A new integrated fuzzy bang–bang relay control system

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ABSTRACT

In this paper, we propose a new fuzzy bang–bang relay controller (FBBRC). The bang–bang control systems use switching relays or hard limiter saturation functions with fixed parameters, and their input does not have the flexibility to control the non-linear system over its entire range of operation. The conventional fuzzy bang–bang controllers have an analog output and an external hard limiting device to convert the output to bang–bang action. The new integrated FBBRC proposed here directly output two-level state. The inputs to the FBBRC are configured on standard fuzzy sets on the basis Mamdani implications. The largest of maxima defuzzification method is used for two-level state output. The stability and optimality of the FBBRC can be established with the Lyapunov stability criterion and Pontrygin minimum principle, respectively. Non-linear bang–bang control action is inherently time-optimal and endows this property to the FBBRC. Because of its design simplicity and cost-effectiveness, the bang–bang control is desired for control applications such as spacecraft–satellite attitude, heating controls, and on/off valve controls. Comparison between the proposed FBBRC and the fuzzy bang–bang controller (FBBC) show that FBBRC gives a better response. Finally, we demonstrate a practical application of FBBRC by controlling the angular position of a single-axis pneumatic rotary actuator in real-time, using the Matlab-Simulink xPC target environment.

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1. Introduction

Bang–bang or on/off action is the basis of simple and inexpensive control systems. Bang–bang control systems usually results when attempting to force continuous-time system from an initial state to its origin in minimum time, under a fixed magnitude control signal. Conventional bang–bang controllers are made from electromechanical relays that are getting obsolete owing to the fact that their parameters are fixed and act slowly. Solid-state relays are fast-acting but are not flexible to control non-linear systems over the entire operating range. The demand for flexible and programmable relays has grown in recent years. Artificial intelligence technique such as fuzzy logic has provided the means to develop flexible fuzzy bang–bang relays. One of the earliest fuzzy bang–bang controller (FBBC) was developed by Chiang and Jang [1]. It made its debut in Cassini spacecraft’s deep space exploration project. The controller proves its superiority over the conventional bang–bang controller. Other applications include minimum time fuzzy satellite attitude controller [2], crane hoisting and lowering operation [3], process control valves operation [4], and in the reduction of harmonic current pollution [5].

Nowadays, digital processors are readily available for the industrial implementation of the fuzzy and neural network controllers. The fuzzy controllers are flexible, simple to build and provide robustness to the bang–bang controller. As the application of fuzzy controller widens, more complex systems are embarked upon and solution area have now expanded to neural networks. As a result, fuzzy neural networks bang–bang controllers have gained the attention of researchers. Pang and Mesbah [6] have demonstrated how a fuzzy bang–bang system can be converted to fuzzy neural network. The fuzzy set parameters, which are based on human logic rules, cannot be trained to meet the specific requirement. Neural networks, on the other hand are provided with learning abilities. The purpose of modifying the fuzzy controller to fuzzy neural network controller is to train the controller to meet the desired response. The applications described in [1–4] use fuzzy bang–bang control. The defuzzified outputs in these works were based on centre of gravity (COG), centre of area (COA) or centroid methods. These methods yield crisp analog output, which is converted to two-level control using hard limiting saturation devices.

The fuzzy bang–bang action is a non-linear control technique that is based on heuristic human logic rules. The idea of fuzzy relay is not new. Kendal et al. [7], and Keichert and Mamdani [8] were first to point out that with mean of maxima (MOM) defuzzification, the fuzzy controller is identical to a multilevel relay. Application of the fuzzy relay in power control was first presented by Panda and Mishra [9]. Hard limiter was used in this work to convert the defuzzified output to two-level control.

New controllers are not acceptable to the control community, unless their stability is proven with the existing stability techniques. In the case of fuzzy controllers, the heuristic approach of
logical rules results in partitioning of decisions space (phase plane) into two semi-planes by means of a sliding (switching) line. Similarity between fuzzy bang–bang controller and sliding mode controller (SMC) can be used to redefine the diagonal form of fuzzy logic controller (FLC) in terms of an SMC, with boundary limits, to verify the stability of the proposed bang–bang controller \[10,11\]. SMC is a robust control method \[12\] and its stability is proven with Lyapunov’s direct method. So in lieu of SMC, the fuzzy bang–bang control stability can be easily established.

In optimal control engineering, the optimal time problem is to find the best possible control technique to transfer the state of the system from a given initial state to a specified final state in the shortest possible time. The minimum time control is highly desirable for bang–bang controller design, especially for satellite attitude control. The Pontrygin's minimum Principle (PMP) has been extensively used to design time-optimal control \[13\]. PMP states that Hamiltonian function described by states and co-state trajectories together with control effort in minimum time, when solved, yield the optimal state trajectory corresponding to optimal control effort. It is not a surprise that the sliding line of SMC and state trajectory solution of PMP optimal control have almost similar control laws as shown by Kulczycki \[14\], thus establishing the fact that fuzzy bang–bang control is indeed a robust control system.

In this paper a new integrated configuration of a fuzzy controller is proposed. This controller directly produces two-level bang–bang crisp output, which is based on the Largest of Maxima (LOM) defuzzification method. The proposed controller does not require any further saturation or hard limiting device as are used in fuzzy bang–bang controller \[1–4,7–8\]. The consequent part of the fuzzy rules has only two linguistic values, while the premise parts are freely chosen. The proposed fuzzy bang–bang relay controller (FBBRC) works exactly like a two-level relay and has flexible output to control non-linear system. It is structurally simple due to parts are freely chosen. The proposed fuzzy bang–bang relay controller (FBBRC) works exactly like a two-level relay and has flexible output to control non-linear system. It is structurally simple due to two membership function in its fuzzy output set and rule matrix. In Section 4, the controllers’ stability and optimality are analyzed. In Section 5, the FBBRC is implemented in a real-time control application of a rotary pneumatic system. Its fuzzy set is also described and its response analyzed. Finally, the paper is concluded in Section 6 with future work propositions.

