The Pari-Mutuel Model

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Abstract

We explore generalizations of the pari-mutuel model (PMM), a formalization of an intuitive way of assessing an upper probability from a precise one. We discuss a naive extension of the PMM considered in insurance and generalize the natural extension of the PMM introduced by P. Walley and other related formulae. The results are subsequently given a risk measurement interpretation: in particular it is shown that a known risk measure, Tail Value at Risk (TVaR), is derived from the PMM, and a coherent risk measure more general than TVaR from its imprecise version. We analyze further the conditions for coherence of a related risk measure, Conditional Tail Expectation. Explicit formulae for conditioning the PMM and conditions for dilation or imprecision increase are also supplied and discussed.

Keywords. Pari-mutuel model, risk measures, natural extension, dilation, 2-monotonicity.

1 Introduction

The *pari-mutuel model* (PMM) formalizes a very intuitive and therefore widely used method of assigning an upper probability starting from a precise probability. To introduce it, consider, following [2], a probability P for event A as a *fair* price for a bet which returns 1 unit to the bettor if A is true, 0 units if A is false, i.e. returns the indicator I_A of A. The bettor's gain is $G = I_A - P(A)$, while that of his opponent, House, is $-G = G_H = P(A) - I_A$.

In most real-world betting schemes House is unwilling to accept such a fair game (the expectation $E(G_H)$ is 0), but asks for a positive gain expectation. It is so when House is a bookmaker, an insurance company, the organizer of a lottery, and so on. A way to achieve this goal is to raise the bettor's price, without altering his reward, and a *naive method* multiplies P by a constant greater than 1, say $1+\delta$, where $\delta > 0$ is a loading constant. The bettor pays $\overline{P}(A) = (1+\delta)P(A)$, while the gain for House is now $\overline{G}_H = (1+\delta)P(A) - I_A$. Alternatively, House might ask the same price to pay a reduced reward $(1-\tau)I_A$, where $0 < \tau < 1$ is interpreted as a commission, or also a taxation. This originates a gain $\overline{G}^*_H = P(A) - (1-\tau)I_A = (1-\tau)(\frac{P(A)}{1-\tau} - I_A) = (1-\tau)\overline{G}_H$ iff $\frac{1}{1-\tau} = 1+\delta$, i.e. iff $\tau = \frac{\delta}{1+\delta}$. Thus, up to a scaling factor, the two methods are equivalent if $\tau = \frac{\delta}{1+\delta}$; the latter is formally more adherent to common betting systems, called in fact *pari-mutuel systems*.

In the theory of imprecise probabilities, $\overline{P}$ is an upper probability, but a slight adjustment to $\overline{P}$ is necessary to achieve coherence. In fact, Walley [11] terms pari-mutuel model the upper probability

$$\overline{P}(A) = \min\{(1+\delta)P(A), 1\}. \qquad (1)$$

Intuitively, the correction should be needed: when $P(A) > \frac{1}{1+\delta}$, it is $\overline{G}_H > 0$ in the naive method, i.e. a bettor suffers from a sure loss no matter whether A is true or false.

This paper investigates further the pari-mutuel model, extending the analysis in [11]. Preliminary issues are recalled in Section 2, very concisely in general, more extensively as for 2-monotone and 2-alternating previsions, since the upper probability $\overline{P}$ in (1) is 2-alternating. In Section 3 we discuss extensions of the PMM. First, we consider alternative expressions for the natural extension $\overline{E}(X)$ of $\overline{P}$, defined on a field $\mathcal{A}$, to any $\mathcal{A}$-measurable gamble X. These expressions were stated in [11], but we make a more detailed analysis of the conditions ensuring that $\overline{E}(X)$ is equal to a certain conditional prevision ($P(X|X > x_\tau)$), which has a risk measurement interpretation. In Section 3.1 we restrict to non-negative gambles and compare the natural extension $\overline{E}$ with the naive extension $\overline{P}_N(X) = \min\{(1+\delta)P(X), \sup X\}$, showing that quite often $\overline{P}_N$ is not coherent, or it sometimes coincides with $\overline{E}$. The motivation for this work is that $\overline{P}_N$ is a premium in insurance, although with different premises: the starting point is not the PMM but a set of non-negative gambles. In Section 3.2 we generalize Walley's approach, obtaining a formula for $\overline{E}(X)$ when the PMM is given on a lattice of events and X is not necessarily measurable.

These results have an interesting and, to the best of our knowledge, so far not considered interpretation in the realm of risk measurement. This is the main topic of Section 4, where the natural extension of the PMM defined on a field is shown to correspond to a coherent risk measure, called Tail Value-at-Risk or TVaR (in [4]; other authors may use a different terminology). When the PMM is defined on a lattice, we obtain a generalization of TVaR (not discussed in the risk literature), which replaces precise with imprecise uncertainty measures; we name it ITVaR. Thus the PMM supplies a motivation for introducing 'imprecise' risk measures: one of them, ITVaR, is the natural extension of a PMM assigned on a lattice. Conditioning the PMM defined on a field is discussed in Section 5. We specialize general formulae for the natural extension of 2-alternating and 2-monotone probabilities to the case of the PMM and discuss the effect on them of *dilation* and of a weaker phenomenon, *imprecision increase*. We obtain a number of conditions for dilation or imprecision increase, and discuss in detail the operationally most relevant cases (when the commission τ is not "too high" and event A is either "common" or "rare"). Section 6 concludes the paper.

2 Preliminaries

Upper ($\overline{P}$) and lower ($\underline{P}$) probabilities are customarily related by the conjugacy relation $\overline{P}(A) = 1 - \underline{P}(A^c)$, which lets one refer to either $\overline{P}$ or $\underline{P}$ only. Applying it to (1), the lower probability in the PMM is [11]

$$\underline{P}(A) = \max\{(1+\delta)P(A) - \delta, 0\}. \tag{2}$$

As noted in the Introduction, the parameter $\tau \in]0;1[$ can, and later will, alternatively describe $\overline{P}$, $\underline{P}$ in the PMM. The relationship between τ and δ is:

$$\tau = \frac{\delta}{1+\delta} \quad ; \quad \delta = \frac{\tau}{1-\tau}. \tag{3}$$

An upper probability $\overline{P}$ defined by (1) for any A in an *arbitrary* set of events $\mathcal{D}$ (or $\underline{P}$ defined by (2)) is *coherent* on $\mathcal{D}$, and probably the simplest way to see it is to apply the later Proposition 2. In general, an *upper prevision* $\overline{P}$ is a mapping from a set $\mathcal{D}$ of *gambles* (bounded random variables) into the real line, and an *upper probability* is its special case that the domain $\mathcal{D}$ is made of (indicators of) events only. The upper prevision $\overline{P}$ is *coherent* on $\mathcal{D}$ iff, $\forall n \in \mathbb{N}$, $\forall s_0, s_1, \dots, s_n \geq 0$, $\forall X_0, X_1, \dots, X_n \in \mathcal{D}$, defining $\overline{G} = \sum_{i=1}^{n} s_i(\overline{P}(X_i) - X_i) - s_0(\overline{P}(X_0) - X_0)$, it holds that $\sup \overline{G} \geq 0$.

There are several necessary conditions for coherence, in particular: *internality*, $\inf X \leq \overline{P}(X) \leq \sup X$, and *subadditivity*, $\overline{P}(X+Y) \leq \overline{P}(X) + \overline{P}(Y)$.

We refer to [11] for a thorough presentation of the theory of coherent upper/lower previsions. One of its most important notions is that of *natural extension* [11, Section 3].

In our framework, the natural extension $\overline{E}$ on $\mathcal{D}'$ of a coherent upper prevision (or probability) $\overline{P}$ defined on $\mathcal{D} \subset \mathcal{D}'$ is the *least-committal* coherent extension of $\overline{P}$ on $\mathcal{D}'$, i.e. $\overline{E}(X) = \overline{P}(X)$, $\forall X \in \mathcal{D}$, and for any coherent $\overline{P}^*$ such that $\overline{P}^* = \overline{P}$ on $\mathcal{D}$, $\overline{E}(X) \geq \overline{P}^*(X)$, $\forall X \in \mathcal{D}'$, i.e. $\overline{E}$ *dominates* $\overline{P}^*$. It can be shown that $\overline{E}$ always exists. Symmetrically, the natural extension $\underline{E}$ on $\mathcal{D}'_L$ of a coherent lower prevision $\underline{P}$ on $\mathcal{D}_L$ is such that $\underline{E} = \underline{P}$ (on $\mathcal{D}_L$), and every coherent extension $\underline{P}^*$ of $\underline{P}$ dominates $\underline{E}$ on $\mathcal{D}'_L$.

