Quantum indistinguishability through exchangeability

Alessio Benavoli[a], Alessandro Facchini[b,*], Marco Zaffalon[b]

[a] *School of Computer Science and Statistics, Trinity College Dublin, Dublin, Ireland*
[b] *Istituto Dalle Molle di Studi sullIntelligenza Artificiale (IDSIA), Lugano, Switzerland*

Abstract

Two particles are identical if all their intrinsic properties, such as spin and charge, are the same, meaning that no quantum experiment can distinguish them. In addition to the well known principles of quantum mechanics, understanding systems of identical particles requires a new postulate, the so called *symmetrization postulate*. In this work, we show that the postulate corresponds to exchangeability assessments for sets of observables (gambles) in a quantum experiment, when quantum mechanics is seen as a normative and algorithmic theory guiding an agent to assess her subjective beliefs represented as (coherent) sets of gambles. Finally, we show how sets of exchangeable observables (gambles) may be updated after a measurement and discuss the issue of defining entanglement for indistinguishable particle systems.

Keywords: Indistinguishability, Exchangeability, Coherence, Entanglement, Quantum mechanics

1. Introduction

In recent works [1, 2], we defined a theory of probability on a continuous space of complex vectors that complies with the two postulates of coherence ("The theory should be logically consistent"), and of computation ("Inferences in the theory should be computable in polynomial time"). We then showed that its deductive closure is tantamount to Quantum Mechanics (QM). Hence QM may be viewed as a normative and algorithmic theory guiding an agent to assess her subjective beliefs represented as (logically consistent, equivalently, coherent) sets of gambles on the results of a quantum experiment. We were then able to derive (in a coherent way) the main postulates of QM from standard operations in probability theory (updating, marginalisation, time coherence). This means we derived a theory of probability which theoretically and empirically agrees with QM experiments. This also allows us to provide a decision-theoretic foundation of QM [3] and to derive the so called Gleason's theorem from the same principles [4].

*Corresponding author. All authors contributed equally to the paper.
Email addresses: `alessio.benavoli@tcd.ie` (Alessio Benavoli), `alessandro.facchini@idsia.ch` (Alessandro Facchini), `zaffalon@idsia.ch` (Marco Zaffalon)

When one considers systems including more than one particle, we must consider the implications of another important empirical observation: in many of these systems the particles of interest belong to distinct classes of indistinguishable (equivalently, identical) particles. Two (or more) particles are said to be identical if all their intrinsic properties (charge, mass, spin, etc.) are exactly the same. In other words, no experiment can distinguish one from the other. Hence, all the electrons in the universe are identical, as are all the protons. This means that, when a physical system contains two identical particles, there is no change in its properties if the roles of these two particles are exchanged.[1]

This law is formulated in QM by the *symmetrization postulate*, which establishes that in a system containing identical particles the only possible configurations of their properties (e.g., spin) are either all symmetrical or all antisymmetrical with respect to permutations of the labels of the particles (we will clarify this definition in Section 4). In the first case, the particles are called *bosons*; in the second case they are called *fermions*.

In this paper, we aim to derive the *symmetrization postulate* from the way a subject accepts gambles on experiments involving indistinguishable particles. We assume that the particles are (finitely) *exchangeable*, meaning roughly that the subject believes that the labels (i.e. electron 1, electron 2,..) used to denote them have no influence on the decisions and inferences she will make regarding the particles.

Exchangeability is a fundamental concept in classical probability theory and statistics [5, 6]. Its assumption, and the analysis of its consequences, goes back to [7] and his famous *Representation Theorem*. In statistics, this theorem is interpreted as stating that "a sequence of random variables is exchangeable if it is conditionally independent and identically distributed." This theorem was generalised to QM by [8] for *quantum-state tomography*, which is a technique to estimate the density matrix of a particle by performing repeated measures (the order of the measures is assumed to be exchangeable).

In this paper, we instead deal with the exchangeability of indistinguishable particles. We show that we can derive the *symmetrization postulate* by using the general framework for exchangeable gambles proposed by [9, 10, 11] for classical (imprecise) probability theory.[2] This confirms, once again, that QM can be seen as a subjective theory of probability.