2. One axis attitude control

A simple one axis attitude control system is described here as an example to develop and demonstrate the fuzzy bang–bang relay controller. This system works on pneumatic and its schematic is shown in Fig. 1.
The fuzzy controller has bang–bang action and acts as a regulator to reset the beam to zero reference, \( \theta = 0^\circ \), by firing thrusters \( T1 \) and \( T2 \). The system operation is practically demonstrated in Section 5.

The equation of motion describing the single-axis attitude control system is given by

\[ M_a u(t) = \ddot{\theta}(t) I + \dot{\theta}(t) C, \]

where \( M_a \) is the moment applied by the thrusters, \( u(t) \) is control input, \( I \) is the moment of inertia of the beam assembly, \( C \) is the coefficient of friction, \( \dot{\theta} \) is the angular rate, and \( \ddot{\theta} \) is the angular acceleration. Eq. (1) is graphically modeled in Fig. 2, and is simulated to establish and analyze the controllers stability and optimality.

A dotted callout in Fig. 2 shows that FBBRC integrates the function of FLC and hard limiter into a single unit. While the fuzzy bang–bang controller (FBBRC) is made of two parts, a FLC and a hard limiting relay. The performance of three different control schemes is compared and the results are presented in Section 4.1. The specifications of one axis satellite system are taken from [15] and are reproduced here in Table 1.

**Table 1**

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( M_a )</td>
<td>Thruster moment</td>
<td>1.89 N m</td>
</tr>
<tr>
<td>( I )</td>
<td>Moment of inertia</td>
<td>0.1035 kg m²</td>
</tr>
<tr>
<td>( C )</td>
<td>Coefficient of friction</td>
<td>0.000453 kg m²/s</td>
</tr>
</tbody>
</table>

### 3. Largest of maximum fuzzy logic controller

The bang–bang fuzzy relay controller is developed in this section. This controller takes advantage of the Largest of Maxima defuzzification (LOM) technique to yield a bang–bang output. For any fuzzy controller, it is necessary to determine the ranges of its state input and output variables, which are considered to be a reasonable representation of all the situations that the controller may face and yield to stability and optimality conditions. The following ranges are selected for simulation purposes, \( \theta(t) = [-30,30] \text{ rad}, \) \( \dot{\theta}(t) = [-30,30] \text{ rad/s} \), and output \( u = [-J, +J] \).

#### 3.1. Linguistic description

The inputs and output of the fuzzy controller are shown in Fig. 2. The inputs and output parameters, as well as the partitions and spread of the controller membership functions are selected to match the dynamic response of a pneumatic rotary system described later. The inputs \( x_i \in \Xi_i \) where \( \Xi_i \) is the universe of discourse of two inputs, \( i = 1, 2 \). For linguistic input variable, \( x_1 = \text{"error angle,"} \) the universe of discourse, \( \Xi_1 = [-30,30] \text{ rad} \), which represents a large range of perturbation angle about the zero reference. For linguistic input variable \( x_2 = \text{"error angle rate,"} \) the universe of discourse, \( \Xi_2 = [-30,30] \text{ rad/s} \). The output universe of discourse \( \Psi = [-J, +J] \) represents the bang–bang output \( y \in \Psi \).

The set \( \tilde{A}_i^j \) defines the \( j \)-th linguistic value of linguistic variable \( x_i \), which in turn is defined over the universe of discourse \( \Xi_i \). The control level of the system operation can be adequately defined for input \( x_i \) by the following \( \tilde{A}_i^{e_j} \), linguistic values:

\[ \tilde{A}_i^{e_1} = [\tilde{A}_i^1 = LN, \tilde{A}_i^2 = SN, \tilde{A}_i^3 = Z, \tilde{A}_i^4 = SP, \tilde{A}_i^5 = LP]. \]

Similar linguistic values are selected for input \( x_2 \). The set \( \tilde{B}_i^j \) that denotes the linguistic values for output linguistic variable \( y_i \) is defined as:

![Fig. 2. Simulink model of single axis attitude system shown here with three different control schemes. The proposed FBBRC scheme, standard FLC centroid scheme, and FLC centroid with external hard limiting bang–bang relay.](image-url)
with five linguistic values, thus there are at most 25 possible rules. These rules are described in matrix form in Tables 2 and 3. In later sections it will be shown that the rules can be interpreted from sliding mode control perspective. The main diagonal entry in the rules given in Table 2 is not used. The rules-partitions are heuristically chosen to reset the beam smoothly over the universe of discourse.

The symmetry of the rules matrix is expected as it arises from the symmetry of the system dynamics. The decomposition of linguistic rules from the FBBRC’s inputs to the output is given by

$$
\mu_{\hat{h}}(y) = \min \left\{ \mu_{\hat{A}_1}(x_1), \mu_{\hat{A}_2}(x_2) \right\},
$$

The index \( g \) refers to the number of rules used in implication. Conventional fuzzy FBBC uses the standard decomposition technique [1,8,16].

### 3.3. Fuzzy set membership functions

The input linguistic variables and values assigned to fuzzy set membership functions are shown in Figs. 3 and 4. Triangular shape membership functions are used in this work. These membership functions are sensitive to small changes that occur in the vicinity of their centers. A small change across the central membership function \( \hat{A}_{11} \), located at the origin, can produce abrupt switching of control command \( u \) between the +ve and –ve halves of the universe of discourse, resulting in chattering. The overlapping of the central membership function \( \hat{A}_{11} \) with its neighboring membership functions \( \hat{A}_{12} \) and \( \hat{A}_{14} \) reduce the sensitivity of the bang–bang control action.

