# Mathematical models and exact algorithms for the min-power symmetric connectivity problem: an overview

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# Chapter 1

# Models and algorithms for the MPSCP: an overview

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## 1.1 Introduction

Ad-hoc wireless networks have received significant attention in recent years due to their potential applications in battlefield, emergency disasters relief, and other application scenarios (see, for example, Blough et al. [2], Chu and Nikolaidis [4], Clementi et al. [5] and [7], Kirousis et al. [10], Lloyd et al. [11], Ramanathan and Rosales-Hain [17], Singh et al. [19], Wan et al. [20] and Wieselthier et al. [22]). Unlike wired networks of cellular networks, no wired backbone infrastructure is installed in ad-hoc wireless networks. A communication session is achieved either through single-hop transmission if the recipient is within the transmission range of the source node, or by relaying through intermediate nodes otherwise.

We consider wireless networks where individual nodes are equipped with omnidirectional

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antennae. Typically these nodes are also equipped with limited capacity batteries and have a restricted communication radius. Topology control is one of the most fundamental and critical issues in multi-hop wireless networks which directly affect the network performance. In wireless networks, topology control essentially involves choosing the right set of transmitter power to maintain adequate network connectivity. Incorrectly designed topologies can lead to higher end-to-end delays and reduced throughput in error-prone channels. In energy-constrained networks where replacement or periodic maintenance of node batteries is not feasible, the issue is all the more critical since it directly impacts the network lifetime.

In a seminal paper on topology control using transmission power control in wireless networks, Ramanathan and Rosales-Hain [17] approached the problem from an optimization viewpoint and showed that a network topology which minimizes the maximum transmitter power allocated to any node can be constructed in polynomial time. This is a critical criterion in battlefield applications since using higher transmitter power increases the probability of detection by enemy radar. In this paper, we attempt to solve the minimum power topology problem in wireless networks. Minimizing the total transmit power has the effect of limiting the total interference in the network. It has been shown in Clementi et al. [6] that this problem is NP-complete. Related work in the area of minimum power topology construction include Wattenhofer et al. [21], Huang et al. [9] and Borbash and Jennings [3], all of which propose distributed algorithms. Specifically, Wattenhofer et al. [21] proposes a cone-based distributed algorithm which relies only on angle-of-arrival estimates to establish a power efficient connected topology. Huang et al. [9] describe a distributed protocol which is designed for sectored antenna systems. The work in Borbash and Jennings [3] explores the use of relative neighborhood graphs (RNG) for topology control and suggests an algorithm for distributed computation of the RNG.

For a given set of nodes, the *min-power symmetric connectivity problem* ($MPSCP$), sometimes also referred to as the *minimum power topology problem*, is to assign transmission powers to the nodes of the network, which are equipped with omnidirectional antennae, in such a way that all the nodes are connected by bidirectional links and the total power consumption over the network is minimized. Having bidirectional links simplifies one-hop transmission protocols by allowing acknowledgement messages to be sent back for every

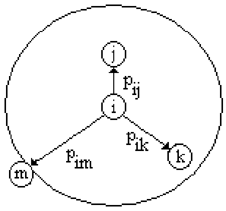


Figure 1.1: Communication model.

packet (see Althaus et al. [1]). It is assumed that no power expenditure is involved in reception/processing activities, that a complete knowledge of pairwise distances between nodes is available, and that there is no mobility.

Unlike in wired networks, where a transmission from $i$ to $m$ generally reaches only node $m$, in wireless networks with omnidirectional antennae it is possible to reach several nodes with a single transmission (this is the so-called *wireless multi-cast advantage*, see Wieselthier et al. [22]). In the example of Figure 1.1 nodes $j$ and $k$ receive the signal originated from node $i$ and directed to node $m$ because $j$ and $k$ are closer to $i$ than $m$, i.e. they are within the transmission range of a communication from $i$ to $m$. This property is used to minimize the total transmission power required to connect all the nodes of the network.

In Section 1.2 the $MPSCP$ is formally described. Section 1.3 is devoted to an overview of the mathematical models end exact algorithms presented so far in the literature. In Section 1.4 a preprocessing rule, useful to reduce problem dimensions, is described. In Section 1.5 a comparison of the performance of the available exact algorithms is proposed, while section 1.6 is devoted to conclusions.

## 1.2 Problem description

In order to represent the problem in mathematical terms, a model for signal propagation has to be selected. We adopt the model presented in Rappaport [18], and used in most of the papers appeared in the literature (see, for example, Wieselthier et al. [22], Montemanni et al.

[14] and Althaus et al. [1]). According to this model, signal power falls as $\frac{1}{d^\kappa}$, where $d$ is the distance from the transmitter to the receiver and $\kappa$ is a environment-dependent coefficient, typically between 2 and 4. Under this model, and adopting the usual convention (see, for example, Althaus et al. [1]) that every node has the same transmission efficiency and the same detection sensitivity threshold, the power requirement for supporting a link from node $i$ to node $j$, separated by a distance $d_{ij}$, is then given by

$$p_{ij} = (d_{ij})^\kappa \tag{1.1}$$

Using the model described above, power requirements are symmetric, i.e. $p_{ij} = p_{ji}$.