If condition '$\forall s_0, s_1, \dots, s_n \geq 0$' is replaced by '$\forall s_0, s_1, \dots, s_n \in \mathbb{R}$' in the definition of coherent upper prevision, we obtain de Finetti's notion of *dF-coherent* (precise) prevision [2]. A dF-coherent prevision P is coherent both as an upper and as a lower prevision. The precise previsions or probabilities in the sequel are meant to be dF-coherent.

Although the domain of an upper prevision may be arbitrary, it will have a special structure in most of the paper, to exploit results on 2-alternating previsions.

More specifically, a set of events $\mathcal{A}$ is a *field* when $\emptyset \in \mathcal{A}$ and $A \vee B, A^c \in \mathcal{A}, \forall A, B \in \mathcal{A}$. If $\mathcal{A}$ is a field, a gamble X is $\mathcal{A}$-*measurable* when the events $X > x$ and $X < x$ are in $\mathcal{A}$, $\forall x \in \mathbb{R}$.

A set of gambles S is a *lattice* if X, $Y \in S$ implies $\max(X,Y) \in S$ and $\min(X,Y) \in S$.

An upper prevision $\overline{P}$ defined on a lattice S is 2-*alternating* iff $\overline{P}(\max(X,Y)) \leq \overline{P}(X) + \overline{P}(Y) - \overline{P}(\min(X,Y))$, $\forall X, Y \in S$. A lower prevision $\underline{P}$ on S is 2-*monotone* iff $\underline{P}(\max(X,Y)) \geq \underline{P}(X) + \underline{P}(Y) - \underline{P}(\min(X,Y))$, $\forall X, Y \in S$.

Results stated for 2-monotone previsions are easily reworded for 2-alternating ones (and vice versa), since the conjugate $\overline{P}(X) = -\underline{P}(-X)$ of a 2-monotone lower prevision is 2-alternating (and vice versa).

When S is a set of (indicators of) events and $\overline{P}$ is therefore an upper probability, S is a lattice iff $A, B \in S$ implies $A \vee B \in S$, $A \wedge B \in S$, and $\overline{P}$ is 2-alternating iff $\overline{P}(A \vee B) \leq \overline{P}(A) + \overline{P}(B) - \overline{P}(A \wedge B)$, $\forall A, B \in S$. With a mild additional condition, 2-alternating upper probabilities are coherent [1]:

Proposition 1. *Let $\overline{P}$ be a 2-alternating upper probability on a lattice S containing $\emptyset$ and Ω. Then $\overline{P}$ is coherent iff $\overline{P}(\emptyset) = 0$ and $\overline{P}(\Omega) = 1$.*

Notation Let S^+ be a lattice of events containing $\emptyset$ and Ω.

One way to obtain coherent 2-alternating upper probabilities defines $\overline{P}$ as a special *distorted probability*, by the following result, adapted from [3], Example 2.1.

Proposition 2. *Let P be a dF-coherent probability on S^+ and ϕ a (weakly) increasing concave function defined on $[0;1]$ with $\phi(0) = 0$, $\phi(1) = 1$. Then the* distorted probability *$\overline{P}(\cdot) = \phi(P(\cdot))$ is a 2-alternating and coherent upper probability.*

Proposition 2 ensures that $\overline{P}$ in (1) is 2-alternating and coherent (put $\phi(x) = \min((1+\delta)x, 1)$), hence its conjugate $\underline{P}$ is 2-monotone and coherent.

To deal with the natural extension of the PMM in Section 3, the following Proposition 3 will be exploited.

Notation The natural extension of interest is that of $\overline{P}$ from S^+ to the set $\mathcal{L} = \mathcal{L}(I\!P_u)$ of all gambles defined on a "universal" partition $I\!P_u$ (termed Ω in [11]). That is, $I\!P_u$ is a set of pairwise disjoint events, whose sum is the sure event Ω, and such that its powerset $2^{I\!P_u}$ contains all the events of interest. In particular $S^+ \subseteq 2^{I\!P_u}$. Given $\overline{P} : S^+ \to \mathbb{R}$, its *outer (set) function* $\overline{P}^*$ is defined on $2^{I\!P_u}$ by $\overline{P}^*(B) = \inf\{\overline{P}(A) : A \in S^+, B \Rightarrow A\}$, $\forall B \in 2^{I\!P_u}$.

Proposition 3. *[1] Let $\overline{P} : S^+ \to \mathbb{R}$ be a coherent 2-alternating upper probability. Its natural extension $\overline{E}$ on $\mathcal{L}$ is given by*

$$\overline{E}(X) = \inf X + \int_{\inf X}^{\sup X} \overline{P}^*(X > x)dx \tag{4}$$

and is 2-alternating too. Further,

(a) The restriction of $\overline{E}$ on $2^{I\!P_u}$ coincides with the outer function $\overline{P}^*$.

(b) If $S^+ = 2^{I\!P_u}$, $\overline{E}$ is the only *2-alternating coherent extension of $\overline{P}$ on $\mathcal{L}$.*

In Section 5 we shall be concerned with natural extensions on conditional events, like $\overline{E}(A|B)$ or $\underline{E}(A|B)$, while precise conditional previsions, like $P(X|X > x_\tau)$, appear in Section 3. Although the paper presentation does not focus on coherence concepts in a conditional environment, our approach employs formally *Williams coherence* or *W-coherence*, in the version presented in [7], Definition 4, which unlike Walley's coherence in [11, Section 7.1.4 (b)] imposes no structure constraints on the domain $\mathcal{D}$ of the upper or lower previsions. However, when finitely many conditioning events are involved in $\mathcal{D}$ (as is always the case in the paper), Williams and Walley's coherence are equivalent (after extending the given W-coherent prevision on a suitable set $\mathcal{D}'$, which can be always done keeping W-coherence, cf. [7]). Thus the results in the paper hold also in terms of Walley's coherence.

Several necessary conditions hold for W-coherence, whenever they are well-defined. Recall *internality*: $\inf(X|B) \leq \overline{P}(X|B) \leq \sup(X|B)$, where for instance $\sup(X|B) = \sup\{X(\omega)|\omega \Rightarrow B\}$, and the *Generalized Bayes Rule* (GBR) $\overline{P}(I_A(X - \overline{P}(X|A))) = 0$, which in the case of precise previsions specialises to

$$P(XI_A) = P(X|A)P(A). \tag{5}$$

3 Extending the pari-mutuel model

The natural extension $\overline{E}$ of $\overline{P}(A) = \min\{(1+\delta)P(A), 1\}$ from a field $\mathcal{A}$ to any $\mathcal{A}$-measurable gamble X was shown in [11] to be

$$\overline{E}(X) = x_\tau + (1+\delta)P((X - x_\tau)^+), \tag{6}$$

where $(X - x_\tau)^+ = \max\{X - x_\tau, 0\}$ and the (upper) *quantile* x_τ is defined as

$$x_\tau = \sup\{x \in \mathbb{R} : P(X \leq x) \leq \tau\}. \tag{7}$$

An alternative expression for $\overline{E}(X)$ is:[1]

$$\begin{aligned} \overline{E}(X) &= (1-\varepsilon)P(X|X > x_\tau) + \varepsilon x_\tau, \\ \varepsilon &= 1 - (1+\delta)P(X > x_\tau). \end{aligned} \tag{8}$$

It is also stated in [11] that $\overline{E}(X) = P(X|X > x_\tau)$ if X has a continuous *distribution function* $F_X(x) \stackrel{\text{def}}{=} P(X \leq x)$.

We shall now explore more thoroughly the relationship between $\overline{E}(X)$ and $P(X|X > x_\tau)$. The results will be exploited also in Section 4, where they will be reinterpreted in a risk measurement perspective.

