

Parameterized Approximation Schemes for Independent Set of Rectangles and Geometric Knapsack

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Abstract

The area of parameterized approximation seeks to combine approximation and parameterized algorithms to obtain, e.g., $(1 + \varepsilon)$ -approximations in $f(k, \varepsilon)n^{O(1)}$ time where k is some *parameter* of the input. The goal is to overcome lower bounds from either of the areas. We obtain the following results on parameterized approximability:

- In the *maximum independent set of rectangles* problem (MISR) we are given a collection of n axis parallel rectangles in the plane. Our goal is to select a maximum-cardinality subset of pairwise non-overlapping rectangles. This problem is NP-hard and also W[1]-hard [Marx, ESA'05]. The best-known polynomial-time approximation factor is $O(\log \log n)$ [Chalermsook and Chuzhoy, SODA'09] and it admits a QPTAS [Adamaszek and Wiese, FOCS'13; Chuzhoy and Ene, FOCS'16]. Here we present a *parameterized approximation scheme* (PAS) for MISR, i.e. an algorithm that, for any given constant $\varepsilon > 0$ and integer $k > 0$, in time $f(k, \varepsilon)n^{g(\varepsilon)}$, either outputs a solution of size at least $k/(1 + \varepsilon)$, or declares that the optimum solution has size less than k .
- In the (*2-dimensional*) *geometric knapsack* problem (2DK) we are given an axis-aligned square knapsack and a collection of axis-aligned rectangles in the plane (*items*). Our goal is to translate a maximum cardinality subset of items into the knapsack so that the selected items do not overlap. In the version of 2DK with rotations (2DKR), we are allowed to rotate items by 90 degrees. Both variants are NP-hard, and the best-known polynomial-time approximation factor is $2 + \varepsilon$ [Jansen and Zhang, SODA'04]. These problems admit a QPTAS for polynomially bounded item sizes [Adamaszek and Wiese, SODA'15]. We show that both variants are W[1]-hard. Furthermore, we present a PAS for 2DKR.

For all considered problems, getting time $f(k, \varepsilon)n^{O(1)}$, rather than $f(k, \varepsilon)n^{g(\varepsilon)}$, would give FPT time $f'(k)n^{O(1)}$ exact algorithms by setting $\varepsilon = 1/(k + 1)$, contradicting W[1]-hardness. Instead, for each fixed $\varepsilon > 0$, our PASs give $(1 + \varepsilon)$ -approximate solutions in FPT time.

For both MISR and 2DKR our techniques also give rise to preprocessing algorithms that take $n^{g(\varepsilon)}$ time and return a subset of at most $k^{g(\varepsilon)}$ rectangles/items that contains a solution of size at least $k/(1 + \varepsilon)$ if a solution of size k exists. This is a special case of the recently introduced notion of a polynomial-size approximate kernelization scheme [Lokshtanov et al., STOC'17].

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1 Introduction

Approximation algorithms and parameterized algorithms are two well-established ways to deal with NP-hard problems. An α -approximation for an optimization problem is a polynomial-time algorithm that computes a *feasible* solution whose cost is within a factor α (that might be a function of the input size n) of the optimal cost. In particular, a *polynomial-time approximation scheme* (PTAS) is a $(1 + \varepsilon)$ -approximation algorithm running in time $n^{g(\varepsilon)}$, where $\varepsilon > 0$ is a given constant and g is some computable function. In parameterized algorithms we identify a parameter k of the input, that we informally assume to be much smaller than n . The goal here is to solve the problem optimally in *fixed-parameter tractable* (FPT) time $f(k)n^{O(1)}$, where f is some computable function. Recently, researchers started to combine the two notions (see, e.g., the survey by Marx [34]). The idea is to design approximation algorithms that run in FPT (rather than polynomial) time, e.g., to get $(1 + \varepsilon)$ -approximate solutions in time $f(k, \varepsilon)n^{O(1)}$. In this paper we continue this line of research on parameterized approximation, and apply it to two fundamental rectangle packing problems.

1.1 Our results and techniques

Our focus is on parameterized approximation algorithms. Unfortunately, as observed by Marx [34], when the parameter k is the desired solution size, computing $(1 + \varepsilon)$ -approximate solutions in time $f(k, \varepsilon)n^{O(1)}$ implies fixed-parameter tractability. Indeed, setting $\varepsilon = 1/(k+1)$ guarantees to find an optimal solution when that value equals to $k \in \mathbb{N}$ and we get time $f(k, 1/(k+1))n^{O(1)} = f'(k)n^{O(1)}$. Since the considered problems are W[1]-hard (in part, this is established in our work), they are unlikely to be FPT and similarly unlikely to have such nice approximation schemes.

Instead, we construct algorithms (for two maximization problems) that, given $\varepsilon > 0$ and an integer k , take time $f(k, \varepsilon)n^{g(\varepsilon)}$ and either return a solution of size at least $k/(1 + \varepsilon)$ or declare that the optimum is less than k . We call such an algorithm a *parameterized approximation scheme* (PAS). Note that if we run such an algorithm for each $k' \leq k$ then we can guarantee that we compute a solution with cardinality at least $\min\{k, \text{OPT}\}/(1 + \varepsilon)$ where OPT denotes the size of the optimal solution. So intuitively, for each $\varepsilon > 0$, we have an FPT-algorithm for getting a $(1 + \varepsilon)$ -approximate solution.

In this paper we consider the following two geometric packing problems, and design PASs for them.

Maximum Independent Set of Rectangles. In the *maximum independent set of rectangles* problem (MISR) we are given a set of n axis-parallel rectangles $\mathcal{R} = \{R_1, \dots, R_n\}$ in the two-dimensional plane, where R_i is the open set of points $(x_i^{(1)}, x_i^{(2)}) \times (y_i^{(1)}, y_i^{(2)})$. A feasible solution is a subset of rectangles $\mathcal{R}' \subseteq \mathcal{R}$ such that for any two rectangles $R, R' \in \mathcal{R}'$ we have $R \cap R' = \emptyset$. Our objective is to find a feasible solution of maximum cardinality $|\mathcal{R}'|$. W.l.o.g. we assume that $x_i^{(1)}, y_i^{(1)}, x_i^{(2)}, y_i^{(2)} \in \{0, \dots, 2n - 1\}$ for each $R_i \in \mathcal{R}$ (see e.g. [1]).