Constraints on maximum transmission powers of nodes can be treated by artificially modifying power requirements. If, for example, node $i$ cannot reach node $j$ even when it is transmitting to its maximum power (i.e. $d_{ij}^\kappa >$ maximum power of node $i$), then $p_{ij}$ can be redefined as $+\infty$.

$MPSCP$ can be formally described as follows. Given the set $V$ of the nodes of the network, a *range assignment* is a function $r : V \to \mathcal{R}^+$. A *bidirectional link* between nodes $i$ and $j$ is said to be established under the range assignment $r$ if $r(i) \geq p_{ij}$ and $r(j) \geq p_{ij}$. Let now $B(r)$ denote the set of all bidirectional links established under the range assignment $r$. $MPSCP$ is the problem of finding a range assignment $r$ minimizing $\sum_{i \in V} r(i)$, subject to the constraint that the graph $(V, B(r))$ is connected.

As suggested in Althaus et al. [1], a graph theoretical description of $MPSCP$ can be given as follows. Let $G = (V, E, p)$ be an edge-weighted complete graph, where $V$ is the set of vertices corresponding to the set of nodes of the network and $E$ is the set of edges containing all the possible pairs $\{i, j\}$, with $i, j \in V$, $i \neq j$. A cost $p_{ij}$ is associated with each edge $\{i, j\}$. It corresponds to the power requirement defined by equation (1.1).

For a node $i$ and a spanning tree $T$ of $G$, let $\{i, i_T\}$ be the maximum cost edge incident to $i$ in $T$, i.e. $\{i, i_T\} \in T$ and $p_{ii_T} \geq p_{ij}$ $\forall \{i, j\} \in T$. The *power cost* of a spanning tree $T$ is then $c(T) = \sum_{i \in V} p_{ii_T}$. Since a spanning tree is contained in any connected graph, $MPSCP$ can be described as the problem of finding the spanning tree $T$ with minimum power cost

$c(T)$.

## 1.3 Mathematical models and exact algorithms

In this section we present four mathematical models for the $MPSCP$ recently appeared in the literature, all based on mixed-integer programming.

For each mathematical formulation discussed, some reinforcing inequalities and an exact algorithm, strongly based on the formulation, are also presented.

### 1.3.1 Althaus et al. [1]

In Althaus et al. [1] a mathematical formulation, with some reinforcing inequalities, and an exact algorithm are presented. They are summarized in this section.

#### Mathematical formulation

A weighted, directed, complete graph $G' = (V, A, p)$ is derived from $G$ by defining $A = \{(i,j)|i,j \in V\}$, i.e. for each edge in $E$ there are the respective two arcs in $A$, and a dummy arc $(i,i)$ with $p_{ii} = 0$ is inserted for each $i \in V$. $p_{ij}$ is defined by equation (1.1) when $i \neq j$.

In formulation $AL$, variables $x$ define the spanning tree $T$ on which the connectivity structure is based. $x_{ij} = 1$ if edge $\{i,j\}$ belongs to the spanning tree $T$, 0 otherwise. $w$ variables represent the transmission range of nodes. $w_{ij} = 1$ if $i_T = j$ (see Section 1.2), $w_{ij} = 0$ otherwise.

$$(AL)\quad \text{Min} \sum_{(i,j)\in A} p_{ij} w_{ij} \tag{1.2}$$

$$\text{s.t.} \sum_{j\in V|(i,j)\in A} w_{ij} = 1 \qquad \forall i \in V \tag{1.3}$$

$$x_{ij} \leq \sum_{\substack{(i,k)\in A,\\ p_{ik}\geq p_{ij}}} w_{ik} \qquad \forall \{i,j\} \in E \tag{1.4}$$

$$x_{ij} \leq \sum_{\substack{(j,k)\in A,\\ p_{jk}\geq p_{ij}}} w_{jk} \qquad \forall \{i,j\} \in E \tag{1.5}$$

$$\sum_{\{i,j\}\in E} x_{ij} = |V| - 1 \tag{1.6}$$

$$\sum_{\substack{i,j\in S,\\ \{i,j\}\in E}} x_{ij} \leq |S| - 1 \qquad \forall S \subset V \tag{1.7}$$

$$x_{ij} \in \{0,1\} \qquad \forall \{i,j\} \in E \tag{1.8}$$

$$w_{ij} \in \{0,1\} \qquad \forall (i,j) \in A \tag{1.9}$$

Constraints (1.3) enforce that exactly one range variable for every node $i \in V$ is selected, i.e. the range of each node is properly defined. Constraints (1.4) and (1.5) enforce that an edge $\{i,j\}$ is included in the tree only if the range of each endpoint is at least the cost of the edge. Constraints (1.6) and (1.7) enforce that the tree variables indeed form a spanning tree. Constraints (1.8) and (1.9) define the domains of variables.