To begin with, we gather some known or anyway elementary, but useful facts in the following proposition.

Proposition 4. *Let X be $\mathcal{A}$-measurable and for $\tau \in]0;1[$ define: x_τ by (7), $F_X(x_\tau^+) = \lim_{x \to x_\tau^+} F_X(x)$, $F_X(x_\tau^-) = \lim_{x \to x_\tau^-} F_X(x)$.*

a) $\tau \in [F_X(x_\tau^-); F_X(x_\tau^+)]$; besides, all values of τ in $[F_X(x_\tau^-); F_X(x_\tau^+)[$ originate by (7) the same (upper) quantile x_τ.

b) $\inf X \leq x_\tau \leq \sup X$.

[1] Equation (8) is stated without proof in [11], Note 3 to Section 3.2. A proof may follow from the later Proposition 9.

c) $(X > x_\tau) = \emptyset$ iff $x_\tau = \sup X$; if $(X \leq x_\tau) = \emptyset$ then $x_\tau = \inf X$.

d) It holds for ε in (8) that $\varepsilon \lesseqqgtr 0$ iff $\tau \gtreqqless F_X(x_\tau)$.[2]

Corollary 1. *If $(X > x_\tau) = \emptyset$, $\overline{E}(X) = \sup X$.*

Proof. Substitute (by Proposition 4, c) $x_\tau = \sup X$ in (6), noting that $P((X - x_\tau)^+) = P(0) = 0$. □

Remark 1. When P is σ-additive, $F_X(x_\tau^+) = F_X(x_\tau)$, i.e. F_X is right-continuous. But an often neglected issue broadens the number of possible alternatives in comparing $\overline{E}(X)$ with $P(X|X > x_\tau)$ (and with another extension in the next Section 3.1): since F_X is originated by a not necessarily σ-additive probability P, there may exist non-zero *adherent probabilities* at x_τ (cf. [2], Section 6.4.11). Precisely,

$$F_X(x_\tau^+) - F_X(x_\tau^-) = P_{x_\tau}^- + P_{x_\tau}^+ + P(X = x_\tau),$$

where $P_{x_\tau}^- = F_X(x_\tau) - F_X(x_\tau^-) - P(X = x_\tau)$ is the *left adherent* probability at x_τ, $P_{x_\tau}^+ = F_X(x_\tau^+) - F_X(x_\tau)$ is the *right adherent* probability at x_τ. Hence,

$$F_X(x_\tau) = F_X(x_\tau^-) + P_{x_\tau}^- + P(X = x_\tau). \qquad (9)$$

While $P_{x_\tau}^+$ is zero iff F_X is right-continuous at x_τ (always if P is σ-additive), from (9), F_X may be left-discontinuous in x_τ also when $P_{x_\tau}^- = 0$, if $P(X = x_\tau) > 0$ (σ-additivity of P implies $P_{x_\tau}^- = 0$). □

Proposition 5. *a) If $P(X|X > x_\tau) = x_\tau$, then $\overline{E}(X) = P(X|X > x_\tau)$.*

b) If $P(X|X > x_\tau) > x_\tau$, then $\overline{E}(X) \lesseqqgtr P(X|X > x_\tau)$ iff $\tau \lesseqqgtr F_X(x_\tau)$.

Proof. Using (8), $\overline{E}(X) \lesseqqgtr P(X|X > x_\tau)$ iff $\varepsilon(x_\tau - P(X|X > x_\tau)) \lesseqqgtr 0$, from which a) follows immediately, b) using also Proposition 4, d). □

Proposition 5, a) considers a really extreme situation. Assuming from now that $P(X|X > x_\tau) > x_\tau$, Proposition 5, b) reduces the comparison between $\overline{E}(X)$ and $P(X|X > x_\tau)$ to comparing τ and $F_X(x_\tau)$ in the further subcases that can be identified. The most notable instances are:

i) F_X is continuous at x_τ. This implies $\tau = F_X(x_\tau)$, and $\overline{E}(X) = P(X|X > x_\tau)$.

ii) F_X is right-continuous, but not continuous at x_τ, and $\tau \neq F_X(x_\tau)$. This implies $F_X(x_\tau) = F_X(x_\tau^+) > \tau$, and $P(X|X > x_\tau) > \overline{E}(X)$.

[2] We write $\lesseqqgtr$ or $\gtreqqless$ to summarize three conditions, here $\varepsilon < 0$ iff $\tau > F_X(x_\tau)$, $\varepsilon = 0$ iff $\tau = F_X(x_\tau)$, $\varepsilon > 0$ iff $\tau < F_X(x_\tau)$.

Case ii) is the most obvious instance that ensures $P(X|X > x_\tau) > \overline{E}(X)$, but not the only one. By Proposition 4, a), it can be $\tau < F_X(x_\tau)$ also when F_X is not right-continuous (while being left-discontinuous). Similarly, there are other cases when $P(X|X > x_\tau) = \overline{E}(X)$, because $\tau = F_X(x_\tau)$, apart from case i), which remains the most important one. And it is also possible that

iii) $P(X|X > x_\tau) < \overline{E}(X)$.

Obviously, case iii) cannot occur when P is σ-additive, since it is equivalent to $\tau > F_X(x_\tau)$, hence $\tau \in]F_X(x_\tau); F_X(x_\tau^+)] = I^>$ and $I^> \neq \varnothing$ iff $P_{x_\tau}^+ > 0$.

When $P(X|X > x_\tau) > \overline{E}(X)$, then $P(X|X > x_\tau)$ is clearly not a coherent extension to X of $\overline{P}$ in the PMM, while it is so when it coincides with $\overline{E}(X)$.

3.1 Comparison with a naive extension

In actuarial applications the upper probability $\overline{P}(A)$ in (1) is the price, determined by increasing P by a loading $\delta > 0$, of an insurance policy which pays 1 unit if and only if event A occurs. In analogy with (1), one could set the price of an insurance policy which refunds x units iff the loss $X = x$ occurs, to $(1+\delta)P(X)$, up to a maximum of $\sup X$. Here $P(X)$ is the expectation of X computed from P. This procedure defines the *naive extension*:

$$\overline{P}_N(X) = \min\{(1+\delta)P(X), \sup X\}.$$

This extension, without the upper bound $\sup X$ (which is however necessary for $\overline{P}_N$ to be coherent), is referred to as *expected value principle* in risk theory literature [5, p. 67]. To fix the framework, suppose (throughout this section only) that P is defined on the field $2^{\mathbb{P}_u}$, and that we are interested in extending it to some set $\mathcal{D}$ strictly contained in the cone $\mathcal{L}^+(\mathbb{P}_u)$ of the non-negative gambles in $\mathcal{L}(\mathbb{P}_u)$. The gambles in $\mathcal{D}$ are non-negative, being refunds to the insured: hence $\inf X \geq 0$, $\forall X \in \mathcal{D}$.

The inclusion $\mathcal{D} \subsetneqq \mathcal{L}^+(\mathbb{P}_u)$ is strict because $\overline{P}_N$ cannot in general be coherent on a set $\mathcal{D}$ containing X, $X + k$, when $k \in \mathbb{R}^+$ is large enough. For instance, if $\overline{P}_N(X) = (1+\delta)P(X) < \sup X$, then $\overline{P}_N(X+k) = \sup X + k > \overline{P}_N(X) + k$ for $k \geq \frac{\sup X - (1+\delta)P(X)}{\delta}$, violating property (c) in [11], Section 2.6.1, which is a necessary condition for coherence.

But even when $\mathcal{D} = \{X\}$, $\overline{P}_N$ may be incoherent with the PMM:

Example 1. Take $\mathbb{P}_u = \{e_0, e_1, e_2, e_3\}$, and let $X(e_i) = i, i = 0, \ldots, 3$, $P(X = 0) = 0$, $P(X = 1) = 0.1$, $P(X = 2) = 0.5$, $P(X = 3) = 0.4$ and $\delta = 1/10$.