The rest of the paper is organised as follows. In Section 2 we will use an example, non-related to QM, to introduce the main concepts. In Section 3 we recall how QM can be formulated, and thus understood, as an algorithmic theory of desirable gambles. After formulating in Section 4 the symmetrisation postulate, in the following Section 5 we derive it in terms of (algorithmic) coherence and exchangeability. Finally, in Section 6 we show how sets of exchangeable observables (gambles) may be updated

[1]Spatially well-separated identical particles can be distinguished. Basically, we can use their location to label them (e.g., particle in place A and particle in place B). If the system of two particles evolves in a way that the two particles do not remain in two distinct regions of space, then we cannot talk anymore of particle A and particle B (we have lost track of their labels). We must treat them as indistinguishable particles.

[2]Exchangeability in the context of imprecise probability was originally proposed by [12, Sec. 9.5]

after a measurement and in Section 7 we discuss the issue of defining entanglement for indistinguishable particle systems. Section 8 deals with partial exchangeability, that is with the case where either only a subset of the particles or a subset of their properties are exchangeable. Finally, Section 9 concludes the paper.

A preliminary version of this work appeared in [13], the present manuscript extends this work in three ways: 1) it provides an analogy example to introduce the framework later used to derive the main results of the paper; 2) it also provides a new result (Proposition 14) to prove the equivalence between two different notions of non-entanglement for identical particles; 3) it briefly discusses the case of partial exchangeability.

2. Algorithmic rationality made simple: an example

In this section, we will use an example to intuitively introduce the concept of algorithmic rationality, which is fundamental to understand QM and the subsequent results on indistinguishable particles. The possibility space and the type of inferences considered in the example are similar to the ones in QM. This will help the reader, who is not familiar with QM, to understand the main results of the paper. Conversely, this example will enable the reader familiar with QM to understand the appropriate way to relate (or contrapose) QM with classical probability theory. The main messages are:

- It is possible to define a finite dimensional theory of probability on an infinite possibility space. Stated otherwise, QM should be more in general compared to continuous probability theory.
- The weirdness of QM is due to algorithmic rationality and it is not related to complex numbers.
- The density matrix ρ, which represents the belief state in QM, is the analogous of a (quasi-)moment matrix.

We stress again that the following example is only introduced to provide an intuition on the possibility space, functions of interest and expectation operators used in QM. We do not claim/imply any general equivalence.

2.1. Where is Bob?

Bob is a troubleshooter for Deliverex, a global shipping and logistics company. Bob travels a lot, is often very busy and, thus, does not always respond to calls or emails right away. In order to plan his next trips, Deliverex must determine his location on the Earth's surface. As depicted in Figure 1, the space of possibility is

$$\Omega = \{\mathbf{x} \in \mathbb{R}^3 : \mathbf{x}^\top \mathbf{x} = 1\}.$$

Not exactly knowing where Bob is, Deliverex expresses its probabilistic beliefs on Bob's location using a probability distribution. It can for instance use the von Mises–Fisher distribution, which is a valid distribution on the sphere, that is

$$p(\mathbf{x}) := \tfrac{1}{C} e^{\kappa \boldsymbol{\mu}^\top \mathbf{x}}, \tag{1}$$