The bottleneck of formulation $AL$ is represented by constraints (1.7), which are in exponential number and make difficult to handle the formulation in case of real problems. An idea to overcome this problem is implemented within the exact algorithm described below.

#### Valid inequalities

In Althaus et al. [1] a set of valid inequalities for formulation $ALT$, which is used within the exact algorithm presented by the authors, is also discussed. These inequalities are summarized in this section.

**Definition 1.** *Given* $W \subset V$, $\forall i \in W$ *we define* $i^W \in V \backslash W$ *such that* $p_{ii^W} \leq p_{ij}$ $\forall j \in V \backslash W$.

**Theorem 1 (Crossing inequalities).** *The set of inequalities*

$$\sum_{i \in W} \sum_{\substack{(i,j) \in A, \\ p_{ij} \geq p_{ii^W}}} w_{ij} \geq 1 \quad \forall W \subset V \tag{1.10}$$

*is valid for formulation* $AL$.

*Proof.* Since $T$ must be a spanning tree, at least one of its edges must cross the cut $W$. Let $\{i, j\}$ be such an edge, with $i \in W$. Then $p_{ij} \geq p_{ii^W}$ and the range of $i$ at least $p_{ij}$. Inequality (1.10) must then be satisfied. □

### Exact algorithm

Althaus et al. [1] proposed a branch and cut algorithm based on formulation $AL$.

Formulation $AL^R_{LR}$ is considered by the algorithm. It is obtained from $AL$ by substituting constraints (1.8) and (1.9) with their linear relaxation, formally

$$0 \leq x_{ij} \leq 1 \quad \forall \{i, j\} \in E \tag{1.11}$$

$$0 \leq w_{ij} \leq 1 \quad \forall (i, j) \in A \tag{1.12}$$

and by adding constraints (1.10).

Formally the branch and cut algorithm works by solving formulation $AL^R_{LR}$. If the solution is integral, the optimal solution has been found, otherwise a variable with a fractional value is picked up and the problem is split into two subproblems by setting the variable to 0 and 1 in the subproblems. The subproblems are solved recursively and disregard a subproblem if the lower bound provided by $AL^R_{LR}$ is worse than the best known solution.

Since there are an exponential number of inequalities of type (1.7), $AL^R_{LR}$ cannot be solved directly at each node of the branching tree. Instead, the algorithm starts with a small subset of these inequalities and algorithmically test whether the solution violates an inequality which is not in the current problem. If so, the inequality is added to it, otherwise the solution of $AL_{LR}^{R}$ has been retrieved. The separation algorithm described in Padberg and Wolsey [15] has been used for these constraints.

A similar approach applies also for inequalities (1.10), which are again in exponential number. Since there was no known separation algorithm for them, the following heuristic was used. Capacity $q_{kl}$ is defined for each edge $\{k, l\}$:

$$q_{kl} = \sum_{\substack{(k,r)\in A, \\ p_{kr} \geq p_{kl}}} w_{kr} \tag{1.13}$$

An arbitrary node $i$ is chosen and for every node $j \in V \backslash \{i\}$ the minimal directed cut from $i$ to $j$ and from $j$ to $i$ (with capacities defined by equation 1.13) is computed and it is tested whether the corresponding inequality is violated or not.

### 1.3.2 Das et al. [8]

In Das et al. [8] a mathematical formulation, with some reinforcing inequalities, and an exact algorithm are presented. They are summarized in this section.

#### Mathematical formulation

The mixed integer programming formulation $DAS$, described in this section is based on a network flow model (see Magnanti and Wolsey [12]). A node $s$ is elected as the source of the flow, and one unit of flow is sent from $s$ to every other node. The meaning of variables in the formulation is as follows. Variable $w_i$ contains the transmission power of node $i$. Variable $t_{ij}$ represents the flow on arc $(i, j)$, while $u_{ij}$ is an indicator variable and assumes value 1 if the flow on arc $(i, j)$ is greater than 0 (i.e. $t_{ij} > 0$), 0 otherwise.

$$(DAS) \quad \text{Min} \sum_{i \in N} w_i \tag{1.14}$$

$$\text{s.t.} \quad \sum_{(i,j)\in A} t_{ij} - \sum_{(k,i)\in A} t_{ki} = \begin{cases} |V|-1 & \text{if } i = s \\ -1 & \text{otherwise} \end{cases} \qquad \forall i \in V \tag{1.15}$$

$$(|V|-1)u_{ij} \geq t_{ij} \qquad \forall (i,j) \in A \tag{1.16}$$

$$w_i \geq p_{ij} u_{ij} \qquad \forall (i,j) \in A \tag{1.17}$$

$$u_{ij} = u_{ji} \qquad \forall \{i,j\} \in E \tag{1.18}$$

$$u_{ij} \in \{0,1\} \qquad \forall (i,j) \in A \tag{1.19}$$

$$t_{ij} \geq 0 \qquad \forall (i,j) \in A \tag{1.20}$$

$$w_i \geq 0 \qquad \forall i \in V \tag{1.21}$$

(1.22)

