $-f \in \mathcal{D}$ and we contradict that $f \in \mathcal{R}$.

(ADD) Given $f, g \in \mathcal{R}$, it follows that $f, g \in \mathcal{D}$, whence (WD4) implies that $f+g \in \mathcal{D}$. If $f+g \notin \mathcal{R}$, then it must be $-f-g \in \mathcal{D}$, and applying (WD4) we deduce that $-f-g+g = -f \in \mathcal{D}$, which contradicts that $f \in \mathcal{R}$. Hence, $f+g \in \mathcal{R}$.

Applying Eq. (27), $\mathcal{D}_{\mathcal{R}}$ is given by $\mathcal{D}_{\mathcal{R}} := \{f : f+g \in \mathcal{R}\ \forall g \in \mathcal{R}\} \supseteq \mathcal{R}$. Given a gamble $f \in \mathcal{D} \setminus \mathcal{R}$ and $g \in \mathcal{R}$, it holds that $f+g \in \mathcal{D}$ because of (WD4); if $f+g \notin \mathcal{R}$, then it must be $-f-g \in \mathcal{D}$, whence $-f-g+f = -g \in \mathcal{D}$, and this contradicts that $g \in \mathcal{R}$. Hence, $\mathcal{D} \subseteq \mathcal{D}_{\mathcal{R}}$. To see that they coincide, consider a gamble $f \in \mathcal{D}_{\mathcal{R}}$; since for every $\delta > 0$ the constant gamble on δ belongs to $\mathcal{R}$ because of (APG), it follows from the definition of $\mathcal{D}_{\mathcal{R}}$ that $f+\delta \in \mathcal{R} \subseteq \mathcal{D}$ for every $\delta > 0$. Applying (WD5), it follows that either $f \in \mathcal{D}$ or $f \lneq 0$; but this second possibility contradicts (WD1), which $\mathcal{D}_{\mathcal{R}}$ satisfies because of [5, Proposition 5]. □

Proof of Proposition 12. Let us denote by μ_1, μ_2 the left- and right-hand sides of Eq. (28), respectively. Since $\mathcal{R} \subseteq \mathcal{D}_{\mathcal{R}}$, it suffices to prove that $\mu_2 \leq \mu_1$. Consider any $\varepsilon > 0$. Then since $\mathcal{D}_{\mathcal{R}}$ satisfies (WD4) and (WD2) from [5, Proposition 5], we deduce that the gamble $B(f-(\mu_2-\varepsilon)) \in \mathcal{D}_{\mathcal{R}}$. Since on the other hand the gamble εB belongs to $\mathcal{R}$ because of (APG), it follows from the definition of $\mathcal{D}_{\mathcal{R}}$ that $B(f-(\mu_2-\varepsilon)) + B\varepsilon = B(f-(\mu_2-2\varepsilon)) \in \mathcal{R}$, whence $\mu_1 \geq \mu_2 - 2\varepsilon$. Since we can do this for every $\varepsilon > 0$, we deduce that $\mu_1 \geq \mu_2$, and as a consequence they are equal. □

References

[1] I. Couso and S. Moral. Sets of desirable gambles and credal sets. In T. Augustin, F. Coolen, S. Moral, and M. C. M. Troffaes, editors, *ISIPTA '09 – Proceedings of the Sixth International Symposium on Imprecise Probability: Theories and Applications*, pages 99–108, Manno, Switzerland, 2009. SIPTA.

[2] L. M. de Campos, M. T. Lamata, and S. Moral. The concept of conditional fuzzy measures. *International Journal of Intelligent Systems*, 5:237–246, 1990.

[3] G. De Cooman and F. Hermans. Coherent immediate prediction: bridging two theories of imprecise probability. *Artificial Intelligence*, 172(11):1400–1427, 2008.

[4] G. de Cooman and E. Miranda. Symmetry of models versus models of symmetry. In W. L. Harper and G. R. Wheeler, editors, *Probability and Inference: Essays in Honor of Henry E. Kyburg, Jr.*, pages 67–149. King's college publications, 2007.