Then $P(X) = 2.3$ and hence $\overline{P}_N(X) = 2.53$. Let us now compute the natural extension in X. We have that $\tau = \frac{\delta}{1+\delta} = 1/11$, hence $x_\tau = 1$, as can be checked using F_X. Applying (6), $\overline{E}(X) = 1 + \frac{11}{10}P(\max\{X-1, 0\}) = 1 + \frac{11}{10}1.3 = 2.43$. □

In Example 1, $\overline{P}_N(X) > \overline{E}(X)$. This is interesting because the natural extension is shown to lead to a price smaller than would be expected from the intuition at the basis of the PMM and also because $\overline{P}_N$ is incoherent with the PMM, being larger than $\overline{E}$.

The dominance relationship between $\overline{P}_N$ and $\overline{E}$ is the object of the following proposition.

Proposition 6. *a) If $(X > x_\tau) = \emptyset$ then $\overline{P}_N(X) \leq \overline{E}(X)$.*

b) If either $(X \leq x_\tau) = \emptyset$ or $\overline{P}_N(X) = \sup X$, then $\overline{P}_N(X) \geq \overline{E}(X)$.

Suppose now $(X > x_\tau) \neq \emptyset$, $(X \leq x_\tau) \neq \emptyset$, $\overline{P}_N(X) < \sup X$, and let ε be given by (8).

c) If $\varepsilon = 0$, then $\overline{P}_N(X) \geq \overline{E}(X)$ and $\overline{P}_N(X) = \overline{E}(X)$ iff $P(X|X \leq x_\tau) = 0$.

d) If $\varepsilon \neq 0$ and $x_\tau = 0$, then $\overline{P}_N(X) = \overline{E}(X)$.

e) If $\varepsilon \neq 0$ and $x_\tau > 0$, condition $F_X(x_\tau) < \tau$ implies $\overline{P}_N(X) > \overline{E}(X)$, while condition $F_X(x_\tau) > \tau$ is necessary, but not sufficient, to ensure $\overline{P}_N(X) \leq \overline{E}(X)$.

Proof. a) follows from Corollary 1. To prove the non-trivial implication in b), put (Proposition 4, c)) $x_\tau = \inf X$ in (6), to get $\overline{E}(X) = \inf X + (1+\delta)P(X - \inf X) = (1+\delta)P(X) - \delta \inf X \leq \min\{(1+\delta)P(X), \sup X\} = \overline{P}_N(X)$.

To prove c), write $(1+\delta)P(X) = P(X|X > x_\tau)(1+\delta)P(X > x_\tau) + (1+\delta)P(X|X \leq x_\tau)P(X \leq x_\tau) = P(X|X > x_\tau)(1-\varepsilon) + (1+\delta)P(X|X \leq x_\tau)(1 - P(X > x_\tau)) = P(X|X > x_\tau)(1-\varepsilon) + (1+\delta)P(X|X \leq x_\tau) - P(X|X \leq x_\tau)(1-\varepsilon) = (1-\varepsilon)P(X|X > x_\tau) + (\delta+\varepsilon)P(X|X \leq x_\tau)$. From here

$$\overline{P}_N(X) = \min\{(1-\varepsilon)P(X|X > x_\tau) + (\delta+\varepsilon)P(X|X \leq x_\tau), \sup X\}.$$

Comparing this equality and (8),

$$\overline{P}_N(X) \gtreqless \overline{E}(X) \text{ iff } (\delta+\varepsilon)P(X|X \leq x_\tau) \gtreqless \varepsilon x_\tau. \quad (10)$$

When $\varepsilon = 0$, c) follows directly from (10).

To prove the remaining cases, we write the right-hand side inequality in (10) in a different form. Since $\delta+\varepsilon = (1+\delta)P(X \leq x_\tau)$ and $\varepsilon x_\tau = ((1+\delta) - (1+\delta)P(X > x_\tau) - \delta)x_\tau = ((1+\delta)P(X \leq x_\tau) - \delta)x_\tau$, using also (5) we get $\overline{P}_N(X) \gtreqless \overline{E}(X)$ iff $P(XI_{X \leq x_\tau}) \gtreqless (P(X \leq x_\tau) - \frac{\delta}{1+\delta})x_\tau$, or equivalently

$$\overline{P}_N(X) \gtreqless \overline{E}(X) \text{ iff } P(XI_{X \leq x_\tau}) \gtreqless (F_X(x_\tau) - \tau)x_\tau.$$

From here and Proposition 4 d), parts d) and e) follow at once (for d), recall that $x_\tau = 0$ implies $P(XI_{X \leq x_\tau}) = 0$). □

It appears from Proposition 6 that $\overline{P}_N$ is only occasionally equal to $\overline{E}$, and may easily be incoherent. Cases a), b), d) treat really extreme situations, while in the common case that F_X is continuous at x_τ, c) ensures that $\overline{P}_N$ is incoherent, unless the limiting evaluation $P(X|X \leq x_\tau) = 0$ applies. Case e) shows that $\overline{P}_N$ can possibly be coherent when $F_X(x_\tau) > \tau$. The most important practical case concerns discrete gambles (with finitely many possible values). However, it should be checked even then whether $\overline{P}_N \leq \overline{E}$, and this makes the use of $\overline{P}_N$ less convenient. For instance, $\overline{P}_N > \overline{E}$ in Example 1.

3.2 A generalization

We shall derive here $\overline{E}$ in the more general framework of Proposition 3, that $\overline{P}$ is defined by the PMM on S^+ and $\overline{E}$ on $\mathcal{L}(\mathbb{P}_u)$. We first obtain an expression for $\overline{E}(B)$, for any event B in $2^{\mathbb{P}_u}$.

Proposition 7. *In the PMM, the natural extension of $\overline{P}: S^+ \to \mathbb{R}$ on $2^{\mathbb{P}_u}$ is*

$$\overline{E}(B) = \min\{(1+\delta)\tilde{P}^*(B), 1\}, \quad (11)$$

where the upper probability $\tilde{P}^*(B) = \inf\{P(A) : A \in S^+, B \Rightarrow A\}$ is the outer function of P.

Proof. By Proposition 3 (a), $\overline{E}(B) = \overline{P}^*(B) = \inf\{\min\{(1+\delta)P(A), 1\} : A \in S^+, B \Rightarrow A\}$. Defining $L_B = \{A \in S^+ : B \Rightarrow A, (1+\delta)P(A) < 1\}$, $L_B = \varnothing$ iff $(1+\delta)\tilde{P}^*(B) \geq 1$.

Two cases may occur: if $L_B = \varnothing$, that is if $(1+\delta)\tilde{P}^*(B) \geq 1$, then $\overline{E}(B) = 1$; if $L_B \neq \varnothing$, that is if $(1+\delta)\tilde{P}^*(B) < 1$, $\overline{E}(B) = \inf\{(1+\delta)P(A) : A \in L_B\} = (1+\delta)\inf\{P(A) : A \in L_B\} = (1+\delta)\tilde{P}^*(B)$. In summary, equation (11) holds. □

We emphasize that $\tilde{P}^*$ in (11) is generally not a precise, but an upper probability. In fact, by Proposition 3 (a), it coincides with the natural extension $\overline{E}_P$ on $2^{\mathbb{P}_u}$ of the probability P, when P is interpreted as a special upper probability.

Proposition 8. *In the PMM, the natural extension of $\overline{P}: S^+ \to \mathbb{R}$ on $\mathcal{L}(\mathbb{P}_u)$ is:*

$$\overline{E}(X) = x_\tau^u + (1+\delta)\overline{E}_P((X - x_\tau)^+) \quad (12)$$

where $\overline{E}_P$ is the natural extension of P (also of $\tilde{P}^*$) on $\mathcal{L}$, and x_τ^u is the (upper) quantile relative to $\tilde{P}^*$

$$x_\tau^u = \sup\{x \in \mathbb{R} : \tilde{P}^*(X \leq x) \leq \tau\}. \quad (13)$$

Proof. Apply (4) and Proposition 3, (a) substituting $\overline{P}^* = \overline{E}$ with its expression in equation (11), getting

$$\overline{E}(X) = \inf X + \int_{\inf X}^{\sup X} \min\{(1+\delta)\tilde{P}^*(X > x), 1\}dx.$$

From here, the derivation of (12) is identical to that sketched in [11], Section 3.2.5, to obtain (6). In fact, $\tilde{P}^*$ is defined on the field $2^{\mathbb{P}_u}$, and every $X \in \mathcal{L}$ is measurable with respect to such a field. □

Clearly, (12) generalizes (6). We might summarize the difference between the natural extension in (12) and that in (6) as follows: computing the natural extension of $\overline{P}$ on gambles which are not necessarily measurable with respect to the domain of $\overline{P}$ introduces imprecision by transforming the precise prevision $P((X - x_\tau)^+)$ in (6) into the upper prevision $\overline{E}_P((X - x_\tau^u)^+)$ in (12). Also the quantile x_τ refers to probability P in (7), while x_τ^u employs the upper probability $\tilde{P}^*$ in (13).