---

**Table 2**

<table>
<thead>
<tr>
<th>( \hat{h} )</th>
<th>( \hat{\theta} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>LN</td>
<td>LN</td>
</tr>
<tr>
<td>SN</td>
<td>SN</td>
</tr>
<tr>
<td>Z</td>
<td>Z</td>
</tr>
<tr>
<td>SP</td>
<td>SP</td>
</tr>
<tr>
<td>LP</td>
<td>LP</td>
</tr>
</tbody>
</table>

**Table 3**

<table>
<thead>
<tr>
<th>( \hat{h} )</th>
<th>( \hat{\theta} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>LN</td>
<td>LN</td>
</tr>
<tr>
<td>SN</td>
<td>SN</td>
</tr>
<tr>
<td>Z</td>
<td>Z</td>
</tr>
<tr>
<td>SP</td>
<td>SP</td>
</tr>
<tr>
<td>LP</td>
<td>LP</td>
</tr>
</tbody>
</table>

\(^aPL = \text{positive large}, PS = \text{positive small}, NL = \text{negative large}.\)

**Fig. 3.** Membership functions of input \( \hat{\theta} \rightarrow \hat{x}_1 \) “error angle” and linguistic values \( \hat{A}_1 \) for both FBBRC and standard FLC.

**Fig. 4.** Membership functions of input \( \hat{\theta} \rightarrow \hat{x}_1 \) “error angle rate” and linguistic values \( \hat{A}_1 \) for both FBBRC and FLC controller.
Arbitrary minimum of five membership functions are selected to control the attitude system in a symmetrical range about the zero reset position. Smooth transition between the adjacent membership functions is achieved with higher percentage of overlap, which is commonly set to 50%. More, odd number symmetrical membership functions can be chosen at the expense of square number rules and processing time without significant improvement in performance of the controller [14]. Triangular membership functions in Figs. 3 and 4 are based on mathematical characteristics given in Table 4.

The output membership functions for standard FLC is shown in Fig. 5 and for FBBRC in Fig. 6. FBBRC has only two membership functions and there is no third central membership function at the origin of the output universe of discourse, as shown in Fig. 6. As a result, there are no diagonal rules in Table 2. For comparison purposes, the standard FLC (centroid output) and FBBRC uses the same input membership functions as shown in Figs. 3 and 4. For comparison between the FBBC and FBBRC, first the FBBC is configured by adding a hard limiting relay to the FLC output [1,8,14,16], as shown in Fig. 2.

### 3.4. Largest of maximum (LOM) aggregation

The output membership functions shown in Fig. 6a and the LOM aggregation, together formulates the fuzzy bang–bang relay controller. Any perturbation of the beam from the zero reference acts on the output membership functions according to the rule matrix in Fig. 2.

The overall output of FBBRC depends upon the maximum value of degree of membership function, \( \mu_\text{overall} \), shown in Fig. 6a. Denoting \( \mu_\text{overall} \) as membership function of the overall implied fuzzy rules \( g = 1, 2, \ldots, G \), is obtained by taking the maximum of aggregation described as

### Table 4

Mathematical characterization of triangular membership functions.

<table>
<thead>
<tr>
<th>Linguistic value</th>
<th>Triangular membership functions</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_1 )</td>
<td>( c = -30 )</td>
</tr>
<tr>
<td>( B_1 )</td>
<td>( x = -30 )</td>
</tr>
<tr>
<td>( A_2 )</td>
<td>( c = -3 )</td>
</tr>
<tr>
<td>( B_2 )</td>
<td>( x = -3 )</td>
</tr>
<tr>
<td>( A_3 )</td>
<td>( c = 0 )</td>
</tr>
<tr>
<td>( B_3 )</td>
<td>( x = 0 )</td>
</tr>
<tr>
<td>( A_4 )</td>
<td>( c = 3 )</td>
</tr>
<tr>
<td>( B_4 )</td>
<td>( x = 3 )</td>
</tr>
<tr>
<td>( A_5 )</td>
<td>( c = 30 )</td>
</tr>
<tr>
<td>( B_5 )</td>
<td>( x = 30 )</td>
</tr>
</tbody>
</table>

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\[
\mu_\text{overall} = \max \{ \mu_1, \mu_2, \ldots, \mu_G \}
\]

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\[
\mu_\text{overall} = \max \{ \mu_1, \mu_2, \ldots, \mu_G \}
\]
\[ \mu_{\text{overall}}(y) = \max_g \left\{ \mu_{g_1}(y), \mu_{g_2}(y), \ldots, \mu_{g_n}(y) \right\}. \]  

The defuzzified crisp output \( y^{\text{crisp}} \) based on Eq. (3) can be evaluated as

\[ y^{\text{crisp}} = \arg \sup \{ \mu_{\text{overall}}(y) \}. \]  

The supremum in Eq. (4) is the Largest of Maximum (LOM) value and occurs at the extremes of the output universe of discourse \( \Psi = [-J, J] \). The argument \( \arg(\sup(\mu)) \) returns \( y^{\text{crisp}} = [-J, J] \). The bang–bang firing action of membership functions \( J_2 \) and \( J_1 \) is shown in Fig. 6b.

4. Controller response and stability

The simulated performance of the proposed controller is compared to that of standard FLC with and without hard limiter device, as shown in Fig. 2. Any proposed control strategy should be supported by stability analysis for acceptance by the control system community. Fuzzy bang–bang relay controller is no exception. As discussed earlier, conventional bang–bang control system has firm stability ground via sliding mode control, which uses Lyapunov-like function to satisfy the stability criteria.

4.1. Controller response

The simulation response of the FBBRC and the standard FLC are shown in Fig. 7. Both controllers use the same input membership functions and initial conditions. However, the output membership functions are different. The FLC is simulated with and without the hard limiting function. The result shows that the overshoot and settling time is less for the FBBRC.

The output of the controllers to reset the beam to \( \theta = 0^\circ \) is shown in Fig. 8. The output universe of discourse for both controllers is \( \Psi = [-5, +5] \). These results are expected because the bang–bang control system is inherently a time-optimal controller [12,13] as discussed in later section.