MISR is very well-studied in the area of approximation algorithms. The problem is known to be NP-hard [24], and the current best polynomial-time approximation factor is

$O(\log \log n)$ for the cardinality case [11] (addressed in this paper), and $O(\log n / \log \log n)$ for the natural generalization with rectangle weights [12]. The cardinality case also admits a $(1 + \varepsilon)$ -approximation with a running time of $n^{\text{poly}(\log \log(n/\varepsilon))}$ [15] and there is a (slower) QPTAS known for the weighted case [1]. The problem is also known to be W[1]-hard w.r.t. the number k of rectangles in the solution [33], and thus unlikely to be solvable in FPT time $f(k)n^{O(1)}$.

In this paper we achieve the following main result:

► **Theorem 1.** *There is a PAS for MISR with running time $k^{O(k/\varepsilon^8)}n^{O(1/\varepsilon^8)}$.*

In order to achieve the above result, we combine several ideas. Our starting point is a polynomial-time construction of a $k \times k$ grid such that each rectangle in the input contains some crossing point of this grid (or we find a solution of size k directly). By applying (in a non-trivial way) a result by Frederickson [21] on planar graphs, and losing a small factor in the approximation, we define a decomposition of our grid into a collection of disjoint *groups* of cells. Each such group defines an independent instance of the problem, consisting of the rectangles strictly contained in the considered group of cells. Furthermore, we guarantee that each group spans only a *constant number* $O_\varepsilon(1)$ of rectangles of the optimum solution. Therefore in FPT time we can guess the correct decomposition, and solve each corresponding subproblem in $n^{O_\varepsilon(1)}$ time. We remark that our approach deviates substantially from prior work, and might be useful for other related problems.

An adaptation of our construction also leads to the following $(1 + \varepsilon)$ -approximative kernelization.

► **Theorem 2.** *There is an algorithm for MISR that, given $k \in \mathbb{N}$, computes in time $n^{O(1/\varepsilon^8)}$ a subset of the input rectangles of size $k^{O(1/\varepsilon^8)}$ that contains a solution of size at least $k/(1 + \varepsilon)$, assuming that the input instance admits a solution of size at least k .*

Similarly as for a PAS, if we run the above algorithm for each $k' \leq k$ we obtain a set of size $k^{O(1/\varepsilon^8)}$ that contains a solution of size at least $\min\{k, \text{OPT}\}/(1 + \varepsilon)$. Observe that any c -approximate solution on the obtained set of rectangles is also a feasible, and $c(1 + \varepsilon)$ -approximate, solution for the original instance if $\text{OPT} \leq k$ and otherwise has size at least $k/(c(1 + \varepsilon))$. Thus, our result is a special case of a *polynomial-size approximate kernelization scheme* (PSAKS) as defined in [32].¹

2-Dimensional Geometric Knapsack. In the (*2-Dimensional*) *Geometric Knapsack* problem (2DK) we are given a square *knapsack* $[0, N] \times [0, N]$, $N \in \mathbb{N}$, and a set of n items I , where each item $i \in I$ is an open rectangle $(0, w_i) \times (0, h_i)$, $N \geq w_i, h_i \in \mathbb{N}$. The goal is to find a feasible *packing* of a subset $I' \subseteq I$ of the items of maximum cardinality $|I'|$. Such packing maps each item $i \in I'$ into a new translated rectangle $(a_i, a_i + w_i) \times (b_i, b_i + h_i)^2$, so that the translated rectangles are fully contained in the knapsack and do not overlap with each other. Here we also consider a variant of 2DK *with rotations* (2DKR) where we can rotate each input rectangle by 90 degrees.

Both 2DK and 2DKR are NP-hard [31] and admit a polynomial-time $(2 + \varepsilon)$ -approximation for any constant $\varepsilon > 0$ [28]. These problems admit a QPTAS if $N = n^{O(1)}$ [2]. Somewhat surprisingly, these problems are not known to be W[1]-hard when parameterized by the

¹ The definition due to Lokshtanov et al. [32] is not restricted to generating a small subset of the input and a dedicated solution lifting algorithm may be used.

² Intuitively, i is shifted by a_i to the right and by b_i to the top.

output number k of items. Note that showing $W[1]$ -hardness is important in our case to motivate the search for a PAS.

► **Theorem 3.** *2DK and 2DKR are $W[1]$ -hard when parameterized by k .*

The result is proved by parameterized reductions from a variant of the $W[1]$ -hard SUBSET SUM problem, where we need to determine whether a set of m positive integers contains a k -tuple of numbers with sum equal to some given value t . The difficulty for reductions to 2DK or 2DKR is of course that rectangles may be freely selected and placed (and possibly rotated) to get a feasible packing.

We complement the $W[1]$ -hardness result by giving a PAS for the case with rotations (2DKR) and a corresponding kernelization procedure like in Theorem 2 (which also yields a PSAKS).

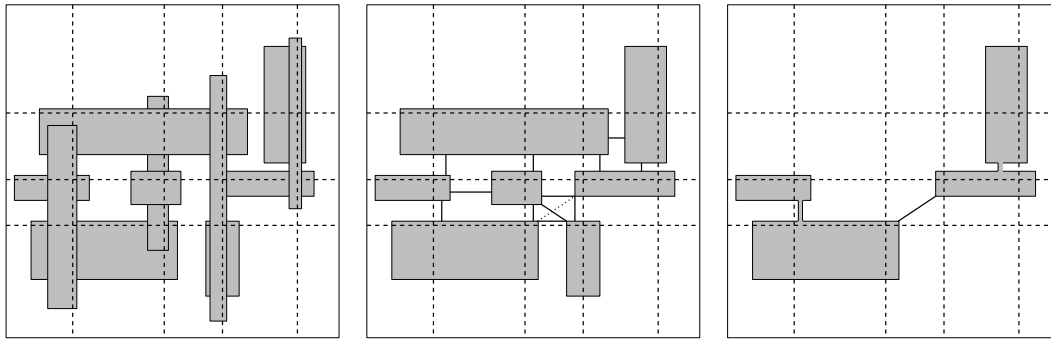
► **Theorem 4.** *For 2DKR there is a PAS with running time $k^{O(k/\epsilon)}n^{O(1/\epsilon^3)}$ and an algorithm that, given $k \in \mathbb{N}$, computes in time $n^{O(1/\epsilon^3)}$ a subset of the input items of size $k^{O(1/\epsilon)}$ that contains a solution of size at least $k/(1 + \epsilon)$, assuming that the input instance admits a solution of size at least k .*

The above result is based on a simple combination of the following two (non-trivial) building blocks: First, we show that, by losing a fraction ϵ of the items of a given solution of size k , it is possible to free a vertical strip of width $N/k^{O_\epsilon(1)}$ (unless the problem can be solved trivially). This is achieved by first sparsifying the solution using the above mentioned result by Frederickson [21]. If this is not sufficient we construct a *vertical chain* of relatively wide and tall rectangles that split the instance into a left and right side. Then we design a *resource augmentation* algorithm, however in an FPT sense: we can compute in FPT time a packing of cardinality k if we are allowed to use a knapsack where one side is enlarged by a factor $1 + 1/k^{O_\epsilon(1)}$. Note that in typical resource augmentation results the packing constraint is relaxed by a constant factor while here this amount is controlled by our parameter.