But there is another attractive interpretation: $\overline{E}(B)$ in (11) can be viewed as a kind of *imprecise PMM*, defined via natural extension on $2^{\mathbb{P}_u}$ starting from a (precise) PMM on a narrower set S^+: then (12) describes the natural extension of this imprecise model.

Some properties of the natural extension of the PMM generalize to the natural extension of the imprecise PMM. The following proposition relaxes (8):

Proposition 9. *If $(X > x_\tau^u) \neq \emptyset$, it holds for the natural extension $\overline{E}$ on $\mathcal{L}(\mathbb{P}_u)$ of $\overline{P} : S^+ \to \mathbb{R}$ that*

$$\overline{E}(X) \leq \varepsilon^u x_\tau^u + (1 - \varepsilon^u)\overline{E}_P(X|X > x_\tau^u) \qquad (14)$$

where $\varepsilon^u \stackrel{\text{def}}{=} 1 - (1+\delta)\overline{E}_P(X > x_\tau^u)$.

Proof. Noting that $(X - x_\tau^u)^+ = (X - x_\tau^u)I_{X > x_\tau^u}$ and by subadditivity of coherent upper previsions and, at the second equality, the GBR,[3] $\overline{E}_P((X - x_\tau^u)^+) = \overline{E}_P((X - x_\tau^u)I_{X>x_\tau^u}) \leq \overline{E}_P(I_{X>x_\tau^u}(X - \overline{E}_P(X|X > x_\tau^u))) + \overline{E}_P(I_{X>x_\tau^u}(\overline{E}_P(X|X > x_\tau^u) - x_\tau^u)) = \overline{E}_P(I_{X>x_\tau^u}(\overline{E}_P(X|X > x_\tau^u) - x_\tau^u)) \stackrel{\text{def}}{=} \lambda$.

Using also the definition of ε^u and λ, we get further $x_\tau^u + (1+\delta)\lambda = x_\tau^u(1 - (1+\delta)\overline{E}_P(X > x_\tau^u)) + (1+\delta)(\lambda + x_\tau^u\overline{E}_P(X > x_\tau^u)) = \varepsilon^u x_\tau^u + (1+\delta)(\overline{E}_P(X > x_\tau^u)(\overline{E}_P(X|X > x_\tau^u) - x_\tau^u) + x_\tau^u\overline{E}_P(X > x_\tau^u)) = \varepsilon^u x_\tau^u + (1 - \varepsilon^u)\overline{E}_P(X|X > x_\tau^u)$.

Finally, by (12) and the expressions above, $\overline{E}(X) = x_\tau^u + (1+\delta)\overline{E}_P((X - x_\tau^u)^+) \leq x_\tau^u + (1+\delta)\lambda = \varepsilon^u x_\tau^u + (1 - \varepsilon^u)\overline{E}_P(X|X > x_\tau^u)$. □

Although the inequality in (14) can be strict (we omit proving this), when $\overline{P}$ is defined on $2^{\mathbb{P}_u}$ then $\overline{E}_P$ is equal to P (or to its extension using (5)), and x_τ^u, ε^u to x_τ, ε respectively. Thus (14) reduces to (8).

The statement corresponding to Proposition 4 d) is $\varepsilon^u \gtreqless 0$ iff $\overline{E}_P(X > x_\tau^u) \lesseqgtr \frac{1}{1+\delta}$, or also $\varepsilon^u \gtreqless 0$ iff $\underline{E}_P(X \leq x_\tau^u) \gtreqless \tau$.

We know that $\varepsilon = 0$ when F_X is continuous at x_τ. When $\overline{F}_X(x) = \overline{E}_P(X \leq x)$ is continuous at x_τ^u, then $\overline{F}_X(x_\tau^u) = \tau$. Hence $\underline{F}_X(x_\tau^u) = \underline{E}_P(X \leq x_\tau^u) \leq \overline{F}_X(x_\tau^u) = \tau$. In terms of ε^u, as seen above, this means that $\varepsilon^u \leq 0$, with $\varepsilon^u = 0$ only when $\underline{F}_X(x_\tau^u) = \overline{F}_X(x_\tau^u)$, a condition obviously warranted when $\underline{F}_X = \overline{F}_X = F_X$. Thus continuity at x_τ^u of $\overline{F}_X$ implies $\varepsilon^u \leq 0$, typically $\varepsilon^u < 0$.

4 Risk measurement interpretations

If Y is a gamble, it is known [6] that $\overline{P}(-Y)$ may be interpreted as a *risk measure* for Y, i.e. a number measuring how risky Y is, or also the amount of money to be reserved to cover potential losses from Y. Several risk measures were introduced in the literature, and there is often no unanimity on the terminology. To ensure comparisons with [4], we shall refer the risk measure to $X = -Y$; this corresponds, when $Y \leq 0$, to thinking in terms of losses and is frequently done in insurance, where X represents the amount to be paid for insurance claims (however, X is not necessarily non-negative in what follows).[4] Thus the upper previsions $\overline{E}(X)$ in (6), (8) and (12) may be seen as risk measures for X, and there is a strong correspondence with measures studied in the literature.

Consider equation (6): x_τ is the *Value-at-Risk* of X at level τ, $VaR_\tau(X)$, while $P((X - x_\tau)^+)$ is the *expected shortfall* $ES_\tau(X)$ (whenever P is replaced by or thought of as an expectation) [4]. In fact, $(X - x_\tau)^+$ measures the shortfall, i.e. the residual loss in absolute value of an agent who reserves an amount of money equal to $VaR_\tau(X) = x_\tau$ to cover losses from X. Also $P(X|X > x_\tau)$ corresponds to a well-known risk measure (when P is an expectation), termed *Conditional Tail Expectation* (CTE_τ) in [4].

Equation (6) corresponds to (2.7) in [4], which defines another measure of risk, $TailVaR_\tau(X)$ or $TVaR_\tau(X)$. This equation is identical to (6), after replacing $\overline{E}$, x_τ, $P((X - x_\tau)^+)$ with, respectively, $TVaR_\tau(X)$, $VaR_\tau(X)$, $ES_\tau(X)$:

$$TVaR_\tau(X) = VaR_\tau(X) + (1+\delta)ES_\tau(X).$$

[3]Recall also that the natural extension $\overline{E}_P$ always exists with W-coherence, cf. [7].

[4]While ensuring compatibility with the prevailing literature and the formulae in [11], the convention of referring to losses modifies the range of the typical values for τ. In this section τ should be fairly close to 1, representing the probability that the loss is not too high, while in the rest of the paper should rather be close to 0, being a taxation or commission.

Analogously, equation (8) corresponds to

$$TVaR_\tau(X) = (1-\varepsilon)CTE_\tau(X) + \varepsilon VaR_\tau(X). \quad (15)$$

The novel fact in our approach (apart from using previsions instead of expectations) is that $TVaR_\tau$ is derived as the natural extension of the PMM, while the starting point in the literature for defining this or other measures is usually a set of random variables, often a linear space equipped with a σ-additive probability measure, using which the various expectations are computed. Recalling also Proposition 3, we deduce the following properties for $TVaR_\tau$:

Proposition 10. *$TVaR_\tau(X)$ is the natural extension on $\mathcal{L}(\mathbb{P}_u)$ of the PMM defined on $2^{\mathbb{P}_u}$. Hence, it is the least-committal risk measure extending the PMM which is coherent. Actually, it is its* only *coherent extension which is 2-alternating.*

CTE_τ complements VaR_τ, in the sense that VaR_τ, unlike CTE_τ, is nearly uninformative about what are the losses, should the threshold x_τ be exceeded. Unfortunately, neither VaR_τ nor CTE_τ is generally coherent, even though their linear combination in (15) originates a coherent risk measure. Conditions for coherence of CTE_τ are discussed in Section 3, and are commoner in practice than those ensuring coherence of VaR_τ.[5] In the classical risk measurement approach using a σ-additive probability, the comparison between CTE_τ and $TVaR_\tau$ is limited to cases i), ii) in Section 3 which, as we pointed out there, are not exhaustive in general.