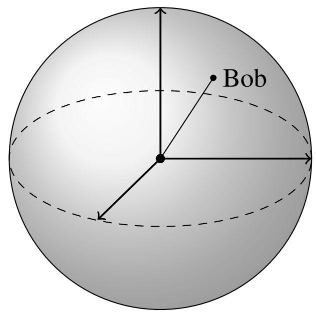

Figure 1: Spherical approximation of the Earth

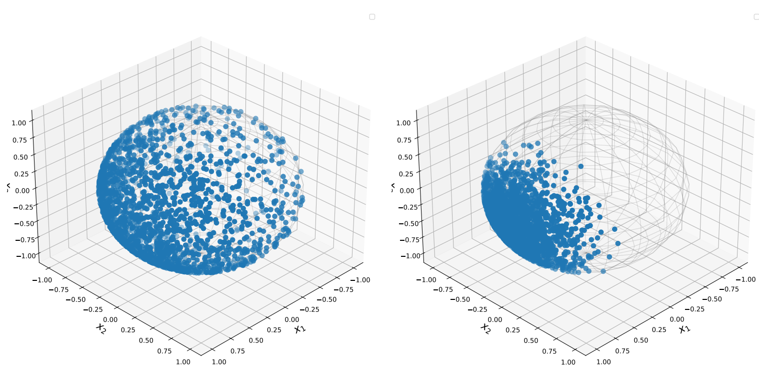

Figure 2: Samples from a von Mises–Fisher distribution on the sphere with $\boldsymbol{\mu} = [1, 0, 0]^\top$, and $\kappa = 2$ (left) and $\kappa = 10$ (right).

where $\kappa \geq 0$ and $\boldsymbol{\mu} \in \Omega$ are called the concentration parameter and mean direction, respectively. $C = \int_\Omega e^{\kappa \boldsymbol{\mu}^\top \mathbf{x}} d\mathbf{x}$ is the normalisation constant. Note that $\kappa = 0$ gives the uniform distribution. Figure 2 shows samples from this distribution for two different κ and $\boldsymbol{\mu} = [1, 0, 0]^\top$.

We assume that Deliverex is interested[3] in inferring polynomial functions of Bob's location $\mathbf{x}$. More specifically, Deliverex is interested in computing probabilistic inferences for the set of functions defined by:

$$\mathscr{F} := \{\mathbf{x}^\top G \mathbf{x} : G \text{ is a real } 3 \times 3 \text{ symmetric matrix}\}. \tag{2}$$

Observe that $\mathscr{F}$ is a vector space of functions including the constant functions, that is $\mathbf{x}^\top c I_d \mathbf{x} = c$ for all $c \in \mathbb{R}$, where I_d is the identity matrix and d denotes its dimension.[4]

[3] For instance, Deliverex may have a device that can only measure polynomial functions of Bob's location and, therefore, they are interested in assessing the expected result of this measure.

[4] Sometimes we omit to indicate the dimension d and simply write I.

This set of polynomial functions $\mathscr{F}$ is interesting for Deliverex because, for instance, the function

$$\mathbf{x}^\top(I - \mathbf{w}\mathbf{w}^\top)\mathbf{x} = 1 - (\mathbf{w}^\top\mathbf{x})^2,$$

which belongs to $\mathscr{F}$, can be interpreted as the "distance" between Bob's position and a location of interest $\mathbf{w} = [1,0,0]^\top$. This function is zero when $\mathbf{x} = [\pm 1,0,0]^\top$. This means that this function is zero whenever Bob is located at two antipodal places along the direction $\mathbf{w} = [1,0,0]^\top$. We will later discuss how to get rid of this phase ($\pm$) ambiguity by using complex numbers (this complex representation is commonly used in QM), so that this can become a proper distance measure. For the moment, we assume that Deliverex is able to solve this ambiguity in some way and we keep working with real-variables for simplicity.

The class of functions $\mathscr{F}$ has the following important property.

Lemma 1. *For any probability distribution $p(\mathbf{x})$, the expectation is given by*

$$E[\mathbf{x}^\top G\mathbf{x}] = \int_\Omega Tr(G\mathbf{x}\mathbf{x}^\top)p(\mathbf{x})d\mathbf{x} = Tr\left(G\int_\Omega \mathbf{x}\mathbf{x}^\top p(\mathbf{x})d\mathbf{x}\right) = Tr(GM), \quad (3)$$

where $Tr(\cdot)$ denotes the trace and

$$M = \int_\Omega \mathbf{x}\mathbf{x}^\top p(\mathbf{x})d\mathbf{x},$$

is the non-central moment matrix of order 2 (a truncated moment matrix). Note that, because of the constraint $\mathbf{x}^\top\mathbf{x} = 1$ in Ω we have that $Tr(M) = 1$. Moreover, the matrix M is positive semi-definite (PSD), $M \geq 0$, because $\mathbf{x}\mathbf{x}^\top \geq 0$ and so $\int_\Omega \mathbf{x}\mathbf{x}^\top p(\mathbf{x})d\mathbf{x} \geq 0$.