4.2. Fuzzy sliding mode controller

The fuzzy rules described in Table 2 can be systematically constructed on the basis of sliding mode control and hitting condition described by Eq. (A.13) in Appendix A. Appendix A provides detailed derivation of the control law and stability condition of the SMC. The state trajectories of the two controllers follow the sliding line to zero (Fig. 9).

The rules in Table 2 can be deduced from Eq. (A.8). Multiplying it with \( s \) yields

\[ ssf(\theta; t)s + b(\theta; t)us + \lambda ds. \]  

For \( b > 0 \), if \( s < 0 \), then increasing \( u \) will result in decreasing \( ss \); and that if \( s > 0 \), then decreasing \( u \) will result in decreasing \( ss \). The control value \( u \) should be selected so that \( ss < 0 \) for \( 0 < s > 0 \). The slope of sliding line is represented by \( \lambda \).

Considering \( s \) as \( \theta \) and \( s \) as \( \dot{\theta} \), then for \( J = 5 \), \( u = [-5, +5] \), the fuzzy rules in Table 2 and the membership functions shown in Fig. 6a agree with the sliding mode condition.

4.3. Time optimal control

Optimal control (OC) of the system described by Eq. (1) can be achieved by using bang–bang action. In optimal control, Pontry-
gin’s Minimum principle (PMP) is extensively used to achieve the minimum time with bounded control \( u \rightarrow \Omega = [-1, 1] \). The optimal control problem is setup to determine the piecewise continuous control \( u : [t_0, t_f] \rightarrow P^m \) and state trajectory \( \theta : [t_0, t_f] \rightarrow P^{m-2} \), that minimize the cost function

\[
Z = \int_{t_0}^{t_f} 1 \cdot dt
\]

and brings the state to zero, \( \theta(t_f) = 0 \), from the initial condition, \( \theta(t_0) = \theta(0) \), in minimum time. The state space representation of Eq. (1) by considering \( \theta'(t) = \theta(t) \) and \( \dot{\theta}'(t) = \dot{\theta}(t) \) is

\[
\begin{align*}
\dot{\theta}_1(t) &= \theta_2(t), \\
\dot{\theta}_2(t) &= -\alpha \theta_2(t) + bu(t)
\end{align*}
\]

or

\[
\begin{bmatrix}
\dot{\theta}_1(t) \\
\dot{\theta}_2(t)
\end{bmatrix}
= \begin{bmatrix}
0 & 1 \\
0 & -\alpha
\end{bmatrix}
\begin{bmatrix}
\theta_1(t) \\
\theta_2(t)
\end{bmatrix}
+ \begin{bmatrix}
0 \\
B
\end{bmatrix} u(t)
\]

and

\[
\dot{\theta}(t) = A \cdot \theta(t) + Bu(t),
\]

\[
\theta(t_0) = \begin{bmatrix}
\theta(0) \\
\theta(0)
\end{bmatrix}^T,
\]

\[
\theta(t_f) = \begin{bmatrix}
0 \\
0
\end{bmatrix}^T.
\]

where \( \alpha = CL/l \) and \( b = M_d/l \).

A new vector of variables \( p(t) = [\theta_1(t), \theta_2(t)]^T \), known as adjoint (co-state) variable is introduced for each state. The Maximum Principle based on the Hamiltonian function in minimum time problem takes on the form [13]

\[
H(t, \theta(t), p(t), u(t)) = p(t)A \cdot \theta(t) + B \cdot u(t),
\]

where \( H \) is called the Hamiltonian, \( A \) and \( B \) are the state matrices from Eq. (7). The minimum principle states that \( u(t) \) and \( \theta(t) \) must satisfy the following three equations:

\[
\begin{align*}
\dot{p}(t) &= \frac{\partial H}{\partial \theta} = -A^T \cdot p(t) \quad \text{Adjoint equations}, \\
\dot{\theta}(t) &= \frac{\partial H}{\partial p} = A \cdot \theta(t) + B \cdot u(t) \quad \text{State equations}, \\
\frac{\partial H}{\partial u} &= 0 \Rightarrow p(t) \cdot B = 0.
\end{align*}
\]

The preceding equations are solved in time \( t \in [t_0, t_f] \) for optimal control \( u(t) \) and corresponding state trajectories \( \dot{\theta}(t) \) as

\[
u'(t) = \arg \min_{u(t)} H(t, \theta(t), p(t), u(t)).
\]

Solving a two-point boundary value problem (TPBVP) in the set of Eq. (9) is not easy, thus the shooting method [17] or the variational approach [3] is used for this purpose. However, for the system in Eq. (7), the standard calculus method with backward integration can be used to solve Eq. (9). The optimal problem, which is solved with standard calculus in Appendix B, results in the following control law:

\[
u'(t) = \text{sgn} \left[ \frac{1}{\alpha} \theta_1'(0) + e^{\alpha t} \left( \frac{1}{2} \theta_1(0) - \alpha \theta_2(0) \right) \right]
\]

(11)

The state trajectories are given in Eqs. (12) and (13)

\[
\begin{align*}
\Theta_1(t) &= \frac{b}{\alpha} \left[ -t + e^{\alpha t} \frac{1}{\alpha} - \frac{1}{\alpha} \right], \quad \Theta_2(t) = \frac{b}{\alpha} (1 - e^{\alpha t}), \\
\Theta_1(t) &= -\frac{b}{\alpha} \left[ -t + e^{\alpha t} \frac{1}{\alpha} - \frac{1}{\alpha} \right], \quad \Theta_2(t) = -\frac{b}{\alpha} (1 - e^{\alpha t}).
\end{align*}
\]

5. Rotary actuator FBBR controller

Fuzzy logic controllers have been widely utilized in industrial processes due to their simplicity and effectiveness [18]. They have proved to be robust and perform well even with disturbances in the input parameters [19,20].