Equations (1.15) define the flow problem on $t$ variables. Constraints (1.16) are the activators for $u$ variables, i.e. they connect $u$ and $t$ variables. Inequalities (1.17) connect $u$ variables to $w$ variables, while equations (1.18) force $u$ variables corresponding to the arcs of a same edge to assume the same value ($u$ have to be symmetric since they regulate transmission powers in (1.17)s). Constraints (1.19), (1.20) and (1.21) define variables' domains.

The main drawback of formulation $DAS$ is represented by constraints (1.16) and (1.17), which tend to push indicator variables $u$ to assume fractional values, making the mixed-integer program very difficult to solve.

### Valid inequalities

In Das et. al [8] some valid inequalities were also used. They all rely on the symmetric nature of indicator variables $u$.

**Theorem 2 (Connectivity inequalities 1).** *The set of inequalities*

$$\sum_{\substack{(i,j)\in A,\\ i\neq j}} u_{ij} \geq 1 \quad \forall i \in V \tag{1.23}$$

*is valid for formulation DAS.*

*Proof.* In order to be connected to the rest of the network, each node $i$ must be able to communicate with at least one other node, i.e. inequality (1.23) must be satisfied. □

**Theorem 3 (Connectivity inequalities 2).** *The set of inequalities*

$$\sum_{\substack{(i,j)\in A,\\ i\neq j}} u_{ji} \geq 1 \quad \forall i \in V \tag{1.24}$$

*is valid for formulation DAS.*

*Proof.* In order to be connected to the rest of the network, each node $i$ must receive the signal of at least one other node, i.e. inequality (1.24) must be satisfied. □

**Theorem 4 (Connectivity inequality 3).** *The inequality*

$$\sum_{\substack{(i,j)\in A,\\ i\neq j}} u_{ij} \geq 2(|V| - 1) \tag{1.25}$$

*is valid for formulation DAS.*

*Proof.* In order to have a topology connected by bidirectional links, there must be at least $2(|V| - 1)$ active indicator variables (i.e. the number of edges of a spanning tree times 2), as stated by constraint (1.25). □

### Exact algorithm

Formulation $DAS^R$ is defined to be formulation $DAS$ reinforced with the inequalities (1.23), (1.24) and (1.25). The exact algorithm described in Das et al. [8] works by directly solving formulation $DAS^R$. As observed in [8], experimental results suggest that solving $DAS^R$ instead of $DAS$ produces shorter computation times.

## 1.3.3 Montemanni and Gambardella [13] (a)

In Montemanni and Gambardella [13] a mathematical formulation, with some reinforcing inequalities, and an exact algorithm are presented. They are summarized in this section.

### Mathematical formulation

In order to describe this mathematical formulation, the following definition is required.

**Definition 2.** *Given* $(i, j) \in A$*, we define the* ancestor *of* $(i, j)$ *as*

$$a_j^i = \begin{cases} i & \textit{if } p_{ij} = \min_{k \in V}\{p_{ik}\} \\ \arg\max_{k \in V} \{p_{ik} | p_{ik} < p_{ij}\} & \textit{otherwise} \end{cases} \tag{1.26}$$

According to this definition, $(i, a_j^i)$ is the arc originated in node $i$ with the highest cost such that $p_{ia_j^i} < p_{ij}$[3]. In case an *ancestor* does not exist for arc $(i, j)$, vertex $i$ is returned, i.e. the dummy arc $(i, i)$ is addressed.

In the example of Figure 1.1, arc $(i, k)$ is the ancestor of arc $(i, m)$, $(i, j)$ is the ancestor of $(i, k)$ and the dummy arc $(i, i)$ is returned as the ancestor of $(i, j)$.

[3] For sack of simplicity, we have considered the (usual) case where $\forall i \in V \nexists k, l \in V$ s.t. $p_{ik} = p_{il}$. In case this is not true, the following formula, which breaks ties, has to be used in place of (1.26):

$$a_j^i = \begin{cases} i & \text{if } p_{ij} = \min_{k \in V}\{p_{ik}\} \\ \mathit{arg}\max_{k \in V} \left\{ p_{ik} \middle| \begin{pmatrix} (p_{ik} < p_{ij} \wedge (\nexists l \in V \text{ s.t. } p_{ik} = p_{il} \wedge l > k)) \\ \vee\ (p_{ij} = p_{ij} \wedge (\nexists l \in V \text{ s.t. } p_{ik} = p_{il} \wedge j > l > k)) \end{pmatrix} \right\} & \text{otherwise} \end{cases}$$

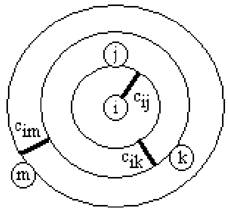


Figure 1.2: Incremental mechanism for costs.