Its proof is immediate by using the fact that $\mathbf{x}^\top G\mathbf{x} = Tr(\mathbf{x}^\top G\mathbf{x}) = Tr(G\mathbf{x}\mathbf{x}^\top)$, and linearity of trace and expectation.

Given the property described in the lemma, assume Deliverex's belief about Bob's location is expressed by the von Mises–Fisher distribution given in Eq. (1) with $\kappa = 10$ and $\boldsymbol{\mu} = [1,0,0]^\top$. The corresponding truncated moment matrix can easily be computed and it turns out to be equal to:

$$M = \begin{bmatrix} 0.82 & 0 & 0 \\ 0 & 0.09 & 0 \\ 0 & 0 & 0.09 \end{bmatrix}.$$

The expectation of the polynomial function

$$\mathbf{x}^\top G\mathbf{x} = \mathbf{x}^\top \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \mathbf{x} = x_1^2 + 2x_1x_3 + x_3^2,$$

can then simply be computed as $Tr(GM) = 0.91$ (no need of integration). This leads us to a first important observation, namely that, when focusing on $\mathscr{F}$, Deliverex can equivalently express their beliefs on Bob's location via M. We can therefore interpret

M as a **belief state; it is the equivalent of $p(\mathbf{x})$ whenever we are interested in computing expectations of functions in $\mathscr{F}$.** Moreover, since M is a matrix, this means that we can use a finite-dimensional object, M, to express beliefs over an infinite possibility space Ω. To sum up:

Remark 1. The knowledge of the truncated moment matrix M is sufficient to compute the expectation of any function in $\mathscr{F}$ w.r.t. any distribution on Ω.

Is this condition also necessary? In other words, can Deliverex define another valid expectation operator $L(\cdot)$ on $\mathscr{F}$, which does not satisfy $L(\cdot) = Tr(\cdot M)$ for some matrix M? To answer this question, we have first to define what a **valid** expectation operator is.

Definition 1 ([12, Sec. 2.8.4]). Let L be a functional $L : \mathscr{F} \to \mathbb{R}$ defined on a vector space $\mathscr{F}$ of functions including the constants. L is a valid expectation operator (aka linear prevision) if it satisfies the following properties:

P1: $L(f+g) = L(f) + L(g)$ for every $f, g \in \mathscr{F}$;

P2: $L(g) \geq \inf g$, for every $g \in \mathscr{F}$.[5]

Note that P1 is simply the linearity of expectation. From P1, $L(0) = L(0+0) = L(0) + L(0)$, whence, by subtracting $L(0)$ from both sides, we obtain that $L(0) = 0$, and therefore that

$$0 = L(0) = L(g-g) \overset{P1}{=} L(g) + L(-g).$$

This means that $L(g) = -L(-g)$. Finally, using P2, we conclude that

$$\inf g \leq L(g) \leq \sup g. \tag{4}$$

Intuitively, this implies that $L(g)$ is a "weighted-average": the weights being the probability measure associated to the expectation operator. Indeed, any weighted-average must satisfy $\inf g \leq \int g(\mathbf{x})p(\mathbf{x})d\mathbf{x} \leq \sup g$. This means we can define expectation as an integral with respect to a probability measure.

Given a linear operator L, it is in general difficult to verify if it is a **valid** expectation operator, that is whether it satisfies properties P1 and P2 in Definition 1 (we call the corresponding decision problem the *validation problem*). In fact, the validation of P2 boils down to solving an optimisation problem (to compute $\inf g$).

Example 1. Let us consider the following operator

$$L(g) = -0.1g(\mathbf{e}_1) + 1.1g(\mathbf{e}_2),$$

where $\mathbf{e}_1, \mathbf{e}_2$ are vector of the canonical basis of $\mathbb{R}^3$. We want to verify if it satisfies P1 and P2. P1 clearly holds. To prove P2, we must verify that for every function g:

$$L(g) \geq \inf_{\mathbf{x}\in\Omega} g(\mathbf{x}).$$

[5] inf is the infimum. With $\inf g$ we mean $\inf_{\mathbf{x}\in\Omega} g(\mathbf{x})$.