1.2 Related work

One of the first fruitful connections between parameterized complexity and approximability was observed independently by Bazgan [3] and Cesati and Trevisan [10]: They showed that EPTASs, i.e., $(1 + \epsilon)$ -approximation algorithms with $f(\epsilon)n^{O(1)}$ time, imply fixed-parameter tractability for the decision version. Thus, proofs for $W[1]$ -hardness of the decision version became a strong tool for ruling out improvements of PTASs, with running time $n^{g(\epsilon)}$, to EPTASs. More recently, Boucher et al. [8] improved this approach by directly proving $W[1]$ -hardness of obtaining a $(1 + \epsilon)$ -approximation, thus bypassing the requirement of a $W[1]$ -hard decision version (see also [17]).

The systematic study of parameterized approximation as a field was initiated independently by three separate publications [9, 13, 19]. A very good introduction to the area including key definitions as well as a survey of earlier results that fit into the picture was given by Marx [34]. In particular, Marx also defined a so-called *standard FPT-approximation algorithm (with performance ratio c)* that, given input (x, k) will run for $f(k)|x|^{O(1)}$ time and return (say, for a maximization problem) a solution of value at least k/c if the optimum is at least k . As mentioned earlier, Marx pointed out that a standard FPT-approximation scheme that finds a solution of value at least $k/(1 + \epsilon)$ in time $f(k, \epsilon)|x|^{O(1)}$ if $\text{OPT} \geq k$ is not interesting to study: By setting $\epsilon = 1/(k + 1)$ we can decide the decision problem “ $\text{OPT} \geq k$?” in FPT time. Thus, such a scheme is not helpful if the decision problem is $W[1]$ -hard and therefore unlikely to have an FPT-algorithm. Nevertheless, PASs can be



■ **Figure 1** (Left) Dashed lines define the grid \mathcal{G} . (Middle) Rectangles from an optimal solution and the edges that form the graph G_1 . Note that in G_1 there is no edge representing the dotted connection since otherwise the graph would not be planar anymore. (Right) The graph G_2 , that captures the missing connections of G_1

useful in this case, as they imply standard FPT-approximation algorithms with ratio $1 + \varepsilon$ for each fixed $\varepsilon > 0$ despite $W[1]$ -hardness.

A central goal of parameterized approximation is to settle the status of problems like DOMINATING SET or CLIQUE, which are hard to approximate and also parameterized intractable. Recently, Chen and Lin [14] made important progress by showing that DOMINATING SET admits no constant-factor approximation with running time $f(k)n^{O(1)}$ unless $FPT = W[1]$. Generally, for problems without exact FPT-algorithms, the goal is to find out whether one can beat inapproximability bounds by allowing FPT-time in some parameter; see e.g. [23, 4, 5, 6, 30, 29, 16, 22, 7]).

For the special case of MISR where all input objects are squares a PTAS is known [20] but there can be no EPTAS [33]. Recently, Galvez et al. [25] found polynomial-time algorithms for 2DK and 2DKR with approximation ratio smaller than 2 (also for the weighted case). For the special case that all input objects are squares there is a PTAS [27] and even an EPTAS [26].

2 A Parameterized Approximation Scheme for MISR

In this section we present a PAS and an approximate kernelization for MISR. We start by showing that there exists an almost optimal solution for the problem with some helpful structural properties (Sections 2.1 and 2.2). The results are then put together in Section 2.3.

2.1 Definition of the grid

We try to construct a non-uniform grid with k rows and k columns such that each input rectangle overlaps a corner of this grid (see Figure 1). To this end, we want to compute $k - 1$ vertical and $k - 1$ horizontal lines such that each input rectangle intersects one line from each set. There are instances in which our routine fails to construct such a grid (and in fact such a grid might not even exist). For such instances, we directly find a feasible solution with k rectangles and we are done.

► **Lemma 5.** *There is a polynomial time algorithm that either computes a set of at most $k - 1$ vertical lines \mathcal{L}_V with x -coordinates $\ell_1^V, \dots, \ell_{k-1}^V$ such that each input rectangle is crossed by one line in \mathcal{L}_V or computes a feasible solution with k rectangles. A symmetric*

statement holds for an algorithm computing a set of at most $k - 1$ horizontal lines \mathcal{L}_H with y -coordinates $\ell_1^H, \dots, \ell_{k-1}^H$.

Proof. Let $\ell_0^V := 0$. Assume inductively that we defined the x -coordinates $\ell_0^V, \ell_1^V, \dots, \ell_{k'}^V$ such that $\ell_1^V, \dots, \ell_{k'}^V$ are the x -coordinates of the first k' constructed vertical lines. We define the x -coordinate of the $(k' + 1)$ -th vertical line by $\ell_{k'+1}^V := \min_{R_i \in \mathcal{R}: x_i^{(1)} \geq \ell_{k'}^V} x_i^{(2)} - 1/2$. We continue with this construction until we reach an iteration k^* such that $\{R_i \in \mathcal{R} : x_i^{(1)} \geq \ell_{k^*-1}^V\} = \emptyset$. If $k^* \leq k$ then we constructed at most $k - 1$ lines such that each input rectangle is intersected by one of these lines. Otherwise, assume that $k^* > k$. Then for each iteration $k' \in \{1, \dots, k\}$ we can find a rectangle $R_{i(k')} := \arg \min_{R_i \in \mathcal{R}: x_i^{(1)} \geq \ell_{k'-1}^V} x_i^{(2)}$. By construction, using the fact that all coordinates are integer, for any two such rectangles $R_{i(k')}, R_{i(k'')}$ with $k' \neq k''$ we have that $(x_{i(k')}^{(1)}, x_{i(k')}^{(2)}) \cap (x_{i(k'')}^{(1)}, x_{i(k'')}^{(2)}) = \emptyset$. Hence, $R_{i(k')}$ and $R_{i(k'')}$ are disjoint. Therefore, the rectangles $R_{i(1)}, \dots, R_{i(k)}$ are pairwise disjoint and thus form a feasible solution.