In this work the practical application of fuzzy bang–bang relay (FBBR) controller is demonstrated by controlling the angular position of a pneumatic rotary actuator. The pneumatic rotary actuator is controlled in real-time with Matlab (2006a) – simulink, xPC target environment. There have been some successful applications on xPC Target since its release as a Matlab toolbox [21–24].

5.1. Hardware and software setup

An experiment based on the model discussed in Section 2 and shown in Fig. 1, is performed here to demonstrate the fuzzy bang–bang relay control system. The pneumatic rotary actuator is shown in Fig. 11. The angle of the rotating beam is determined from the pulses generated by the inductive proximity encoder. The gear has 36 teeth/rev, and gives physical resolution of 10°/teeth.

For xPC target operation, two personnel computers connected via RS232 serial port communication cable is used, Fig. 1. The host computer compiles the Real Time Window application program
using Microsoft ‘VC98’ compiler and produces an executable file. The executable file is used for booting and running the executable file on Target computer. The rotor system in Fig. 11 is connected to target computer using a data acquisition card as shown in Fig. 1.

The resolution of the inductive proximity sensor is coarse, but it provides latency for fuzzy controller, so that the output commands, J1 and J2 are available for a period of time greater than the solenoid’s response time [25]. The latency time is also necessary to build necessary pressure at the nozzles (thrusters) to overcome the inertia and friction forces.

The real-time workshop environment is set to fixed step time of 0.01 s or 100 Hz that is sufficient for the operational function of this system, and the interrupt driven scheduler of xPC Target kernel [26]. The inlet pressure used is 3 bars. The airflow rate at the nozzle outlet is determined from the characteristic graph provided by leading pneumatic product manufacturer ‘Festo – AG & Co. KG’ [25]. The air is treated as incompressible, since the air density changes slightly at velocities less than the speed of sound. The force of 2.2 N produced by the nozzle, is calculated from the general thrust equation [27]. The beam angular moment is then:

\[
F = m_2(V_2 - V_1) + (P_2 - P_1)A_2,
\]

or

\[
M_0 = F \cdot (HBL)
\]

and

\[
M_a = 2.2(0.143),
\]

\[
= 0.314 \text{ N m}.
\]

The force equation parameters are described in glossary and the specifications of the pneumatic rotary system are summarized in Table 5. The moment of inertia of the beam shown in Fig. 11, and evaluated using the parallel axis theorem is 0.00244 kg m².

### 5.2. Real-time xPC controller

The simulink real time control program is shown in Fig. 12. The state flow is used to calculate the angle, the direction, and the angular rate of the pneumatic rotary actuator based on the pulses from the single inductive proximity sensor.

The pulse shaping filter is a relay which is switched on/off on encoder pulse transitions. The relay replaces the noisy corrupted encoder’s pulse with its own 0–5 V output pulses. The state flow chart is used for evaluating the direction and angle of the beam. The state flow chart can be eliminated by using two proximity sensors in quadrature configuration. The instantaneous angular velocity is evaluated in the speed block. The delay block provides the initial value and the direction of rotation required by the state flow chart. The FBBRC outputs control signal \( y_{\text{ crisp}} \) for each thruster-solenoid (\( J_1, J_2 \)). The relays R1 and R2 convert \( u \) to \( U \equiv [J2, J1] \) as illustrated in Fig. 13.

The controller output \( u \) has two states; \(-1 \) or \(+1\), thus the DAQ system requires two-level voltage, high (+5 V) and common (0 V or –5 V) for operation of each thruster-solenoid (\( T1, T2 \)). The relays R1 and R2 are configured such as to provide two-level output for each state of \( u \). The –5 V instead of 0 V provides better isolation of common from high (+5 V) in case of noisy channels.

The solenoid output is \( U \equiv F \), where \( F \) is the nozzle force. In order to reset the beam angle to zero reference, the FBBRC controls the firing sequence and duration (\( \Delta t \)) of the jets. The selection of FBBRC membership functions is based upon the perturbation of the beam from ±90° to the horizontal position, as \( \Xi = [-90°, 90°] \). To determine \( \Xi_2 \), the angular momentum law [28] is used. The law states that the sum of all the moments acting on the body
Fig. 13. Relays logic and solenoids on/off command.

rotating about a point is equal to the time rate change of the total angular momentum
\[ M_a = I \dot{\omega} \quad \text{(kg m}^2/\text{s}) \]  
(15)

To reset the beam from initial conditions of ±90° to the horizontal position, the maximum thrusters firing time is selected to be 5 s. Then \( \dot{\theta} \) is evaluated as
\[ \int_0^5 M_a \, dt = I \dot{\omega} \quad \text{(kg m}^2/\text{s}), \]
\[ \int_0^5 (0.00314) \, dt = 0.0024 \dot{\omega}, \]
\[ 0.0157 = 0.0024 \dot{\omega}, \]
\[ \dot{\theta} = 374^\circ/\text{s}. \]

The angular velocity \( \dot{\theta} \) in Eq. (16) is a large quantity and therefore requires scaling down before being input to the FBBR. A derivative gain [29] block \( G_d \) in Fig. 12, is set to 0.01, thus reducing \( \dot{\theta} \) to 3.74°/s and setting the velocity requirement for \( \Xi_2 \). However, to compensate for the nonlinearities of the real system, \( \dot{\theta} \) is set to \( \Xi_2 = [-6.5\text{s}^{-1}; 6.5\text{s}^{-1}] \).

The system update requirement is based on the maximum angular velocity
\[ \dot{\theta} = 374^\circ/\text{s} \]
or
\[ f_{hz} \equiv 1 \text{ Hz}. \]

The system sampling frequency is set to 100 Hz, which is much higher than the \( \sim 2 \text{ Hz} \) Nyquist rate.