In particular this means that $L(1) = 1$. For any g we can easily compute the left hand side of the inequality, but its right hand side requires to solve an optimisation problem. In this case, we can use a function which we know is non-negative to prove that $L(g)$ is not a valid expectation. Consider for instance the non-negative function $\mathbf{x}^\top \mathbf{e}_1 \mathbf{e}_1^T \mathbf{x}$, then $L(g) = -0.1 < 0$. This violates P2.

For the class of functions $\mathscr{F}$, the validation problem is a computationally tractable decision problem. Indeed, the following is common knowledge.[6]

Lemma 2. *The infimum (minimum) of $g(x) = \mathbf{x}^\top G\mathbf{x}$ is equal to the minimum eigenvalue of G, and can therefore be computed in **P**-time.*

From this result, we can prove the following.

Proposition 1. *Let $\mathscr{F}$ be as in (2), and consider $L : \mathscr{F} \to \mathbb{R}$. Then L satisfies P1 and P2 (is a valid expectation) if and only if there exists a **symmetric PSD matrix** M **with trace one** such that*

$$L(\mathbf{x}^\top G\mathbf{x}) = Tr(GM),$$

for $g(\mathbf{x}) = \mathbf{x}^\top G\mathbf{x} \in \mathscr{F}$.

PROOF. Assume $L(\mathbf{x}^\top G\mathbf{x}) = Tr(GM)$, for some symmetric PSD matrix M with trace one. P1 follows by linearity of trace, and P2 by Lemma 2. The other direction follows from Lemma 1.

Any expectation operator can thus be written as $L(g) = Tr(GM)$ for some matrix M. In particular, Proposition 1 means that the set of **belief states** is

$$\{M \text{ is a real } 3 \times 3 \text{ symmetric matrix} : M \geq 0, \;\; Tr(M) = 1\},$$

and that we do not need a probability distribution to define an expectation operator. Indeed, in general, infinitely many probability measures have M as truncated moment matrix (for instance the case $M = I$). Also note that if $p_1(\mathbf{x}), p_2(\mathbf{x})$ have M as moment matrix, then $p_3(\mathbf{x}) = \alpha p_1(\mathbf{x}) + (1 - \alpha)p_2(\mathbf{x})$ for any $\alpha \in (0, 1)$ has M as moment matrix. Note instead that, rank-one moment matrices uniquely define a probability measure.

Going back to our running scenario, Deliverex can therefore express their beliefs on Bob's location through a truncated moment matrix, as this finite-dimensional object takes the place of $p(\mathbf{x})$ for inferences on $\mathscr{F}$.

2.2. *Where are Alice and Bob?*

Deliverex has another troubleshooter, Alice, who also travels around the globe. Assume Deliverex is also interested in the location $\mathbf{y} \in \Omega$ of Alice and, therefore, they

[6] Consider an eigenvalue-eigenvector decomposition (it may not be unique) of $G = \sum_{i=1}^{3} \lambda_i \mathbf{v}_i \mathbf{v}_i^\top$ and observe that $\mathbf{x}^\top G\mathbf{x} = \sum_{i=1}^{3} \lambda_i (\mathbf{x}^\top \mathbf{v}_i)^2$. Since $\sum_{i=1}^{3} (\mathbf{x}^\top \mathbf{v}_i)^2 = 1$, this proves that the minimum of g corresponds to the minimum eigenvalue. The eigenvalues of a $N \times N$ matrix can be computed in $O(N^3)$ time.

also consider the functions:

$$\mathscr{H} = \{\mathbf{y}^\top H\mathbf{y} : H \text{ is a real } 3\times 3 \text{ symmetric matrix}\}. \tag{5}$$

How can we put together the set of functions $\mathscr{F}, \mathscr{H}$ so that they can make inferences on Bob and Alice's location separately but also jointly? For instance, Bob and Alice may sometimes travel independently, while in some cases they may need to work closely. In the latter case, their locations are dependent.