[5] G. De Cooman and E. Quaeghebeur. Exchangeability and sets of desirable gambles. *International Journal of Approximate Reasoning*, 2011. In press.

[6] G. de Cooman and M. Zaffalon. Updating beliefs with incomplete observations. *Artificial Intelligence*, 159(1–2):75–125, 2004.

[7] R. Fagin and J. Y. Halpern. A new approach to updating beliefs. In P. P. Bonissone, M. Henrion, L. N. Kanal, and J. F. Lemmer, editors, *Uncertainty in Artificial Intelligence*, volume 6, pages 347–374. North-Holland, Amsterdam, 1991.

[8] J.-Y. Jaffray. Bayesian updating and belief functions. *IEEE Transactions on Systems, Man and Cybernetics*, 22:1144–1152, 1992.

[9] D. Kikuti, F. G. Cozman, and R. Shirota Filhoa. Sequential decision making with partially ordered preferences. *Artificial Intelligence*. In press.
[10] I. Levi. Potential surprise: its role in inference and decision making. In L. J. Cohen and M. Hesse, editors, *Applications of Inductive Logic*, pages 1–27. Clarendon Press, Oxford, 1980.
[11] E. Miranda. A survey of the theory of coherent lower previsions. *International Journal of Approximate Reasoning*, 48(2):628–658, 2008.
[12] E. Miranda. Updating coherent previsions on finite spaces. *Fuzzy Sets and Systems*, 160(9):1286–1307, 2009.
[13] E. Miranda and G. de Cooman. Coherence and independence in non-linear spaces. 2005. Technical Report. Downloadable at http://bellman.ciencias.uniovi.es/∼emiranda/wp05-10.pdf.
[14] E. Miranda and M. Zaffalon. Conditional models: coherence and inference through sequences of joint mass functions. *Journal of Statistical Planning and Inference*, 140(7):1805–1833, 2010.
[15] S. Moral. Epistemic irrelevance on sets of desirable gambles. *Annals of Mathematics and Artificial Intelligence*, 45:197–214.
[16] R. Pelessoni and P. Vicig. Williams coherence and beyond. *International Journal of Approximate Reasoning*, 50(4):612–626, 2009.
[17] C. A. B. Smith. Consistency in statistical inference and decision. *Journal of the Royal Statistical Society, Series A*, 23:1–37, 1961.
[18] P. Vicig, M. Zaffalon, and F. Cozman. Notes on 'Notes on conditional previsions'. *International Journal of Approximate Reasoning*, 44(3):358–365, 2007.
[19] P. Walley. *Statistical Reasoning with Imprecise Probabilities*. Chapman and Hall, London, 1991.
[20] P. Walley. Inferences from multinomial data: learning about a bag of marbles. *Journal of the Royal Statistical Society, Series B*, 58:3–57, 1996. With discussion.
[21] P. Walley. Towards a unified theory of imprecise probability. *International Journal of Approximate Reasoning*, 24(2–3):125–148, 2000.
[22] P. Walley, R. Pelessoni, and P. Vicig. Direct algorithms for checking consistecy and making inferences for conditional probability assessments. *Journal of Statistical Planning and Inference*, 126(1):119–151, 2004.
[23] P. M. Williams. Coherence, strict coherence and zero probabilities. In *Proceedings of the Fifth International Congress in Logic, Methodology, and Philosophy of Science*, pages 29–33. 1975.
[24] P. M. Williams. Notes on conditional previsions. Technical report, School of Mathematical and Physical Science, University of Sussex, UK, 1975. Reprinted in [25].
[25] P. M. Williams. Notes on conditional previsions. *International Journal of Approximate Reasoning*, 44(3):366–383, 2007.

University of Oviedo, Department of Statistics and Operations Research, C-Calvo Sotelo, s/n, 33007 Oviedo, Spain
E-mail address: mirandaenrique@uniovi.es

IDSIA, Galleria 2, CH-6928 Manno (Lugano), Switzerland
E-mail address: zaffalon@idsia.ch