The instantaneous angular velocity is calculated in the speed block of Fig. 12 using the following equation:
\[ \dot{\theta}(kT_e) = \frac{[\theta(kT_e) - \theta(k - 1)T_e]}{T_e}, \]
(17)
where \( k, ..., 36 \) is the incremental encoder index, \( T_e \) is the encoder cycle time, which is equal to the triggering time between the rising transitions of the encoder cycle, and \( \Psi \) is \([-1, 1]\). The structure of the fuzzy logic controller used here is the same as that described in Section 3. This includes the fuzzy set membership functions rules, the partitions overlap, the aggregation, and the defuzzification. The changes in the spread of the membership functions are due to different universe of discourse of \( \Xi_1, \Xi_2 \), and \( \Psi \) as shown in Table 6. For practical implementation the FBBRC is formulated as follows:

(1) Define the input fuzzy sets freely over the desired universe of discourse.
(2) Define the output fuzzy set as described in Fig. 6a.
(3) The fuzzy rules in Table 2 should be of two-level. Heuristic approach should confirm with the SMC requirement in Eq. (5).
(4) Fuzzy implication should be ‘minimum’.
(5) Rules aggregation should be ‘maximum’.
(6) Defuzzification should be ‘LOM’.
(7) Two-level \( u = [-J, J] \) should be converted to three level output with ‘common’ signal for DAQ output.

5.3. Rotary actuator control

The parameters for rotary actuator system are given in Table 5. Initial angle of 30° and zero angular velocity are used for system excitation. The beam resets to zero degree and chatters with 10° encoder resolution as can be seen in Fig. 14. A curve fitting is added to the discrete output pulses of the encoder, shown in Fig. 14 to approximate the continuous angle convergence to 0°. The encoder output has a resolution of 10°/teeth as shown in this figure. The angular velocity based on Eq. (17) can be inferred from Fig. 14 as
\[ \dot{\theta}(kT_e) = \frac{[15 - 5]}{0.166}, \quad k = 4, T_e = 0.166 \text{ s} \]
(18)

The angular velocity in Eq. (18) is well below the maximum range found in Eq. (16). Which after the derivative gain \( G_d = 0.01 \) reduces to 0.6°/s.

One thruster firing cycle is required to reset the beam in less than 1.5 s, as shown in figure Fig. 15. The phase plane plots of the states are shown in Fig. 16. The discontinuities at 10° interval are due to encoder resolution. The time response of the phase plot in Fig. 16a is shown in Fig. 14; where it is illustrated that \( k = 4 \) steps are required to reach the origin in one thruster cycle. The trajectories cannot slide on sliding line due to the 10° (−5° to +5°) hardware limitation.

Table 6

<table>
<thead>
<tr>
<th>Triangular membership functions</th>
<th>Angle</th>
<th>Angle rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c^L )</td>
<td>−90</td>
<td>−6.5</td>
</tr>
<tr>
<td>( c^S )</td>
<td>−4.5</td>
<td>−0.8658</td>
</tr>
<tr>
<td>( c^R )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( c^M )</td>
<td>4.5</td>
<td>0.8658</td>
</tr>
<tr>
<td>( c^P )</td>
<td>90</td>
<td>6.5</td>
</tr>
</tbody>
</table>

Fig. 14. Encoder output – Euler angle of rotary actuator system. Reset to 0° angle with 10°/teeth resolution with.
6. Conclusion

This paper proposes an integrated fuzzy bang–bang relay controller. We demonstrate its operation with a simulated model and a real system application. The simulation model, representing a linear model without uncertainties, was used for the development, stability, and optimality analysis. The real system parameters were not used in the simulation model because its uncertainties were not modeled. Fuzzy controllers are known for absorbing the non-linearity of the systems and, as the results show, it works well for the real system.

The bang–bang control is inherently time–optimal, and this property is an important feature of the FBBRC. Comparison between the FBBRC and the standard FLC showed that FBBRC could reset the beam from any initial condition in optimal time and with a smaller overshoot. FBBRC uses fewer output membership functions in comparison to the standard FLC and thus simplifying the unification process of the fuzzy rule matrix. In the FBBRC, the Largest of Maxima defuzzification method yields direct bang–bang output from the fuzzy controller. The bang–bang output is similar to conventional hardware relay output and flexible to meet the changing demands of the non-linear system dynamics. The stability and optimality of FBBRC were satisfied by the SMC–Lyapunov criterion and optimal bang–bang control theory, respectively.

The FBBRC developed in this work was successfully applied to the real-time control of a pneumatic rotary actuator. The real-time control was achieved by using the Matlab-simulink xPC target environment. The real-time application program demonstrated how the controller output is divided into the two-level voltage required for the operation of pneumatic solenoid valves. We provide a simple procedure to implement the FBBRC. FBBRC response time could be studied further for arresting high-velocity beam motion by changing the fuzzy rule polarity of diagonals immediately adjacent to the main diagonal. Adaptive fuzzy tuning of FBBRC membership function can be further investigated to enhance the robust performance of the proposed controller.

Appendix A

A.1. Sliding mode control (SMC)

A general 2nd order non-linear single-input–single-output (SISO) control system could be described [12] as

$$\dot{\theta}(t) = f(\theta; t) + b(\theta; t)u,$$  (A.1)

where $\dot{\theta}(t)$ is the output of interest, $u(t)$ is the scalar input, and $\theta = [\theta, \dot{\theta}]$ is state vector. In general, $f(\theta; t)$ is not precisely known, but upper bounded by a known continuous function of $\theta$. Similarly $b(\theta; t)$ is not known, but is of known sign and is bounded by a known continuous function of $x$ as
\[ |f - \hat{f}| \leq F(\theta; t), \]
\[ \frac{1}{\dot{\beta}(\theta; t)} \leq \frac{\dot{b}}{b} \leq \beta(\theta; t), \]  
(A.2)

where \( f \) and \( b \) are nominal values of \( f \) and \( b \), respectively, without the function argument for brevity purpose.

Comparing Eqs. (1) and (A.1), the system becomes:
\[ f(\theta; t) = \frac{C}{T} \dot{\theta}(t), \]
\[ b(\theta; t)u = \frac{M}{T} u(t) \quad \cdot \quad b(\theta; t) = \frac{M}{T}, \]  
(A.3)

where \( u(t) \) is a unit step input. The system blocks based on Eq. (A.3) are shown in Fig. 2.