A way to put together these sets is by considering the set of of functions:

$$\mathscr{G} = \{(\mathbf{x}\otimes\mathbf{y})^\top G(\mathbf{x}\otimes\mathbf{y}) : G \text{ is a real } 9\times 9 \text{ symmetric matrix}\}, \tag{6}$$

where $\otimes$ is the Kronecker product.[7] This is a vector space, which includes the constants. Moreover, we have that $\mathscr{F}, \mathscr{H} \subset \mathscr{G}$, which allows Deliverex to make marginal inferences on Bob and Alice's location. In fact, notice that for the *mixed-product property* of the Kronecker product.

$$\begin{aligned}\mathbf{x}^\top G\mathbf{x}\mathbf{y}^\top H\mathbf{y} &= (\mathbf{x}^\top G\mathbf{x})\otimes(\mathbf{y}^\top H\mathbf{y}) = (\mathbf{x}^\top\otimes\mathbf{y}^\top)(G\mathbf{x}\otimes H\mathbf{y})\\ &= (\mathbf{x}\otimes\mathbf{y})^\top(G\mathbf{x}\otimes H\mathbf{y}) = (\mathbf{x}\otimes\mathbf{y})^\top(G\otimes H)(\mathbf{x}\otimes\mathbf{y}).\end{aligned}$$

For $H = I$, one has $\mathbf{x}^\top G\mathbf{x}\mathbf{y}^\top H\mathbf{y} = \mathbf{x}^\top G\mathbf{x}$ and vice versa. Independence judgements can be expressed by Deliverex as $L(\mathbf{x}^\top G\mathbf{x}\mathbf{y}^\top H\mathbf{y}) = L(\mathbf{x}^\top G\mathbf{x})L(\mathbf{y}^\top H\mathbf{y})$.

Deliverex wishes to define an expectation operator $L : \mathscr{G} \to \mathbb{R}$. However, to establish if a given linear operator L is a **valid** expectation, that is in order to verify P2, Deliverex must be able to find the infimum of a gamble $g \in \mathscr{G}$.

Proposition 2 (Section 6 in [14]). *Computing the minimum of a function in* (6) *is in general* ***NP****-hard.*

This means that it is also hard to define the equivalent of Proposition 1. Indeed, in general, given $(\mathbf{x}\otimes\mathbf{y})^\top G(\mathbf{x}\otimes\mathbf{y})$, we can find a PSD 9×9 symmetric matrix M of trace one such that

$$Tr(GM) < \inf g,$$

and which thus violates P2, meaning that in this case $L(\cdot) = Tr(\cdot M)$ is not a valid expectation operator for every M.[8] The morality is that M **must satisfy some additional constraints besides being PSD and trace one to satisfy both P1 and P2.** Determining these constraints, and thus solving the corresponding validation problem, is **NP**-hard.

This is where **algorithmic rationality** comes to the rescue. The idea at the core of algorithmic rationality is to re-define property P2 so that the corresponding validation problem can be solved in **P**-time.

[7] Given two vectors $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$, their Kronecker product is $a\otimes b := (a_1b_1, a_1b_2, a_1b_3, a_2b_1, a_2b_2, a_2b_3, a_3b_1, a_3b_2, a_3b_3)$.

[8] For instance, this is the case for the matrix M^{xy} in Example 4.

Definition 2. We say that an operator $\widetilde{L}$ is an *A-expectation*[9] if it is linear (P1) and satisfies the following property:

P2*: if $g(\mathbf{x}, \mathbf{y}) = (\mathbf{x} \otimes \mathbf{y})^\top G(\mathbf{x} \otimes \mathbf{y})$ and $\underline{\lambda}_G$ is the minimum eigenvalue of G, then $\widetilde{L}(g) \geq \underline{\lambda}_G$.

The notation $\widetilde{L}$ aims at distinguishing A-expectation from the classical ones (which instead satisfy P1 and P2).

From P2*, we can also derive

$$\underline{\lambda}_G \leq \widetilde{L}(g) \leq \overline{\lambda}_G, \tag{7}$$

where $\overline{\lambda}_G$ is the maximum eigenvalue of G, which is the equivalent of the validity constraints (4) in case of algorithmic rationality.