The algorithm for constructing the horizontal lines works symmetrically. \blacktriangleleft

We apply the algorithms due to Lemma 5. If one of them finds a set of k independent rectangles then we output them and we are done. Otherwise, we obtain the sets \mathcal{L}_V and \mathcal{L}_H . For convenience, we define two more vertical lines with x -coordinates $\ell_0^V := 0$ and $\ell_{|\mathcal{L}_V|+1}^V = 2n - 1$, resp., and similarly two more horizontal lines with y -coordinates $\ell_0^H = 0$ and $\ell_{|\mathcal{L}_H|+1}^H = 2n - 1$, resp.. We denote by \mathcal{G} the set of grid cells formed by these lines and the lines in $\mathcal{L}_V \cup \mathcal{L}_H$: for any two consecutive vertical lines (i.e., defined via x -coordinates ℓ_j^V, ℓ_{j+1}^V with $j \in \{0, \dots, |\mathcal{L}_V|\}$) and two consecutive horizontal grid lines (defined via y -coordinates $\ell_{j'}^H, \ell_{j'+1}^H$ with $j' \in \{0, \dots, |\mathcal{L}_H|\}$) we obtain a grid cell whose corners are the intersection of these respective lines. We interpret the grid cells as closed sets (i.e., two adjacent grid cells intersect on their boundary).

► **Proposition 6.** *Each input rectangle R_i contains a corner of a grid cell of \mathcal{G} . If a rectangle R intersects a grid cell g then it must contain a corner of g .*

2.2 Groups of rectangles

Let \mathcal{R}^* denote a solution to the given instance with $|\mathcal{R}^*| = k$. We prove that there is a special solution $\mathcal{R}' \subseteq \mathcal{R}^*$ of large cardinality that we can partition into $s \leq k$ groups $\mathcal{R}'_1 \dot{\cup} \dots \dot{\cup} \mathcal{R}'_s$ such that each group has constant size $O(1/\epsilon^8)$ and no grid cell can be intersected by rectangles from different groups. The remainder of this section is devoted to proving the following lemma.

► **Lemma 7.** *There is a constant $c = O(1/\epsilon^8)$ such that there exists a solution $\mathcal{R}' \subseteq \mathcal{R}^*$ with $|\mathcal{R}'| \geq (1 - \epsilon)|\mathcal{R}^*|$ and a partition $\mathcal{R}' = \mathcal{R}'_1 \dot{\cup} \dots \dot{\cup} \mathcal{R}'_s$ with $s \leq k$ and $|\mathcal{R}'_j| \leq c$ for each j and such that if any two rectangles in \mathcal{R}' intersect the same grid cell $g \in \mathcal{G}$ then they are contained in the same set \mathcal{R}'_j .*

Given the solution \mathcal{R}^* we construct a planar graph $G_1 = (V_1, E_1)$. In V_1 we have one vertex v_i for each rectangle $R_i \in \mathcal{R}^*$. We connect two vertices $v_i, v_{i'}$ by an edge if and only if there is a grid cell $g \in \mathcal{G}$ such that R_i and $R_{i'}$ intersect g and

- R_i and $R_{i'}$ are crossed by the same horizontal or vertical line in $\mathcal{L}_V \cup \mathcal{L}_H$ or if
- R_i and $R_{i'}$ contain the top left and the bottom right corner of g , resp.

Note that we do not introduce an edge if R_i and $R_{i'}$ contain the bottom left and the top right corner of g , resp. (see Fig. 1): this way we preserve the planarity of the resulting graph, however we will have to deal with the missing connections in a later stage.

► **Lemma 8.** *The graph G_1 is planar.*

Next, we use a result by Frederickson [21] to obtain a subgraph G'_1 of G_1 in which each connected component has constant size.

► **Lemma 9.** *Let $\epsilon' > 0$. There exists a value $c' = O(1/(\epsilon')^2)$ such that the following holds: let $G = (V, E)$ be a planar graph. There exists a set of vertices $V' \subseteq V$ with $|V'| \geq (1 - \epsilon')|V|$ such that in the graph $G' := G[V']$ each connected component has at most c' vertices.*

Let G'_1 be the graph obtained when applying Lemma 9 to G_1 with $\epsilon' := \epsilon/2$ and let $c_1 = O((1/\epsilon)^2)$ be the respective value c' . Now we would like to claim that if two rectangles $R_i, R_{i'}$ intersect the same grid cell $g \in \mathcal{G}$ then $v_i, v_{i'}$ are in the same component of G'_1 . Unfortunately, this is not true. It might be that there is a grid cell $g \in \mathcal{G}$ such that R_i and $R_{i'}$ contain the bottom left corner and the top right corner of g , resp., and that v_i and $v_{i'}$ are in different components of G'_1 . We fix this in a second step. We define a graph $G_2 = (V_2, E_2)$. In V_2 we have one vertex for each connected component in G'_1 . We connect two vertices $w_i, w_{i'} \in V_2$ by an edge if and only if there are two rectangles $R_i, R_{i'}$ such that their corresponding vertices $v_i, v_{i'}$ in V_1 belong to the connected components of G'_1 represented by w_i and $w_{i'}$, resp., and there is a grid cell g whose bottom left and top right corner are contained in R_i and $R_{i'}$, resp.

► **Lemma 10.** *The graph G_2 is planar.*

Similarly as above, we apply Lemma 9 to G_2 with $\epsilon' := \frac{\epsilon}{2c_1}$ and let $c_2 = O((1/\epsilon')^2) = O(1/\epsilon^6)$ denote the corresponding value of c' . Denote by G'_2 the resulting graph. We define a group \mathcal{R}'_q for each connected component \mathcal{C}_q of V'_2 . The set \mathcal{R}'_q contains all rectangles R_i such that v_i is contained in a connected component C_j of G'_1 such that $w_j \in \mathcal{C}_q$. We define $\mathcal{R}' := \dot{\cup}_q \mathcal{R}'_q$.

► **Lemma 11.** *Let $R_i, R_{i'} \in \mathcal{R}'$ be rectangles that intersect the same grid cell $g \in \mathcal{G}$. Then there is a set \mathcal{R}'_q such that $\{R_i, R_{i'}\} \subseteq \mathcal{R}'_q$.*

Proof. Assume that in G_1 there is an edge connecting $v_i, v_{i'}$. Then the latter vertices are in the same connected component $C_{j'}$ of G'_1 and thus they are in the same group \mathcal{R}'_q . Otherwise, if there is no edge connecting $v_i, v_{i'}$ in G_1 then R_i and $R_{i'}$ contain the bottom left and top right corners of g , resp. Assume that v_i and $v_{i'}$ are contained in the connected components C_j and $C_{j'}$ of G'_1 , resp. Then $w_j, w_{j'} \in V'_2$, $\{w_j, w_{j'}\} \in E_2$ and $w_j, w_{j'}$ are in the same connected component of V'_2 . Hence, $R_i, R_{i'}$ are in the same group \mathcal{R}'_q . ◀

It remains to prove that each group \mathcal{R}'_q has constant size and that $|\mathcal{R}'| \geq (1 - \epsilon)|\mathcal{R}^*|$.