The formulation is based on an incremental mechanism over the variables representing transmission powers. The costs associated with these variable in the objective function (1.28) will be given by the following formula:

$$c_{ij} = p_{ij} - p_{ia^i_j} \quad \forall (i,j) \in A \tag{1.27}$$

$c_{ij}$ is equal to the power required to establish a transmission from nodes $i$ to node $j$ ($p_{ij}$) minus the power required by nodes $i$ to reach node $a^i_j$ ($p_{ia^i_j}$). In Figure 1.2 the costs arising from the example of Figure 1.1 are depicted.

It is important to observe that the incremental mechanism is the most important element of the formulation. It will allow us to define very strong reinforcing inequalities, which are at the basis of the good performance of the exact algorithm based on the formulation (see Section 1.5).

The mixed integer programming formulation $MGa$, described in this section is based on a network flow model (see Magnanti and Wolsey [12]). A node $s$ is elected as the source of the flow, and one unit of flow is sent from $s$ to every other node. Variable $x_{ij}$ represents the flow on arc $(i, j)$. Variable $y_{ij}$ is 1 when node $i$ has a transmission power which allows it to reach node $j$, $y_{ij} = 0$ otherwise.

$$(MGa) \quad \text{Min} \sum_{(i,j)\in A} c_{ij} y_{ij} \tag{1.28}$$

$$\text{s.t.} \quad y_{ij} \le y_{ia^i_j} \qquad \forall (i,j) \in A, a^i_j \neq i \tag{1.29}$$

$$x_{ij} \le (|V|-1) y_{ij} \qquad \forall (i,j) \in A \tag{1.30}$$

$$x_{ij} \le (|V|-1) y_{ji} \qquad \forall (i,j) \in A \tag{1.31}$$

$$\sum_{(i,j)\in A} x_{ij} - \sum_{(k,i)\in A} x_{ki} = \begin{cases} |V|-1 & \text{if } i = s \\ -1 & \text{otherwise} \end{cases} \qquad \forall i \in V \tag{1.32}$$

$$x_{ij} \in \mathcal{R} \qquad \forall (i,j) \in A \tag{1.33}$$

$$y_{ij} \in \{0,1\} \qquad \forall (i,j) \in A \tag{1.34}$$

Constraints (1.29) realize the incremental mechanism by forcing the variables associated with arc $(i, a^i_j)$ to assume value 1 when the variable associated with arc $(i,j)$ has value 1, i.e. the arcs originated in the same node are activated in increasing order of $p$. Inequalities (1.30) and (1.31) connect the flow variables $x$ to $y$ variables. Equations (1.32) define the flow problem, while (1.33)s and (1.34)s are domain definition constraints. We refer the interested reader to Magnanti and Wolsey [12] for a more detailed description of the spanning tree formulation behind the formulation presented above.

In formulation $MGa$ the bottleneck is represented by constraints (1.30) and (1.31), which tend to push $y$ variables to be fractional. Fortunately, the incremental mechanism on which the mixed-integer program is based, allows us to define strong reinforcing inequalities that help to overcome this problem (see below).

**Valid inequalities**

The valid inequalities presented in this section were proposed in Montemanni and Gambardella [13] to reinforce mathematical formulation $MGa$.

In the remainder of this section we will refer to the subgraph of $G'$ defined by the $y$ variables with value 1 as $G_y$. Formally, $G_y = (V, A_y)$, where $A_y = \{(i,j) \in A | y_{ij} = 1 \text{ in the solution of } MPSC\}$.

**Theorem 5 (Connectivity inequalities).** *The set of inequalities*

$$y_{ij} = 1 \quad \forall (i,j) \in A \ s.t.\ a^i_j = i \tag{1.35}$$

*is valid for formulation MGa.*

*Proof.* In order to have the graph $G_y$ connected, each node must be able to communicate with at least one other node. Then its transmission power must be sufficient to reach at least the node which is closest to it, i.e. $y_{ia^i_j} = 1$. □

**Theorem 6 (Bidirectional inequalities 1).** *The set of inequalities*

$$y_{a^i_j i} \geq y_{ia^i_j} - y_{ij} \quad \forall (i,j) \in A \ s.t.\ a^i_j \neq i \tag{1.36}$$

*is valid for formulation MGa.*

*Proof.* If $y_{ij} = 1$ then $y_{ia^i_j} = 1$ because of inequalities (1.29) and consequently in this case the constraint does not give any new contribution.

If $y_{ij} = 0$ and $y_{ia^i_j} = 0$ then again the constraint does not give any new contribution.