The control problem is to get the state \( \theta \) to track \( \theta_d = [\theta_d^T \dot{\theta}_d^T]^T \) in minimum time and in the presence of imprecise friction. The initial \( \theta_d \) should be the following in view of finite control \( u \)
\[ \dot{\theta}_d(0) = \dot{\theta}(0). \]  
(A.4)

The tracking error between the actual and desired state would be
\[ e = \theta - \theta_d = [e^T]^T. \]  
(A.5)

A sliding–switching line \( s(\theta, t) \) in the second order state space \( \mathbb{R}^2 \) is defined such that \( e \) follows the line \( s(\theta, t) = 0 \). The sliding line \( s(\theta, t) \) is determined by
\[ s(\theta, t) = \left( \frac{d}{dt} + \lambda \right) e. \]  
(A.6)

Eq. (A.6) can be expanded with binomial expansion and \( \lambda \) is positive constant. For \( n = 2 \)
\[ s = e + \lambda e \quad \cdot \quad e = 0 \quad \text{and} \quad e = 0. \]  
(A.7)

Then from Eq. (A.1)
\[ s = f(\theta; t) + b(\theta, t)u + \lambda \dot{e}. \]  
(A.8)

A.2. SMC control law

Let \( u_{eq} \) be the equivalent control law that will keep the states on the sliding trajectory, computed by \( s = 0 \) for \( u \equiv u_{eq} \), then from Eqs. (A.4), (A.5) and (A.7)
\[ s = \dot{\theta} + \lambda \dot{\theta}, \]
\[ s = \dot{\theta} + \lambda \dot{\theta}. \]  
(A.9)

Then from Eq. (A.8) with uncertainties
\[ s_{\dot{u}=u_{eq}} = f(\theta; t) + b(\theta, t)u_{eq} + \lambda \dot{e} = 0. \]

Solving the above equation
\[ u_{eq} = \frac{b}{b - 1} \dot{u}, \]  
(A.10)

where
\[ \dot{u} = [\dot{f}(\theta; t) - \lambda \dot{e}] \]  
(A.11)

or
\[ \dot{e} = -\dot{f}(\theta; t) - \dot{u} \]
is the nominal control input in presence of uncertainties.

A.3. SMC – reaching condition

The control input \( u \) to get the state \( \theta \) to track \( \theta_d \) is then made to satisfy the Lyapunov-like function \( V = (1/2)\dot{e}^2 \), if there exist \( \eta > 0 \) and by the following sliding condition [12]:
\[ 1 \quad \text{or} \]
\[ \frac{d}{dt} \dot{e}^2(\theta, t) \leq -\eta |s| \]  
(A.12)

Which is reduced to the so-called sliding mode ‘reaching condition’ for Eq. (1)
\[ s \cdot \text{sgn}(s) \leq -\eta |s|, \quad \eta > 0. \]  
(A.13)

The control law that satisfies the sliding mode reaching conditions, Eq. (A.13) can be obtained as
\[ u = u_{eq} + u_s, \]  
(A.14)

where
\[ u_s = -K \text{sgn}(s) \]  
(A.15)

and
\[ \text{sgn}(s) = \begin{cases} +1, & \text{if } s > 0, \\ -1, & \text{if } s < 0. \end{cases} \]

Substituting Eqs. (A.1) and (A.8) in Eq. (A.13)
\[ s\dot{s} = s(f + bu + \lambda \dot{e}) \leq -\eta |s|. \]

Note: here we have dropped the function argument for brevity purpose. Then equivalently we can write:
\[ s\dot{s} = \text{sgn}(s)(f + \lambda \dot{e} + bu \text{sgn}(s)) \leq -\eta |s|. \]  
(A.16)

Substituting Eqs. (A.14) and (A.15) into Eq. (A.16)
\[ \dot{s} = \text{sgn}(s)(f + \lambda \dot{e} + bu_{eq} + \lambda \dot{e}) \text{sgn}(s) \leq -\eta |s|. \]

Substituting from Eqs. (A.10) and (A.11) in above, we get
\[ \dot{s} = \text{sgn}(s)(f + (-\dot{f} - \dot{\theta}) + b[b^{-1}\dot{u} + b^{-1}K \text{sgn}(s)] \text{sgn}(s) \leq -\eta |s|. \]

Simplifying we get
\[ \text{sgn}(s)(f - \dot{f} + b[b^{-1}\dot{u} + b^{-1}K \text{sgn}(s)] \text{sgn}(s) \leq -\eta |s|. \]  
(A.17)

Then for upper bounds from Eq. (A.1) need
\[ K \geq \beta|F + \eta + (\beta - 1)|. \]  
(A.18)

to satisfies the reaching or hitting condition.

Appendix B. Solving optimal problem

To solve the optimal problem with standard calculus, consider \( u(t) : [\bar{t}_u, \bar{t}_f] \rightarrow \Omega \) as optimal control and \( \dot{\theta}(t) \) be the corresponding trajectory, then there exists a non-zero solution \( p(t) \) of
\[ \begin{bmatrix} p_1'(t) \\ p_2'(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & -\lambda \end{bmatrix} \begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix}. \]  
(B.1)

Such that Eq. (9) is satisfy and Eq. (10) holds. The transition matrix of homogeneous part of Eq. (7) is
\[ \Phi(t) = \begin{bmatrix} 1 & \frac{1}{2}(1 - e^{\lambda t}) \\ 0 & e^{\lambda t} \end{bmatrix}. \]

Then the solution of Eq. (B.1) becomes
\[ \begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{1}{2}(1 - e^{\lambda t}) & e^{\lambda t} \end{bmatrix} \begin{bmatrix} p_1(0) \\ p_2(0) \end{bmatrix}. \]