► **Lemma 12.** *There is a constant $c = O(1/\epsilon^8)$ such that for each group \mathcal{R}'_q it holds that $|\mathcal{R}'_q| \leq c$.*

Proof. For each group \mathcal{R}'_q there is a connected component \mathcal{C}_q of G'_2 such that \mathcal{R}'_q contains all rectangles R_i such that v_i is contained in a connected component C_j of G'_1 and $w_j \in \mathcal{C}_q$. Each connected component of G'_1 contains at most $c_1 = O(1/\epsilon^2)$ vertices of V'_1 and each component of G'_2 contains at most $c_2 = O(1/\epsilon^6)$ vertices of V'_2 . Hence, $|\mathcal{R}'_q| \leq c_1 \cdot c_2 =: c$ and $c = O((1/\epsilon^2)(1/\epsilon^6)) = O(1/\epsilon^8)$. ◀

► **Lemma 13.** *We have that $|\mathcal{R}'| \geq (1 - \epsilon)|\mathcal{R}^*|$.*

Proof. At most $\frac{\epsilon}{2} \cdot |V_1|$ vertices of G_1 are deleted when we construct G'_1 from G_1 . Each vertex in G'_1 belongs to one connected component C_j , represented by a vertex $w_j \in G_2$. At most $\frac{\epsilon}{2c_1} |V_2|$ vertices are deleted when we construct G'_2 from G_2 . These vertices represent at most $c_1 \cdot \frac{\epsilon}{2c_1} |V_2| \leq \frac{\epsilon}{2} |V'_1| \leq \frac{\epsilon}{2} |V_1|$ vertices in G_1 (and each vertex in G_1 represents one rectangle in \mathcal{R}^*). Therefore, $|\mathcal{R}'| \geq |\mathcal{R}^*| - \frac{\epsilon}{2} \cdot |V_1| - \frac{\epsilon}{2} \cdot |V_1| = (1 - \epsilon)|\mathcal{R}^*|$. ◀

This completes the proof of Lemma 7.

2.3 The algorithm

In our algorithm, we compute a solution that is at least as good as the solution \mathcal{R}' as given by Lemma 7. For each group \mathcal{R}'_j we define by \mathcal{G}_j the set of grid cells that are intersected by at least one rectangle from \mathcal{R}'_j . Since in \mathcal{R}' each grid cell can be intersected by rectangles of only one group, we have that $\mathcal{G}_j \cap \mathcal{G}_q = \emptyset$ if $j \neq q$. We want to guess the sets \mathcal{G}_j . The next lemma shows that the number of possibilities for one of those sets is polynomially bounded in k .

► **Lemma 14.** *Each \mathcal{G}_j belongs to a set \mathcal{G} of cardinality at most $k^{O(1/\epsilon^8)}$ that can be computed in polynomial time.*

Proof. The cells \mathcal{G}_j intersected by \mathcal{R}'_j are the union of all cells $\mathcal{G}(R)$ with $R \in \mathcal{R}'_j$ where for each rectangle R the set $\mathcal{G}(R)$ denotes the cells intersected by R . Each set $\mathcal{G}(R)$ can be specified by indicating the 4 *corner* cells of $\mathcal{G}(R)$, i.e., top-left, top-right, bottom-left, and bottom-right corner. Hence there are at most k^4 choices for each such R . The claim follows since $|\mathcal{R}'_j| = O(1/\epsilon^8)$. ◀

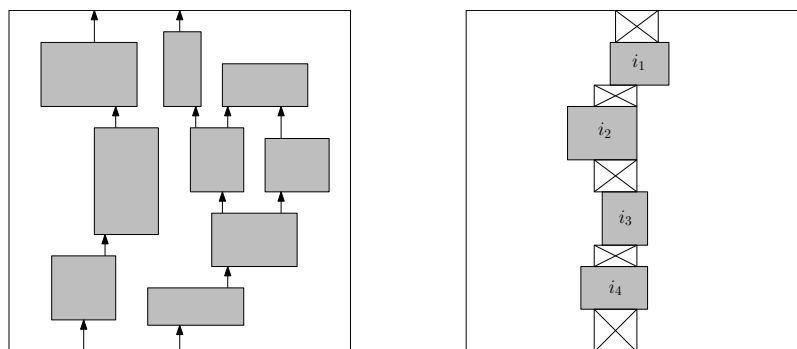
We hence achieve the main result of this section.

Proof of Theorem 1. Using Lemma 14, we can guess by exhaustive enumeration all the sets \mathcal{G}_j in time $k^{O(k/\epsilon^8)}$. We obtain one independent problem for each value $j \in \{1, \dots, s\}$ which consists of all input rectangles that are contained in \mathcal{G}_j . For this subproblem, it suffices to compute a solution with at least $|\mathcal{R}'_j|$ rectangles. Since $|\mathcal{R}'_j| \leq c = O(1/\epsilon^8)$ we can do this in time $n^{O(1/\epsilon^8)}$ by complete enumeration. Thus, we solve each of the subproblems and output the union of the computed solutions. The overall running time is as in the claim. If all the computed solutions have size less than $(1 - \epsilon)k$, this implies that the optimum solution is smaller than k . Otherwise we obtain a solution of size at least $(1 - \epsilon)k \geq k/(1 + 2\epsilon)$ and the claim follows by redefining ϵ appropriately. ◀

Essentially the same construction as above also gives an approximate kernelization algorithm as claimed in Theorem 2, see the full version of this work for details.

3 A Parameterized Approximation Scheme for 2DKR

In this section we present a PAS and an approximate kernelization for 2DKR. W.l.o.g., we assume that $k \geq \Omega(1/\epsilon^3)$, since otherwise we can optimally solve the problem in time $n^{O(1/\epsilon^3)}$ by exhaustive enumeration. In Section 3.1 we show that, if a solution of size k exists, there is a solution of size at least $(1 - \epsilon)k$ in which no item intersects some horizontal strip $(0, N) \times (0, (1/k)^{O(1/\epsilon)}N)$ at the bottom of the knapsack. In Section 3.2 we show that, if there exists a solution of size k' that does not use the mentioned strip, then we can compute in polynomial time a set of size $(k')^{O(1/\epsilon)}$ that contains a solution of size k' (where we are allowed to use the full knapsack). Combining these two results gives Theorem 4.