If $y_{ij} = 0$ and $y_{ia^i_j} = 1$ then the transmission power of node $i$ is set to reach node $a^i_j$ and nothing more. The only reason for node $i$ to reach node $a^i_j$ and nothing more is the existence of a bidirectional link on edge $\{i, a^i_j\}$ in $G_y$. Consequently $y_{a^i_j i}$ must be equal to 1, as stated by the constraint. □

**Theorem 7 (Bidirectional inequalities 2).** *The set of inequalities*

$$y_{ji} \geq y_{ij} \ \forall (i,j) \in A \ s.t.\ \nexists (i,k) \in A, a^i_k = j \tag{1.37}$$

*is valid for formulation MGa.*

*Proof.* If $y_{ij} = 0$ the constraint does not give any new contribution.

If $y_{ij} = 1$ then the transmission power of node $i$ is set in such a way to reach node $j$, which is the farthest node from $i$ in $G$. The only reason for node $i$ to reach node $j$ is the existence of a bidirectional link on edge $\{i, j\}$ in $G_y$. Consequently $y_{ji}$ must be equal to 1, as stated by the constraint. □

**Theorem 8 (Tree inequality).** *The inequality*

$$\sum_{(i,j)\in A} y_{ij} \geq 2(|V| - 1) \tag{1.38}$$

*is valid for formulation MGa.*

*Proof.* In order to be strongly connected, the directed graph $G_y$ must have at least $2(|V|-1)$ arcs, as stated by constraint (1.38). □

**Definition 3.** $G_a = (V, A_a)$ *is the subgraph of the complete graph* $G'$ *such that* $A_a = \{(i, j)|\ a^i_j = i\}$.

Notice that $|A_a| = |V|$ by definition.

**Definition 4.** $\mathcal{R}_i = \{j \in V|\ j$ *can be reached from* $i$ *in* $G_a\}$.

**Theorem 9 (Reachability inequalities 1).** *The set of inequalities*

$$\sum_{\substack{(k,l)\in A \\ s.t.\ k\in\mathcal{R}_i, l\in V\setminus\mathcal{R}_i}} y_{kl} \geq 1 \quad \forall i \in V \tag{1.39}$$

*is valid for formulation MGa.*

*Proof.* Since graph $G_y$ must be strongly connected, it must be possible to reach every node $j$ starting from each node $i$. This implies that at least one arc must exist between the nodes which is possible to reach from $i$ in $G_a$ (i.e. $\mathcal{R}_i$) and the other nodes of the graph (i.e. $V\setminus\mathcal{R}_i$). □

**Definition 5.** $\mathcal{Q}_i = \{j \in V|\ i$ *can be reached from* $j$ *in* $G_a\}$.

**Theorem 10 (Reachability inequalities 2).** *The set of inequalities*

$$\sum_{\substack{(k,l)\in A \\ s.t.\ k\in\mathcal{Q}_i, l\in V\setminus\mathcal{Q}_i}} y_{kl} \geq 1 \quad \forall i \in V \tag{1.40}$$

*is valid for formulation $MGa$.*

*Proof.* Since graph $G_y$ must be strongly connected, it must be possible to reach every node $i$ from every other node $j$ of the graph. This means that at least one arc must exist between the nodes which cannot reach $i$ in $G_a$ (i.e. $V\setminus\mathcal{Q}_i$) and the other nodes of the graph (i.e. $\mathcal{Q}_i$). □

#### Exact algorithm

Formulation $MGa^R$ is defined to be formulation $MGa$ reinforced with the inequalities (1.35), (1.36), (1.37), (1.38), (1.39) and (1.40). The exact algorithm described in Montemanni and Gambardella [13] works by directly solving formulation $MGa^R$. In [13] it is shown that solving $MGa^R$ instead of $MGa$ produces computation times which are shorter up to a factor of 1920 for some problems.

### 1.3.4 Montemanni and Gambardella [13] (b)

In Montemanni and Gambardella [13] also a second mathematical formulation and a second exact algorithm are presented. They are summarized in this section together with a new valid inequality.

#### Mathematical formulation

The mathematical model described in this section is based on the same incremental mechanism discussed in Section 1.3.3. In formulation $MGb$ a spanning tree is defined by $z$ variables.

Variable $z_{ij}$ is 1 if edge $\{i,j\}$ is on the spanning tree, $z_{ij} = 0$ otherwise. Variable $y_{ij}$ is 1 when node $i$ has a transmission power which allows it to reach node $j$, $y_{ij} = 0$ otherwise.