Note that there are other ways to define an A-expectation operator, which are not based on P2* (that is on the minimum eigenvalue). P2* is how A-expectation is defined in QM. Justifying why P2* is an important aspect of the research effort that goes under the name of "quantum reconstruction" (see [15, 16] for a review). In this paper, we will not address this issue and simply assume P2* as defined above.

Given that assessing P2* is computationally tractable, we can now prove an analogous statement as to Proposition 1.

Proposition 3. *Let $\mathscr{H}$ be as in (6), and consider $\widetilde{L} : \mathscr{H} \to \mathbb{R}$. Then $\widetilde{L}$ satisfies P1 and P2*, and thus is a valid A-expectation, if and only if there exists a* ***symmetric positive semi-definite (PSD) matrix*** M ***with trace one*** *such that*

$$\widetilde{L}((\mathbf{x} \otimes \mathbf{y})^\top G(\mathbf{x} \otimes \mathbf{y})) = Tr(GM)$$

for all $g \in \mathscr{H}$.

PROOF. The direction from right to left is immediate, see proof of Proposition 1. For the other direction, the argument can be proven by using duality, similarly to what done in [2].

Notice that since

$$\underline{\lambda}_G \leq \inf g \leq \sup g \leq \overline{\lambda}_G, \tag{8}$$

and the external inequalities can be strict for some g, this implies that $\widetilde{L}(g)$ cannot be a "weighted average", that is we cannot define $\widetilde{L}$ as an integral with respect to a probability measure. In other words, $\widetilde{L}$ is not a classical expectation operator. In order to write $\widetilde{L}(g)$ as $\int_\Omega g(\mathbf{x}, \mathbf{y})\, p(\mathbf{x}, \mathbf{y}) d\mathbf{x} d\mathbf{y}$ and satisfy (7), we need to introduce some negative values in $p(\mathbf{x}, \mathbf{y})$.[10] Negative probabilities are indeed a manifestation of algorithmic rationality. A more direct manifestation of algorithmic rationality is a phenomenon we call entanglement in QM.

[9] "A" comes from algorithmic rationality.

[10] The fact that this is sufficient come from the fact that any element $\mathbf{z} \in \mathbb{R}^9$ can be obtained as the span of the elements $\mathbf{e}_i \otimes \mathbf{e}_j$ with $\mathbf{e}_i, \mathbf{e}_j$ belonging to a basis of $\mathbb{R}^3$.

Example 2. Consider the joint belief moment matrix

$$M^{xy} = \widetilde{L}((\mathbf{x} \otimes \mathbf{y})(\mathbf{x} \otimes \mathbf{y})^\top) = \frac{1}{3} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

This moment matrix has rank one. A rank-one moment matrix in classical probability is uniquely associated to an atomic probability measure. This means that if M^{xy} was a classical moment matrix, it would represent a deterministic belief state (Deliverex knows with certainty the joint position of Alice and Bob).

Note that $M^{xy}_{11} = \widetilde{L}(x_1^2 y_1^2), M^{xy}_{22} = \widetilde{L}(x_1^2 y_2^2), M^{xy}_{33} = \widetilde{L}(x_1^2 y_3^2)$ and, therefore, $M^{xy}_{11} + M^{xy}_{22} + M^{xy}_{33} = \widetilde{L}(x_1^2)$ (because $y_1^2 + y_2^2 + y_3^2 = 1$). By exploiting these equalities, we can compute the marginal moment matrix on $\mathbf{x}$, which is

$$M^x = \widetilde{L}(\mathbf{x}\mathbf{x}^\top) = \frac{1}{3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which for instance corresponds to the moment matrix of the uniform distribution on the sphere. By symmetry, it results that $M^y = M^x$. This moment matrix does not have rank 1, meaning Deliverex is marginally uncertain about their individual positions.

How can Deliverex be totally sure about the joint position of Alice and Bob without knowing their individual positions? There is no classical expectation operator (that is, one satisfying P1 and P2) on two variables, which can have the above M^{xy} as joint moment matrix and the above $M^y = M^x$ as marginal moment matrices. For a classical expectation operator, the marginal of a rank 1 joint moment matrix M^{xy} is also rank 1. So in classical probability, if we know the joint position of Alice and Bob with certainty, then we also know their marginal (individual) positions with certainty.