The generalization in Section 3.2 forms a basis for further considerations on the risk measurement side. This time, $\overline{E}(X)$ in (12) is the natural extension of the PMM defined on $S^+(\subset 2^{\mathbb{P}_u})$, and may again be interpreted as a risk measure, let us name it *Imprecise TailVar* or $ITVaR_\tau$. Using Proposition 3, $ITVaR_\tau$ is coherent and also 2-alternating. However, $ITVaR_\tau$ has no analogue in the risk measurement literature. The reason lies in the standard way of defining risk measures from an underlying precise probability, which rules out potentially interesting risk measures which are functions of imprecise measures. And looking at (12), we notice that $ITVaR_\tau$ is a linear combination of other two measures which are imprecise versions of VaR_τ and ES_τ: x^u_τ is defined in (13) as a function of the upper probability $\tilde{P}^*$, the shortfall $(X - x^u_\tau)^+$ is evaluated by the upper prevision $\overline{E}_P$. We may conclude that the PMM provides a formal justification for the existence of a new, and still largely not investigated, kind of risk measures, those defined in terms of imprecise uncertainty measures.

[5] For VaR_τ, see the discussion in [6].

5 Conditioning the pari-mutuel model

Reconsider the basic PMM, with $\overline{P}(A)$, $\underline{P}(A)$ given by (1), (2), $A \in \mathcal{D}$, and $\mathcal{D}$ is now a *field* of events. We shall compute the natural extensions $\overline{E}(A|B)$, $\underline{E}(A|B)$ of $\underline{P}$ and $\overline{P}$ on $A|B$, with $B \in \mathcal{D}$, $B \neq \emptyset$. Since $\overline{P}$ and $\underline{P}$ are, respectively, 2-alternating and 2-monotone, from a well-known result ([10], Thm. 7.2; see also [8]), when $\underline{P}(B) > 0$:

$$\overline{E}(A|B) = \frac{\overline{P}(A\wedge B)}{\overline{P}(A\wedge B) + \underline{P}(A^c\wedge B)}, \qquad \underline{E}(A|B) = \frac{\underline{P}(A\wedge B)}{\underline{P}(A\wedge B) + \overline{P}(A^c\wedge B)}. \quad (16)$$

When $\underline{P}(B) = 0$, equations (16) do not apply, but it can be shown (directly, using Williams coherence, or alternatively from results in [11]) that

Lemma 1. *Given a coherent lower probability $\underline{P}$ on a set $\mathcal{D}$ of (unconditional) events, let $B \in \mathcal{D}$, $\underline{P}(B) = 0$. The natural extension $\underline{E}$ of $\underline{P}$ on $\mathcal{D} \cup \{A_1|B, \dots, A_n|B\}$ is $\underline{E}(A_i|B) = 1$ if $B \Rightarrow A_i$, $\underline{E}(A_i|B) = 0$ otherwise, for $i = 1, \dots, n$.*

Applying Lemma 1 for $n = 2$, $A_1 = A$, $A_2 = A^c$ and using conjugacy, it follows that, when $\underline{P}(B) = 0$ in the PMM, then $\underline{E}(A|B) = 0$, $\forall A$ such that $B \not\Rightarrow A$, and $\overline{E}(A|B) = 1$, $\forall A$ such that $A \wedge B \neq \emptyset$.

We assume in the sequel $\underline{P}(B) > 0$; note that by (2) $\underline{P}(B) > 0$ iff $P(B) > \frac{\delta}{\delta+1} = \tau$. Further, $\underline{P}(B) > 0$ ensures that the denominators in (16) are non-zero. Take $\overline{E}(A|B)$: using property 2.7.4 (d) in [11], $\overline{P}(A \wedge B) + \underline{P}(A^c \wedge B) \geq \underline{P}(B) > 0$. Similarly for $\underline{E}(A|B)$.

To derive $\overline{E}(A|B)$, from (16), two alternatives occur:

a) $\underline{P}(A^c \wedge B) = \max\{(1+\delta)P(A^c \wedge B) - \delta, 0\} = 0$. Hence $\overline{E}(A|B) = 1$.

b) $\max\{(1+\delta)P(A^c \wedge B) - \delta, 0\} > 0$. This happens iff $P(A^c \wedge B) > \frac{\delta}{1+\delta} = \tau$ and implies $\min\{(1+\delta)P(A \wedge B), 1\} < 1$ (otherwise $P(A \wedge B) \geq \frac{1}{1+\delta}$ and $P(B) > \frac{\delta}{\delta+1} + \frac{1}{1+\delta} = 1$). Hence $\overline{E}(A|B) = \frac{(1+\delta)P(A\wedge B)}{(1+\delta)(P(A\wedge B)+P(A^c\wedge B))-\delta} = \frac{P(A\wedge B)}{P(B)-\tau}$.

The derivation of $\underline{E}(A|B)$ is analogous:

a) If $\underline{P}(A \wedge B) = \max\{(1+\delta)P(A \wedge B) - \delta, 0\} = 0$, $\underline{E}(A|B) = 0$.

b) If $\max\{(1+\delta)P(A \wedge B) - \delta, 0\} > 0$, this implies $\tau < P(A \wedge B)$ and $\min\{(1+\delta)P(A^c \wedge B), 1\} < 1$; then $\underline{E}(A|B) = \frac{P(A\wedge B)-\tau}{P(B)-\tau}$.

$$
\begin{array}{lcll}
\overline{P}(A) & = & \begin{cases} \frac{P(A)}{1-\tau} & \text{if} \quad \tau < P(A^c) \\ 1 & \text{if} \quad \tau \geq P(A^c) \end{cases} \\
\underline{P}(A) & = & \begin{cases} \frac{P(A)-\tau}{1-\tau} & \text{if} \quad \tau < P(A) \\ 0 & \text{if} \quad \tau \geq P(A) \end{cases} \\
\overline{E}(A|B) & = & \begin{cases} \frac{P(A\wedge B)}{P(B)-\tau} & \text{if} \quad \tau < P(A^c \wedge B) \\ 1 & \text{if} \quad \tau \geq P(A^c \wedge B) \end{cases} \\
\underline{E}(A|B) & = & \begin{cases} \frac{P(A\wedge B)-\tau}{P(B)-\tau} & \text{if} \quad \tau < P(A \wedge B) \\ 0 & \text{if} \quad \tau \geq P(A \wedge B) \end{cases}
\end{array}
$$

Table 1: Values of $\overline{P}(A)$, $\underline{P}(A)$, $\overline{E}(A|B)$, $\underline{E}(A|B)$.

Table 1 lists the values of $\overline{P}(A)$, $\underline{P}(A)$, $\overline{E}(A|B)$, $\underline{E}(A|B)$. They are written as functions of τ, to simplify the inequalities in the 'if' clauses (referring to δ, the clauses involve ratios of probabilities instead of probabilities). Note that the expressions for $\overline{E}(A|B)$, $\underline{E}(A|B)$ reduce to those for $\overline{P}(A)$, $\underline{P}(A)$ when $B = \Omega$.

5.1 Dilation and imprecision increase

How does imprecision in the evaluations vary when conditioning in the PMM model? To supply some answers, we first recall two concepts.