$$(MGb) \quad \text{Min} \sum_{(i,j)\in A} c_{ij} y_{ij} \tag{1.41}$$

$$\text{s.t.} \quad y_{ij} \leq y_{ia^i_j} \qquad \forall (i,j) \in A, a^i_j \neq i \tag{1.42}$$

$$z_{ij} \leq y_{ij} \qquad \forall \{i,j\} \in E \tag{1.43}$$

$$z_{ij} \leq y_{ji} \qquad \forall \{i,j\} \in E \tag{1.44}$$

$$\sum_{\substack{\{i,j\}\in E,\\ i\in S, j\in V\setminus S}} z_{ij} \geq 1 \qquad \forall S \subset V \tag{1.45}$$

$$z_{ij} \in \{0,1\} \qquad \forall \{i,j\} \in E \tag{1.46}$$

$$y_{ij} \in \{0,1\} \qquad \forall (i,j) \in A \tag{1.47}$$

Constraints (1.42) realize the incremental mechanism by forcing the variables associated with arc $(i, a^i_j)$ to assume value 1 when the variable associated with arc $(i,j)$ has value 1, i.e. the arcs originated in the same node are activated in increasing order of $p$. Inequalities (1.43) and (1.44) connect the spanning tree variables $z$ to $y$ variables. Equations (1.45) state that all the vertices have to be mutually connected in the subgraph induced by $z$ variables, while (1.46) and (1.47) are domain definition constraints.

The main problem of formulation $MGb$ is related to the exponential number of constraints (1.45). In the exact algorithm described below, a technique to overcome this bottleneck is implemented.

### Valid inequalities

Since all the inequalities presented in Section 1.3.3 use $y$ variables only, and since the role of $y$ variables is the same in both formulations $MGa$ and $MGb$ (i.e. $y$ variables implement the incremental mechanism described in Section 1.3.3), the results presented in Section 1.3.3 for formulation $MGa$ are valid also for formulation $MGb$.

In addition, the following inequality is also considered. It will be used within the exact algorithm described in the next section.

**Theorem 11 ($z$ tree inequalities).**

$$\sum_{\{i,j\} \in E} z_{ij} \geq |V| - 1 \tag{1.48}$$

*Proof.* Inequality (1.48) forces the number of active $z$ variables to be at least $|V| - 1$. This condition is necessary in order to have a spanning tree. □

**Exact algorithm**

The integer program $MGb^R$ is defined as $MGb$ without constraints (1.45) but with the inequalities (1.35), (1.36), (1.37), (1.38), (1.39), (1.40) and (1.48). Notice that constraint (1.48) forces the active $z$ variables to be at least $|V| - 1$ already during the very first iterations of the method we are describing in this section. This will contribute to speed up the algorithm.

The idea at the basis of the method is that it is very difficult to deal directly with constraints (1.45) of formulation $MGb_R$ in case of large problems. For this reason some techniques which leave some of them out have to be considered. In this section we present an iterative approach which in the beginning does not consider any constraint (1.45), and adds them step by step in case they are violated. Formulation $MGb$ is solved and the values of the $z$ variables in the solution are examined. If the edges corresponding to variables with value 1 form a spanning tree then the problem has been solved to optimality, otherwise constraints (1.49), described below, are added to the integer program and the process is repeated.

At the end of each iteration, if edges corresponding to $z$ variables with value 1 in the last solution generate a set $\mathcal{CC}$ of connected components, with $|\mathcal{CC}| > 1$, then the following inequalities are added to the formulation:

$$\sum_{\substack{\{i,j\}\in E,\\ i\in C, j\in V\setminus C}} z_{ij} \geq 1 \quad \forall C \in \mathcal{CC} \tag{1.49}$$

Inequalities (1.49) force $z$ variables to connect the (elsewhere disjoint) connected components of $\mathcal{CC}$.

## 1.4 Preprocessing procedure

The results described in this section are used to delete some arcs of graph $G'$ and consequently to speed up the exact algorithms previously presented. They were originally presented in Montemanni and Gambardella [13].

We suppose an heuristic solution for the problem, *heu*, is available, and its cost is $cost(heu)$. All the variables that, if active, would induce a cost higher than $cost(heu)$ can be deleted from the problem.

**Theorem 12.** *If the following inequality holds*

$$p_{ij} + p_{ji} + \sum_{\substack{k\in V\setminus\{i,j\},\\ a_l^k = k}} p_{kl} > cost(heu) \tag{1.50}$$

*then arc $(i, j)$ can be deleted from $A$.*

*Proof.* Using the same intuition at the basis of the proofs of Theorems 6 and 7, we have that if $p_{ij}$ is the power of node $i$ in a solution, this means that the power of node $j$ must be greater than or equal to $p_{ji}$ (i.e. arc $(j, i)$ must be in the solution), because otherwise there would be no reason for node $i$ to reach node $j$. The left hand side of inequality (1.50) represents then a lower bound for the total power required in order to maintain the network connected in case node $i$ transmits to a power which allows it to reach node $j$ and nothing farther. For this reason, if inequality (1.50) holds, arc $(i, j)$ can be deleted from $A$. □

Table 1.1: Average computation times (sec) on the problems described in Althaus et al. [1].