Then
\[ p_1(t) = p_1(0) \]
and
\[ p_2(t) = \frac{1}{2} \begin{bmatrix} 1 - e^{\lambda t} \\ 1 - e^{\lambda t} \end{bmatrix} \begin{bmatrix} p_1(0) \\ p_2(0) \end{bmatrix}. \]  
(B.2)
From Eqs. (A.18) and (10) the minimization of Hamiltonian is

\[ H = [p_1(t)p_2(t)]^T \begin{bmatrix} 0 & 1 \\ 0 & -\alpha \end{bmatrix} \begin{bmatrix} \dot{\theta}_1(t) \\ \dot{\theta}_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ b \end{bmatrix} u(t) \]  

(B.3)

\[ \begin{align*} 
\dot{p}_1(t) &= \frac{1}{\alpha} p_1(t) + p_1(t) \alpha \dot{\theta}_2(t) + p_2(t) \cdot b \cdot u(t) \\
\dot{p}_2(t) &= e^{\alpha t} p_1(0) - \alpha p_2(0) \dot{\theta}_2(t) + p_2(0) bu(t) 
\end{align*} \quad \text{Eq. (B.2)}. \]

The necessary condition for optimality in above equation is to choose \( u(t) = u(t) \) which minimize \( H \). The \( H \) in Eq. (B.3) is minimized by reducing the factor.

Then from Eq. (B.2)

\[ p_2(t) = \frac{1}{\alpha} p_1(0) \quad \text{or} \quad p_1(t) - \alpha p_2(t) = 0. \]

To minimize \( H \) in Eq. (B.3), \( p(t) \neq 0 \) from Eq. (B.1), then \( u(t) \) behave in only two ways:

\[ u(t) \equiv \begin{cases} +1 & \text{and} \quad p_2(t) = \frac{1}{\alpha} p_1(0) > 0, \\ -1 & \text{and} \quad p_2(t) = \frac{1}{\alpha} p_1(0) < 0. \end{cases} \]

(B.4)

The optimal control from Eq. (B.3) and Eq. (B.4) is then

\[ u(t) = \text{sgn} p_1(t), \]

\[ = \text{sgn} \left[ \frac{1}{\alpha} p_1(t) + e^{\alpha t} \left( \frac{1}{\alpha} p_1(0) - \alpha p_2(0) \right) \right]. \]

(B.5)

The optimal control Eq. (B.5) for the system Eq. (A.17) can now be rephrase as

\[ \dot{\theta}_1(t) = \theta_1(t), \]

\[ \dot{\theta}_2(t) = -\alpha \dot{\theta}_2(t) + b \text{sgn} \left[ \frac{1}{\alpha} p_1(0) + e^{\alpha t} \left( \frac{1}{\alpha} p_1(0) - \alpha p_2(0) \right) \right], \]

\[ (\theta_1(t) = \left[ 360^\circ / s, 360^\circ / s \right]^T, \]

\[ (\theta_1(t) = [0, 0]^T. \]

The TPBVP Eq. (B.6) is solved here with backward integration in time. The initial and final states are known. Backward integration starts from the final state toward the initial state with the guess time. The initial and final states are known. Backward integration in time is reverse.

\[ \dot{\theta}_1(t) = -\theta_1(t), \]

\[ \dot{\theta}_2(t) = -\frac{b}{\alpha} \theta_2(t) - \theta_2(t), \]

\[ (\dot{\theta}_2(t) = \left[ 360^\circ / s, 360^\circ / s \right]^T, \]

\[ (\dot{\theta}_2(t) = [0, 0]^T. \]

The Matlab command to solve Eq. (B.7) is

\[ [\Theta_1(t), \Theta_2(t)] = \text{dsolve}('DTheta1= -Theta2', 'DTheta2 = \alpha^*Theta2 - b^*', 'Theta1(0)=0', 'Theta2(0)=0') \]

\[ \Theta_1(t) = -\frac{b}{\alpha} \left( -t + e^{\alpha t} - 1 \right), \quad \Theta_2(t) = \frac{b}{1 - e^{\alpha t}}. \]

(B.8)

Choosing \( p'(0) \) such that \(-p_1(0) + \alpha p_2(0) > 0 \) and \( p_2(0) < 0 \). Then \( u(t) \equiv -1 \), and by backward integration of the trajectory \( \Theta(t) \) is

\[ \dot{\theta}_1(t) = -\Theta_2(t), \]

\[ \dot{\theta}_2(t) = -\frac{b}{\alpha} \theta_2(t) + \theta_2(t) \]

(B.9)

The Matlab command to solve Eq. (B.8) is

\[ [\Theta_{final}, \Theta_{final}] = \text{dsolve}('DTheta1= -Theta2', 'DTheta2 = \alpha^*Theta2 - b^*', 'Theta1(0)=0', 'Theta2(0)=0') \]

\[ \Theta_1(t) = -\frac{b}{\alpha} \left( -t + e^{\alpha t} - 1 \right), \quad \Theta_2(t) = \frac{b}{1 - e^{\alpha t}}. \]

(B.10)

Choosing \( p'(0) \) such that \(-p_1(0) + \alpha p_2(0) > 0 \) and \( p_2(0) < 0 \). Then \( Eq. (B.2) \) will be negative for \( t \in (0, t') \) and positive for \( t \in (t', \infty) \), where \( t' \) is the time for the trajectory to reach to sliding trajectory from any initial values. Then by backward integration of Eq. (B.6) the trajectory \( \Theta(t) \) is

\[ \dot{\theta}_1(t) = -\Theta_2(t), \]

\[ \dot{\theta}_2(t) = -\frac{b}{\alpha} \theta_2(t) + \left\{ \begin{array}{ll}
- \frac{b}{\alpha} \theta_1(t) & \text{for } t < t' \\
\theta_1(t) & \text{for } t > t'
\end{array} \right\} \quad \text{with } \Theta_1(t) = \Theta_2(t). \]

(B.11)

References