Definition 1. Given a partition of non-impossible events $\mathbb{P}$, we say that *(weak) dilation* occurs (with respect to A and $\mathbb{P}$) when

$$\underline{P}(A|B) \leq \underline{P}(A) \leq \overline{P}(A) \leq \overline{P}(A|B), \forall B \in \mathbb{P}, \quad (17)$$

while there is an *imprecision increase* when

$$\overline{P}(A) - \underline{P}(A) \leq \overline{P}(A|B) - \underline{P}(A|B), \forall B \in \mathbb{P}. \quad (18)$$

Dilation is a so far little investigated phenomenon (see [9]), which implies that our a posteriori opinions on A will be *vaguer* and hence also *more imprecise* (at least in a weak sense, if the first or last weak inequalities in (17) are equalities) than the a priori ones, *no matter* which $B \in \mathbb{P}$ is true. Even though dilation is $\mathbb{P}$-dependent (so that we may hope that a well-chosen partition $\mathbb{P}$ avoids dilation), it is a puzzling phenomenon. Clearly, dilation implies the weaker concept of imprecision increase, which captures one of the two basic features of dilation, the growth in the degree of imprecision.

To discuss the occurrence of dilation or of imprecision increase in the PMM, we assume that $\mathbb{P} = \{B, B^c\}$ and the conditional probabilities are the natural extensions. The formulas for $\overline{E}(A|B^c)$, $\underline{E}(A|B^c)$ are obtained from those for $\overline{E}(A|B)$, $\underline{E}(A|B)$ in Table 1 (when $\tau < P(B^c)$) replacing B with B^c.

We present now a number of results, whose operational relevance is discussed in Section 5.2.

Notation We write A' to denote, indifferently, either A or A^c. For instance, $\min\{P(A' \wedge B')\}$ is a short form for $\min\{P(A \wedge B), P(A^c \wedge B), P(A \wedge B^c), P(A^c \wedge B^c)\}$.

Proposition 11. *Each of the following conditions is necessary for dilation (of A, relative to $\{B, B^c\}$), whenever the denominator is positive:*

$$\tau < P(A \wedge B) \quad \Rightarrow \tau \geq \frac{P(A\wedge B)-P(A)P(B)}{P(A^c\wedge B^c)} \quad (19)$$

$$\tau < P(A^c \wedge B) \quad \Rightarrow \tau \geq \frac{P(A)P(B)-P(A\wedge B)}{P(A\wedge B^c)} \quad (20)$$

$$\tau < P(A \wedge B^c) \quad \Rightarrow \tau \geq \frac{P(A\wedge B^c)-P(A)P(B^c)}{P(A^c\wedge B)} \quad (21)$$

$$\tau < P(A^c \wedge B^c) \quad \Rightarrow \tau \geq \frac{P(A)P(B^c)-P(A\wedge B^c)}{P(A\wedge B)} \quad (22)$$

Proof. Impose either $\underline{E}(A|B') \leq \underline{P}(A)$ or $\overline{E}(A|B') \geq \overline{P}(A)$ in (17), and use Table 1 to choose the appropriate values of $\underline{E}$, $\overline{E}$, $\underline{P}$, $\overline{P}$.

To exemplify, Equation (19) implements the condition $\underline{E}(A|B) \leq \underline{P}(A)$, which is written as $\frac{P(A\wedge B)-\tau}{P(B)-\tau} \leq \frac{P(A)-\tau}{1-\tau}$. Multiply by $(P(B)-\tau)(1-\tau) > 0$ and solve the ensuing linear inequality in τ to get (19). □

Proposition 12. *Define* $m = \min\{P(A' \wedge B')\}$, $M = \max\{P(A' \wedge B')\}$, $M_\tau = \max\{(P(A \wedge B) - P(A)P(B))/P(A^c \wedge B^c), (P(A)P(B) - P(A \wedge B))/P(A \wedge B^c), (P(A \wedge B^c) - P(A)P(B^c))/P(A^c \wedge B), (P(A)P(B^c) - P(A \wedge B^c))/P(A \wedge B)\}$

(a) If $\tau < m$, dilation occurs if and only if $\tau \geq M_\tau$.

(b) The condition $\tau \geq M$ is sufficient for dilation.

Proof. (a): when $\tau < m = \min\{P(A' \wedge B')\}$, (17) holds iff τ satisfies the weak inequalities in (19÷22) i.e. iff $\tau \geq M_\tau$.

(b): when $\tau \geq M$, $\underline{E}(A|B') = 0$ and $\overline{E}(A|B') = 1$,[6] so dilation occurs no matter what are $\underline{P}(A)$, $\overline{P}(A)$. □

Remark 2. At most two of the four weak inequalities in (19÷22) need to be checked. In fact, let A and B be *positively correlated under* P, hence A and B^c are negatively correlated and $P(A)P(B) - P(A \wedge B) < 0$, $P(A \wedge B^c) - P(A)P(B^c) < 0$. Thus, (20) and (21) trivially hold ($\tau > 0$) and $M_\tau = \max\left\{\frac{P(A\wedge B)-P(A)P(B)}{P(A^c\wedge B^c)}, \frac{P(A)P(B^c)-P(A\wedge B^c)}{P(A\wedge B)}\right\}$. Similarly, (19) and (22) trivially hold when A and B are *negatively correlated* under P. □

Let us point out some special instances of dilation.

[6]This ensues from Table 1 when $\tau \leq \min P(B')$, if not use also Lemma 1.

Corollary 2. *Dilation occurs if:*

(a) $P(A' \wedge B') = P(A')P(B')$ *and* $\tau < m$.

(b) P *is* uniform *on* $\mathbb{P}_{A,B} = \{A \wedge B, A \wedge B^c, A^c \wedge B, A^c \wedge B^c\}$, $\forall \tau \in]0, 1[$.

Proof. Condition (a) ensures dilation, as it implies $M_\tau = 0$ and hence (a) of Proposition 12. As for (b), it implies $P(A' \wedge B') = P(A')P(B')$ and $m = M = 0.25$: hence dilation occurs by (a) when $\tau < M = m$, by Proposition 12, (b) when $\tau \geq M$. □

Concerning imprecision increase, it holds that

Proposition 13. *Imprecision increases, i.e., Equation* (18) *holds, if the following system holds for* τ*:*

$$\begin{cases} (\tau - P(A \wedge B))(\tau - P(A^c \wedge B)) > 0 \\ (\tau - P(A \wedge B^c))(\tau - P(A^c \wedge B^c)) > 0 \end{cases} \tag{23}$$

Proof. Check that (18) holds, using Table 1. □

Remark 3. Note that (23) holds in particular when $\tau < m = \min\{P(A' \wedge B')\}$. Therefore imprecision *always increases* in this case. □

5.2 Imprecision variation in practice

As a general remark, the existence and relevance of dilation and imprecision increase in the PMM should be investigated distinguishing more cases, according to the relative ordering of $P(A' \wedge B')$, $P(A')$, and τ in $[0, 1]$. However, the importance of each case varies greatly in the applications. We present in detail the most significant ones, while the remaining may be analyzed using Table 1 and the preceding results to check (17) and (18), as demonstrated in Example 3.

Case i) $\tau < m = \min\{P(A' \wedge B')\}$;

Case ii) $P(A) \leq \tau < \min\{P(A^c \wedge B')\}$.

Case i) is probably the most important: τ will often be rather low, recalling that it has the meaning of a commission or taxation (this happens for instance with Internet betting). In such circumstances case i) applies if none among $P(A' \wedge B')$ is too low.

Case i) is completely solved by the results in Section 5.1: dilation occurs iff $\tau \geq M_\tau$ (Proposition 12, (a)), imprecision always increases (Remark 3).

We do not necessarily meet case i) when A is a rare event, or $P(A)$ is anyway smaller than the commission τ in favour of House or of an insurer (these cases are relatively frequent in non-life insurance). If τ is also smaller than $\min\{P(A^c \wedge B')\}$, *case ii)* occurs. We discuss it in the next example.