| Algorithms | $\|V\|$ | | | | | | |
|---|---|---|---|---|---|---|---|
| | 10 | 15 | 20 | 25 | 30 | 35 | 40 |
| *AL* | 2.144 | 18.176 | 71.040 | 188.480 | 643.200 | 2278.400 | 15120.000 |
| *MGa* | 0.192 | 0.736 | 8.576 | 33.152 | 221.408 | 1246.304 | 9886.080 |
| Preprocessing + *MGa* | 0.078 | 0.289 | 0.715 | 4.924 | 28.908 | 87.357 | 583.541 |
| Preprocessing + *MGb* | **0.052** | **0.196** | **0.601** | **2.181** | **13.481** | **28.172** | **79.544** |

It is important to notice that once arc $(i, j)$ is deleted from $A$, the value of the ancestor of node $k$ (see Section 1.3.3), with $a_k^i = j$, has to be updated to $a_j^i$.

## 1.5 Computational results

Computational tests have been carried out on two different families of problems, randomly generated as described in Althaus et al. [1] and in Das et al. [8] respectively. In Althaus et al. [1] $\kappa = 4$ and a problem with $|V|$ nodes is obtained by choosing $|V|$ points uniformly at random from a grid of size 10000 $\times$ 10000. For the problems described in Das et al. [8] the procedure is the same, but the grid has dimension 5 $\times$ 5. In addition, for these last problems, a maximum transmission power, depending on the number of nodes of the network, is fixed. The following pairs (*number of nodes, maximum transmission power*) have been adopted: $(15, 3.00)$, $(20, 3.00)$, $(30, 2.50)$, $(40, 1.50)$, $(50, 0.75)$. ILOG CPLEX[4] 6.0 has been used to solve integer programs.

In the remainder of this section we will refer to the algorithm presented in Althaus et al. [1] (see Section 1.3.1) as *AL*, to the one described in Das et al. [8] (see Section 1.3.2) as *DAS*, and to those proposed in Montemanni and Gambardella [13] (see Sections 1.3.3 and 1.3.4) as *MGa* and *MGb* respectively.

In Table 1.1 we present the average computation times required (on a SUNW Ultra-30 machine) by some of the exact algorithms on the problems described in Althaus et al. [1], for different values of $V$. Fifty instances are considered for each value of $|V|$.

[4]http://www.cplex.com.

Table 1.2: Average computation times (sec) on the problems described in Das et al. [8].

| Algorithms | $\|V\|$ | | | | |
|---|---|---|---|---|---|
| | 15 | 20 | 30 | 40 | 50 |
| *DAS* | 0.014 (0.018) | 7.511 (36.697) | - | - | - |
| *MGa* | **0.008** (0.006) | **0.027** (0.013) | 1.518 (4.401) | 24.723 (111.378) | **12.233** (18.025) |
| *MGb* | 0.019 (0.010) | 0.058 (0.038) | **0.795** (1.093) | **9.906** (20.312) | 47.756 (136.234) |

Table 1.1 shows that the $MGa$ and $MGb$ outperform $AL$. $MGb$ also performs clearly better than $MGa$.

In Table 1.1 also the benefit derived from the use of the preprocessing technique described in Section 1.4 is highlighted. In order to apply this preprocessing procedure, a heuristic solution to the problem has to be available. For this purpose we use one of the simplest algorithms available, which works by calculating the *Minimum Spanning Tree* (see Prim [16]) on the weighted graph with costs defined by equation (1.1), and by assigning the power of each transmitter $i$ to $p_{ii_T}$, as described near the end of Section 1.2. The computational times of the algorithm $MGa$ are improved up to 17 times (for $|V| = 40$) when this technique is used (on average 79 % of the arcs were deleted for $|V| = 40$ - see Montemanni and Gambardella [13]).

In Table 1.2 we present the average computation times required (on a Pentium 4 1.5GHz machine) by some of the exact algorithms on the problems described in Das et al. [8], for different values of $V$. In brackets we also report the average standard deviation on solving times. Twentyfive instances are considered for each value of $|V|$. Some entries are marked with '-'. This means that the corresponding algorithms failed to solve some of the corresponding instances in less than 3600 seconds.

Table 1.2 suggests that again $MGa$ and $MGb$ obtain the best performance. For these problems the algorithms highlight also that all the algorithms are not extremely robust (see large standard deviation on solution times), i.e. there are very different performance on instances of the same family. This could depend on the small grid adopted, which tends to flatten down power requirements, and this causes many almost equivalent solutions. On the other hand, average computation times are much shorter than those reported in Table 1.1, and this depends on the maximum transmission power constraints, that substantially contribute to reduce the number of variables of the problems.

## 1.6 Conclusion

We have presented an overview of the mathematical formulation presented so far for the min-power symmetric connectivity problem in wireless networks. Some exact algorithms, strongly based on these formulations and on some reinforcing inequalities developed for them, have been discussed, together a preprocessing rule.

Computational results have finally been presented, aiming to compare the performance of the different exact approaches discussed in the paper.

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