■ **Figure 2** The left figure shows the arcs of the graph G . Each item corresponds to one vertex of the graph. The right figure shows the items i_1, \dots, i_K and the deletion rectangles between them.

3.1 Freeing a Horizontal Strip

In this section, we prove the following lemma that shows the existence of a near-optimal solution that leaves a sufficiently tall empty horizontal strip in the knapsack (assuming $k \geq \Omega(1/\epsilon^3)$). W.l.o.g., $\epsilon \leq 1$. Since we can rotate the items by 90 degrees, we can assume w.l.o.g. that $w_i \geq h_i$ for each item $i \in I$.

► **Lemma 15.** *Let $k \in \mathbb{N}$, $k = \Omega(1/\epsilon^3)$, and $\epsilon > 0$. Given an instance of 2DKR with a solution of size k , there exists a solution of size at least $(1 - \epsilon)k$ in which no packed item intersects $(0, N) \times (0, (1/k)^c N)$, for a proper constant $c = O(1/\epsilon)$.*

We classify items into large and thin items. Via a shifting argument, we get the following lemma.

► **Lemma 16.** *There is an integer $B \in \{1, \dots, \lceil 8/\epsilon \rceil\}$ such that by losing a factor of $1 + \epsilon$ in the objective we can assume that the input items are partitioned into*

- large items L such that $h_i \geq (1/k)^B N$ (and thus also $w_i \geq (1/k)^B N$) for each item $i \in L$,
- thin items T such that $h_i < (1/k)^{B+2} N$ for each item $i \in T$.

Let B be the integer due to Lemma 16 and we work with the resulting item classification. If $|T| \geq k$ then we can create a solution of size k satisfying the claim of Lemma 15 by simply stacking k thin items on top of each other: any k thin items have a total height of at most $k \cdot (1/k)^{B+2} N \leq (1/k)^2 N$. Thus, from now on assume that $|T| < k$.

Sparsifying large items. Our strategy is now to delete some of the large items and move the remaining items. This will allow us to free the area $[0, N] \times [0, (1/k)^{O(1/\epsilon)} N]$ of the knapsack. Denote by OPT' the almost optimal solution obtained by applying Lemma 16. We remove the items in $\text{OPT}'_T := \text{OPT}' \cap T$ temporarily; we will add them back later.

We construct a directed graph $G = (V, A)$ where we have one vertex $v_i \in V$ for each item $i \in \text{OPT}'_L := \text{OPT}' \cap L$. We connect two vertices $v_i, v_{i'}$ by an arc $a = (v_i, v_{i'})$ if and only if we can draw a vertical line segment of length at most $(1/k)^B N$ that connects item i with item i' without intersecting any other item such that i' lies above i , i.e., the bottom coordinate of i' is at least as large as the top coordinate of i , see Figure 2 for a sketch. We obtain the following proposition since for each edge we can draw a vertical line segment and these segments do not intersect each other.

► **Proposition 17.** *The graph G is planar.*

Next, we apply Lemma 9 to G with $\epsilon' := \epsilon$. Let $G' = (V', A')$ be the resulting graph. We remove from OPT'_L all items $i \in V \setminus V'$ and denote by OPT''_L the resulting solution. We push up all items in OPT''_L as much as possible. If now the strip $(0, N) \times (0, (1/k)^B N)$ is not intersected by any item then we can place all the items in T into the remaining space. Their total height can be at most $k \cdot (1/k)^{B+2} N \leq (1/k)^{B+1} N$ and thus we can leave a strip of height $(1/k)^B N - (1/k)^{B+1} N \geq (1/k)^{O(1/\epsilon)} N$ and width N empty. This completes the proof of Lemma 15 for this case.

Assume next that the strip $(0, N) \times (0, (1/k)^B N)$ is intersected by some item: the following lemma implies that there is a set of $c' = O(1/\epsilon^2)$ vertices whose items intuitively connect the top and the bottom edge of the knapsack.

► **Lemma 18.** *Assume that in OPT''_L there is an item i_1 intersecting $(0, N) \times (0, (1/k)^B N)$. Then G contains a path $v_{i_1}, v_{i_2}, \dots, v_{i_K}$ with $K \leq c' = O(1/\epsilon^2)$, such that the distance between i_K and the top edge of the knapsack is less than $(1/k)^B N$.*

Proof. Let C denote all vertices v in G' such that there is a directed path from v_{i_1} to v in G' . The vertices in C are contained in the connected component C' in G' that contains v_{i_1} . Note that $|C| \leq |C'| \leq c'$. We claim that C must contain a vertex v_j whose corresponding item j is closer than $(1/k)^B N$ to the top edge of the knapsack. Otherwise, we would have been able to push up all items corresponding to vertices in C by $(1/k)^B N$ units: first we could have pushed up all items such that their corresponding vertices have no outgoing arc, then all items such that their vertices have outgoing arcs pointing at the former set of vertices, and so on. By definition of C , there must be a path connecting v_{i_1} with v_j . This path $v_{i_1}, v_{i_2}, \dots, v_{i_K} = v_j$ contains only vertices in C and hence its length is bounded by c' . The claim follows. ◀

Our goal is now to remove the items i_1, \dots, i_K due to Lemma 18 and $O(K) = O(1/\epsilon^2)$ more large items from OPT''_L . Since we can assume that $k \geq \Omega(1/\epsilon^3)$ this will lose only a factor of $1 + O(\epsilon)$ in the objective. To this end we define $K + 1$ *deletion rectangles*, see Figure 2. We place one such rectangle R_ℓ between any two consecutive items $i_\ell, i_{\ell+1}$. The height of R_ℓ equals the vertical distance between i_ℓ and $i_{\ell+1}$ (at most $(1/k)^B N$) and the width of R_ℓ equals $(1/k)^B N$. Since $v_{i_\ell}, v_{i_{\ell+1}}$ are connected by an arc in G' , we can draw a vertical line segment connecting i_ℓ with $i_{\ell+1}$. We place R_ℓ such that it is intersected by this line segment. Note that for the horizontal position of R_ℓ there are still several possibilities and we choose one arbitrarily. Finally, we place a special deletion rectangle between the item i_K and the top edge of the knapsack and another special deletion rectangle between the item i_1 and the bottom edge of the knapsack. The heights of these rectangles equal the distance of i_1 and i_K with the bottom and top edge of the knapsack, resp. (which is at most $(1/k)^B N$), and their widths equal $(1/k)^B N$. They are placed such that they touch the bottom edge of i_1 and the top edge of i_K , resp.