Example 2. When $P(A) \leq \tau < \min\{P(A^c \wedge B')\}$, then (see Table 1) $\overline{P}(A) = P(A)/(1 - \tau)$, $\overline{E}(A|B) = P(A \wedge B)/(P(B) - \tau)$, $\overline{E}(A|B^c) = P(A \wedge B^c)/(P(B^c) - \tau)$, $\underline{E}(A|B') = \underline{P}(A) = 0$. Imposing either (17) or (18) originates the same system of inequalities, i.e. *in this case* there is dilation iff there is imprecision increase. The system is

$$\begin{cases} \frac{P(A \wedge B)}{P(B) - \tau} \geq \frac{P(A)}{1 - \tau} \\ \frac{P(A \wedge B^c)}{P(B^c) - \tau} \geq \frac{P(A)}{1 - \tau} \end{cases} \tag{24}$$

and its inequalities are easily seen to be equivalent to (20) and (22). Thus dilation arises iff both (20) and (22) hold (and the lower bound they supply for τ is not greater than $\min\{P(A^c \wedge B')\}$). In practice, only one of them (at most) has to be checked, depending on the correlation of A and B, by Remark 2. For instance, if $P(A) = 0.02$, $P(A \wedge B) = 0.005$, $P(A \wedge B^c) = 0.015$, $P(B) = 0.4$, then $P(A|B) = 0.0125 < P(A)$ and (20) gives the bound $\tau \geq 0.2$. Since $\min\{P(A^c \wedge B')\} = 0.395 > 0.2$, the bound is effective: there is dilation (and imprecision increase) for $\tau \in [0.2; 0.395]$, none of them for $\tau \in [0.02; 0.2[$. □

Discussion We point out that dilation occurs in both case i) and ii) when A' and B' are judged *stochastically independent* or at least *not correlated* by P (as follows from Corollary 2 (a) and Example 2).

Further, dilation occurs when τ is too "large": Proposition 12 (b) ensures it when $\tau \geq M = \max\{P(A' \wedge B')\}$. This happens merely because $\overline{E}$, $\underline{E}$ are then vague, but dilation may occur also when $\tau < M$, as in the next example. □

Example 3. Assign P on $\mathbb{P}_{A,B}$ as follows: $P(A \wedge B) = \frac{1}{10}$, $P(A \wedge B^c) = P(A^c \wedge B^c) = \frac{2}{10}$,$P(A^c \wedge B) = \frac{1}{2}$. Consequently $P(A) = \frac{3}{10}$, $P(B) = \frac{6}{10}$, $P(A|B) = \frac{1}{6}$, $P(A|B^c) = \frac{1}{2}$.

Dilation occurs when $\tau \geq \frac{1}{2}$, by Proposition 12, (b). When $\tau < \frac{1}{2} = P(A^c \wedge B)$, use the necessary condition in (20), which requires that $\tau \geq \frac{4}{10}$, to rule out dilation for $\tau \in]0; \frac{4}{10}[$. If $\tau \in [\frac{4}{10}; \frac{1}{2}[$, (20) ensures that $\overline{E}(A|B) \geq \overline{P}(A)$, and the other inequalities in (17) hold too, because $\overline{E}(A|B^c) = 1$ and $\underline{E}(A|B') = 0$. Thus there is dilation for $\tau \in [\frac{4}{10}, \frac{1}{2}[$ too.

As for imprecision increase, it is ensured by Proposition 13 (Remark 3) when $\tau < \frac{1}{10}$. For $\tau \in [\frac{1}{10}; \frac{4}{10}[$, we have to check whether the inequalities (18) hold, distinguishing more subcases according to the different expressions for $\overline{P}$, $\underline{P}$, $\overline{E}(A|B')$, $\underline{E}(A|B')$. Conditioning on B^c, we should check whether

$$\overline{E}(A|B^c) - \underline{E}(A|B^c) \geq \overline{P}(A) - \underline{P}(A). \tag{25}$$

Now, $\overline{E}(A|B^c) - \underline{E}(A|B^c) = 1$ and (25) therefore holds if $\tau \in [\frac{2}{10}; \frac{4}{10}[$, while (25) specialises into $\frac{\tau}{P(B^c) - \tau} \geq$

$\frac{\tau}{1-\tau}$ when $\tau \in [\frac{1}{10}; \frac{2}{10}[$, and this inequality is true. Therefore (25) is verified for $\tau \in [\frac{1}{10}; \frac{4}{10}[$, and imprecision increase in this interval depends only on whether the inequality $\overline{E}(A|B) - \underline{E}(A|B) \geq \overline{P}(A) - \underline{P}(A)$ holds. Noting that $\overline{E}(A|B) - \underline{E}(A|B) = \frac{P(A \wedge B)}{P(B)-\tau} = \frac{1}{6-10\tau}$, $\forall \tau \in [\frac{1}{10}; \frac{4}{10}[$, we have to check whether:

$$\begin{array}{llll} \frac{1}{6-10\tau} & \geq & \frac{P(A)}{1-\tau} = \frac{3}{10(1-\tau)} & \text{if } \tau \in [\frac{3}{10}; \frac{4}{10}[\\ \frac{1}{6-10\tau} & \geq & \frac{\tau}{1-\tau} & \text{if } \tau \in [\frac{1}{10}; \frac{3}{10}[. \end{array}$$

The former inequality has no solution in $[\frac{3}{10}; \frac{4}{10}[$, the latter is true for $\tau \in [\frac{1}{10}; \frac{2}{10}]$. Conclusions: dilation occurs iff $\tau \in [\frac{4}{10}; 1[$, imprecision increase (without dilation) iff $\tau \in]0; \frac{2}{10}]$, neither of them iff $\tau \in]\frac{2}{10}; \frac{4}{10}[$. □

Limiting dilation or imprecision increase in the PMM is not straightforward. This may be achieved by an appropriate choice of τ in some, but not all cases (for instance, $\tau \in [\frac{2}{10}; \frac{4}{10}[$ might be too high a percentage in Example 3). More generally, choosing a coherent extension other than the natural extension often shrinks imprecision, by the dominance properties of the natural extension, but finding a computationally simple such extension may be not so easy in practice.

6 Conclusions

The pari-mutuel model represents a simple and natural way of eliciting upper/lower probabilities, and can be extended in more directions, thanks to the availability of standard procedures for 2-monotone and 2-alternating previsions. We computed explicitly its natural extension $\overline{E}$ starting from a PMM assignment on a lattice of events, generalizing the approach in [11], which is anyway discussed, focusing on comparing the different formulae available for $\overline{E}$. While a naive extension, considered in insurance premium pricing, does not seem to be a valuable alternative to the natural extension, being generally not coherent, the various formulae for the natural extension have a notable meaning in risk measurement. In fact, they correspond to known measures of risk or generalize them. We discussed also how to use the natural extension when conditioning, delimiting the influence of dilation and imprecision increase for the PMM.

A tempting new direction would, in a sense, merge our analysis in the conditional and unconditional framework, studying the natural extension to conditional gambles. Here a difficulty arises: available generalisations of equations (16), studied in [8], are lower/upper bounds for the natural extension and might not be reached, even when $\overline{P}$ is 2-alternating. In other words, the available procedures seem to give weaker results. This and the considerations at the end of Section 5.2 on how to limit dilation or imprecision increase might motivate investigating coherent extensions of the PMM alternative to the natural extension.

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References

[1] G. de Cooman, M. C. M. Troffaes and E. Miranda. n-Monotone lower previsions. *Journal of Intelligent & Fuzzy Systems*, 16: 253–263, 2005.

[2] B. de Finetti. *Theory of Probability*, volume 1, Wiley, 1974.

[3] D. Denneberg. *Non-Additive Measure and Integral.* Kluwer, 1994.

[4] M. Denuit, J. Dhaene, M. Goovaerts and R. Kaas. *Actuarial Theory for Dependent Risks: Measures, Orders and Models.* Wiley, 1994.

[5] H.U. Gerber. *An Introduction to Mathematical Risk Theory.* Huebner Foundation, 1979.

[6] R. Pelessoni and P. Vicig. Imprecise Previsions for Risk Measurement. *International Journal of Uncertainty, Fuzziness and Knowledge-based Systems*, 11: 393–412, 2003.

[7] R. Pelessoni and P. Vicig. Williams coherence and beyond. *International Journal of Approximate Reasoning*, 50(4):612–626, 2009.

[8] R. Pelessoni and P. Vicig. Bayes' theorem bounds for convex lower previsions. *Journal of Statistical Theory and Practice*, 3(1):85–101, 2009.

[9] T. Seidenfeld and L. Wasserman. Dilation for sets of probabilities. *The Annals of Statistics*, 21(3):1139–1154, 1993.

[10] P. Walley. Coherent lower (and upper) probabilities. *Research Report*, University of Warwick, Coventry, 1981.

[11] P. Walley. *Statistical Reasoning with Imprecise Probabilities.* Chapman and Hall, 1991.