► **Lemma 19.** *Each deletion rectangle can intersect at most 4 large items in its interior. Hence, there can be only $O(K) \leq O(c') = O(1/\epsilon^2)$ large items intersecting a deletion rectangle in their interior.*

Observe that the deletion rectangles and the items in $\{i_1, \dots, i_K\}$ separate the knapsack into a left and a right part with items OPT''_{left} and OPT''_{right} , resp. We delete all items in i_1, \dots, i_K and all items intersecting the interior of a deletion rectangle. Each deletion rectangle and each item in $\{i_1, \dots, i_K\}$ has a width of at least $(1/k)^B N$. Thus, we can move all items in OPT''_{left} simultaneously by $(1/k)^B N$ units to the right. After this, no large item intersects the area $(0, (1/k)^B N) \times (0, N)$. We rotate the resulting solution by 90 degrees,

hence getting an empty horizontal strip $(0, N) \times (0, (1/k)^B N)$. The total height of items in OPT'_T is at most $k \cdot (1/k)^{B+2} N \leq (1/k)^{B+1} N$. Therefore, the items in OPT'_T can be stacked (one on top of the other) inside a horizontal strip of height $(1/k)^{B+1} N$ that can be placed right below the rectangles in $OPT''_{left} \cup OPT''_{right}$. This leaves an empty horizontal strip of height $(1/k)^B N - (1/k)^{B+1} N \geq (1/k)^{O(1/\epsilon)} N$ at the bottom of the knapsack. This completes the proof of Lemma 15.

3.2 FPT-algorithm with resource augmentation

We now compute a packing that contains as many items as the solution due to Lemma 15. However, it might use the space of the entire knapsack. In particular, we use the free space in the knapsack in the latter solution in order to round the sizes of the items. In the following lemma the reader may think of $k' = (1 - \epsilon)k$ and $\tilde{k} = k^{O(1/\epsilon)}$.

► **Lemma 20.** *Let $k', \tilde{k} \in \mathbb{N}$. There is an algorithm for 2DKR with a running time of $(\tilde{k}k')^{O(k')} n^{O(1)}$ that computes a solution of size k' or asserts that there is no solution of size k' fitting into a restricted knapsack $[0, N] \times [0, (1 - 1/\tilde{k})N]$. Also, in time $n^{O(1)}$ we can compute a set of size $O(\tilde{k}(k')^2)$ that contains a solution of size k' if there is such a solution that fits into the latter knapsack.*

Note that Lemma 20 yields an FPT algorithm if we are allowed to increase the size of the knapsack by a factor $1 + O(1/\tilde{k})$ where \tilde{k} is a second parameter.

In the remainder of this section, we prove Lemma 20 and we do not differentiate between large and thin items anymore. Assume that there exists a solution OPT'' of size k' that leaves the area $[0, N] \times [0, N/\tilde{k}]$ of the knapsack empty. We want to compute a solution of size k' . We use the empty space in order to round the heights of the items in the packing of OPT'' to integral multiples of $N/(k'\tilde{k})$. Note that in OPT'' an item i might be rotated. Thus, depending on this we actually want to round its height h_i or its width w_i . To this end, we define rounded heights and widths by $\hat{h}_i := \left\lceil \frac{h_i}{N/(k'\tilde{k})} \right\rceil N/(k'\tilde{k})$ and $\hat{w}_i := \left\lceil \frac{w_i}{N/(k'\tilde{k})} \right\rceil N/(k'\tilde{k})$ for each item i .

► **Lemma 21.** *There exists a feasible packing for all items in OPT'' even if for each rotated item i we increase its width w_i to \hat{w}_i and for each non-rotated item $i' \in OPT''$ we increase its height $h_{i'}$ to $\hat{h}_{i'}$.*

To visualize the packing due to Lemma 21 one might imagine a container of height \hat{h}_i and width w_i for each non-rotated item i and a container of height $h_{i'}$ and width $\hat{w}_{i'}$ for each rotated item i' . Next, we group the items according to their values \hat{h}_i and \hat{w}_i . We define $I_h^{(j)} := \{i \in I \mid \hat{h}_i = jN/(k'\tilde{k})\}$ and $I_w^{(j)} := \{i \in I \mid \hat{w}_i = jN/(k'\tilde{k})\}$ for each $j \in \{1, \dots, k'\tilde{k}\}$. The crucial observation is now that from each set $I_h^{(j)}$ it suffices to consider only the k' items with smallest width. If OPT'' uses an item from $I_h^{(j)}$ with larger width then we can replace it by one of the k' thinner items that is not contained in OPT'' . A symmetric statement holds for the sets $I_w^{(j)}$.

► **Lemma 22.** *We can assume that from each set $I_h^{(j)}$ the solution OPT'' contains only items among the k' items in $I_h^{(j)}$ with smallest width. Similarly, from each set $I_w^{(j)}$ the solution OPT'' contains only items among the k' items in $I_w^{(j)}$ with smallest height.*

We eliminate from each set $I_h^{(j)}$ and $I_w^{(j)}$ the items that are not among the k' items with smallest width and height, resp. At most $2k' \cdot k'\tilde{k} = O(\tilde{k}(k')^2)$ items remain, denote them by \bar{I} . Then, in time $(\tilde{k}k')^{O(k')}$ we can solve the remaining problem by completely enumerating

over all subsets of \bar{I} with at most k' elements. For each enumerated set we check within the given time bounds whether its items can be packed into the knapsack (possibly via rotating some of them) by guessing sufficient auxiliary information. Therefore, if a solution of size k' for a knapsack of width N and height $(1 - 1/\tilde{k})N$ exists, then we will find a solution of size k' that fits into a knapsack of width and height N .

Now the proof of Theorem 4 follows by using Lemma 15 and then applying Lemma 20 with $k' = (1 - \epsilon)k$ and $\tilde{k} = k^{O(1/\epsilon)}$. The set \bar{I} is the claimed set (which intuitively forms the approximative kernel), we compute a solution of size at least $(1 - \epsilon)k \geq k/(1 + 2\epsilon)$ and we can redefine ϵ appropriately.

4 Hardness of Geometric Knapsack

We show that 2DK and 2DKR are both $W[1]$ -hard for parameter k by reducing from a variant of SUBSET SUM. Recall that in SUBSET SUM we are given m positive integers x_1, \dots, x_m as well as integers t and k , and have to determine whether some k -tuple of the numbers sums to t ; this is $W[1]$ -hard with respect to k [18]. In the variant MULTI-SUBSET SUM it is allowed to choose numbers more than once. It is easy to verify that the proof for $W[1]$ -hardness of SUBSET SUM due to Downey and Fellows [18] extends also to MULTI-SUBSET SUM. In our reduction to 2DKR we prove that rotations are not required for optimal solutions, making $W[1]$ -hardness of 2DK a free consequence.

Proof sketch for Theorem 3. We give a polynomial-time parameterized reduction from MULTI-SUBSET SUM to 2DKR with output parameter $k' = O(k^2)$; this establishes $W[1]$ -hardness of 2DKR.

Observe that, for any packing of items into the knapsack, there is an upper bound of N on the total width of items that intersect any horizontal line through the knapsack, and similarly an upper bound of N for the total height of items along any vertical line. We will let the dimensions of some items depend on numbers x_i from the input instance (x_1, \dots, x_m, t, k) of MULTI-SUBSET SUM such that, using these upper bound inequalities, a correct packing certifies that $y_1 + \dots + y_k = t$ for some k of the numbers. The key difficulty is that there is a lot of freedom in the choice of which items to pack and where in case of a no instance.

To deal with this, the items corresponding to numbers x_i from the input are all *almost* squares and their dimensions are incomparable. Concretely, an item corresponding to some number x_i has height $L + S + x_i$ and width $L + S + 2t - x_i$; we call such an item a *tile*. (The exact values of L and S are immaterial here, but $L \gg S \gg t > x_i$ holds.) Thus, when using, e.g., a tile of smaller width (i.e., smaller value of x_i) it will occupy “more height” in the packing. The knapsack is only slightly larger than a k by k grid of such tiles, implying that there is little freedom for the placement. Let us also assume for the moment, that no rotations are used.

Accordingly, we can specify k vertical lines that are guaranteed to intersect all tiles of any packing that uses k^2 tiles, by using pairwise distance $L - 1$ between them. Moreover, each line is intersecting exactly k private tiles. The same holds for a similar set of k horizontal lines. Together we get an upper bound of N for the sum of the widths (heights) along any horizontal (vertical) line. Since the numbers x_i occur negatively in widths, we effectively get lower bounds for them from the horizontal lines. When the sizes of these tiles (and the auxiliary items below) are appropriately chosen, it follows that all upper bound equalities must be tight. This in turn, due to the exact choice of N , implies that there are k numbers y_1, \dots, y_k with sum equal to t .

Unsurprisingly, using just the tiles we cannot guarantee that a packing exists when given a yes-instance. This can be fixed by adding a small number of flat/thin items that can be inserted between the tiles (see Figure 3, but note that it does not match the size ratios from this proof); these have dimension $L \times S$ or $S \times L$. Because one dimension of these items is large (namely L) they must be intersected by the above horizontal or vertical lines. Thus, they can be proved to enter the above inequalities in a uniform way, so that the proof idea goes through.

Finally, let us address the question of why we can assume that there are no rotations. This is achieved by letting the width of any tile be larger than the height of any tile, and adding a final auxiliary item of width N and small height, called the *bar*. To get the desired number of items in a solution packing, it can be ensured that the bar must be used as no more than k^2 tiles can fit into $N \times N$ and there is a limited supply of flat/thin items. W.l.o.g., the bar is not rotated. It can then be checked that using at least one tile in its rotated form will violate one of the upper bounds for the height. This completes the proof sketch. ◀

5 Open Problems

This paper leaves several interesting open problems. A first obvious question is whether there exists a PAS also for 2DK (i.e., in the case without rotations). We remark that the algorithm from Lemma 20 can be easily adapted to the case without rotations. Unfortunately, Lemma 15 does not seem to generalize to the latter case. Indeed, there are instances in which we lose up to a factor of 2 if we require a strip of width $\Omega_{\varepsilon,k}(1) \cdot N$ to be emptied, see Figure 4. We also note that both our PASs work for the cardinality version of the problems: an extension to the weighted case is desirable. Unlike related results in the literature (where extension to the weighted case follows relatively easily from the cardinality case), this seems to pose several technical issues.

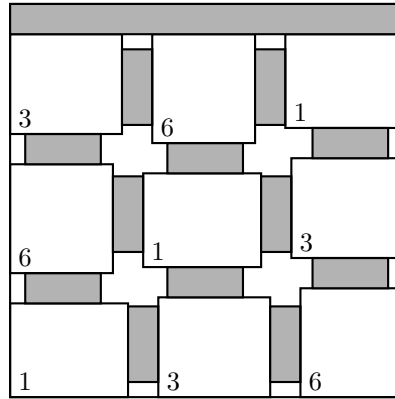
We remark that all the problems considered in this paper might admit a PTAS in the standard sense, which would be a strict improvement on our PASs. Indeed, the existence of a QPTAS for these problems [1, 2, 15] suggests that such PTASs are likely to exist. However, finding those PTASs is a very well-known and long-standing problem in the area. We hope that our results can help to achieve this challenging goal.

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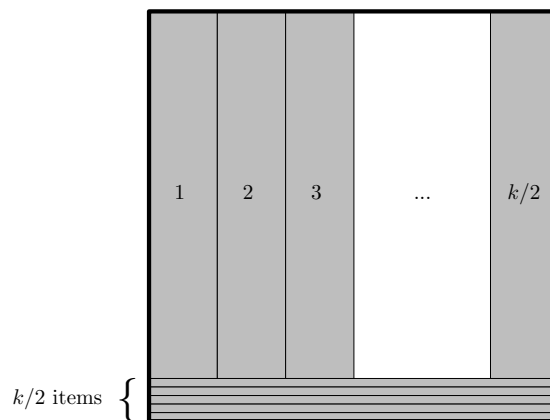
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■ **Figure 3** A sketch of the packing used in Theorem 3 for a solution with $k = 3$ and $1 + 3 + 6 = 10$. Items corresponding to the same number have the same size. The figure is not to scale: The gray items should be much flatter and the clear ones should look like squares of almost identical size.



■ **Figure 4** Example showing that Lemma 15 cannot be generalized to 2DK (without rotations). The total height of the $k/2$ items on the bottom of the knapsack can be made arbitrarily small. Suppose that we wanted to free up an area of height $f(k) \cdot N$ and width N or of height N and width $f(k) \cdot N$ (for some fixed function f). If the total height of the items on the bottom is smaller than $f(k) \cdot N$ then we would have to eliminate the $k/2$ items on the bottom or the $k/2$ items on top. Thus, we would lose a factor of $2 > 1 + \varepsilon$ in the approximation ratio.