Algorithmic rationality violates this common sense rationality constraint. Therefore, M^{xy} is an example of what we call entanglement matrix in QM.

2.3. Exchangeable employees

We now introduce the concept of exchangeability which we will use to prove the main results of this paper. For Deliverex, Bob and Alice are exchangeable: they are both skilled troubleshooters. Therefore, Deliverex does not care to know if Alice is in Dublin and Bob is in Lugano, they only need to know that one of their troubleshooters is solving a problem in Dublin and the other in Lugano.

Let us change the notation for Bob's and Alice's position from $\mathbf{x}, \mathbf{y}$ to $\mathbf{x}_1, \mathbf{x}_2$ respectively. The fact that Alice and Bob are exchangeable employees means the labels in $\mathbf{x}_1, \mathbf{x}_2$ are not important for Deliverex.

In other words, Deliverex's belief model on Bob's and Alice's position satisfies a set of symmetry constraints, such as:

$$\widetilde{L}((\mathbf{x}_1 \otimes \mathbf{x}_2)^\top G(\mathbf{x}_1 \otimes \mathbf{x}_2)) = \widetilde{L}((\mathbf{x}_2 \otimes \mathbf{x}_1)^\top G(\mathbf{x}_2 \otimes \mathbf{x}_1)) \tag{9}$$

$$\widetilde{L}((\mathbf{x}_1 \otimes \mathbf{x}_2)^\top G(\mathbf{x}_1 \otimes \mathbf{x}_2)) = \widetilde{L}\left(\tfrac{1}{2}[(\mathbf{x}_2 \otimes \mathbf{x}_1)^\top G(\mathbf{x}_1 \otimes \mathbf{x}_2) + (\mathbf{x}_1 \otimes \mathbf{x}_2)^\top G(\mathbf{x}_2 \otimes \mathbf{x}_1)]\right). \tag{10}$$

This simply means that $\widetilde{L}$ should be invariant to label permutations.

These equalities can be translated into constraints on the valid Ms under exchangeability. The main goal of this work is to show how particle indistinguishability can be expressed through exchangeability.

2.4. Phase problem

Previously, we discussed that the distance between Bob's position and some location of interest $\mathbf{w} = [1, 0, 0]^\top \in \Omega$ can be inferred by computing the expectation of the function

$$\mathbf{x}^\top (I - \mathbf{w}\mathbf{w}^\top)\mathbf{x} = 1 - (\mathbf{w}^\top \mathbf{x})^2,$$

which belongs to $\mathscr{F}$. This function is zero when $\mathbf{x} = [\pm 1, 0, 0]$. This means that the distance is zero whenever Bob is located at two antipodal places. We are now going to solve this ambiguity.

We can describe any vector in $\mathbf{x} \in \mathbb{R}^3$ on the sphere in spherical coordinates:

$$\begin{cases} x_1 = \sin(\theta)\sin(\phi), \\ x_2 = \sin(\theta)\cos(\phi), \\ x_3 = \cos(\theta), \end{cases} \tag{11}$$

with $\theta \in [0, \pi]$ and $\phi \in (0, 2\pi]$. To make the coordinates unique, we use the convention that when $\theta \in \{0, \pi\}$ then ϕ is zero.

We can then perform the following transformation to convert spherical coordinates into complex numbers:

$$\mathbf{z} = \begin{bmatrix} \cos\left(\frac{\theta}{2}\right) \\ e^{i\phi}\sin\left(\frac{\theta}{2}\right) \end{bmatrix}, \tag{12}$$

with $\mathbf{z}^\dagger \mathbf{z} = 1$, where $\dagger$ means conjugate-transpose. By doing that, the canonical directions $\pm\mathbf{e}_1, \pm\mathbf{e}_2, \pm\mathbf{e}_3$ are transformed into the complex vectors:

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \to \mathbf{z}_{e_3} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \to \mathbf{z}_{-e_3} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \to \mathbf{z}_{e_1} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ i\frac{\sqrt{2}}{2} \end{bmatrix},$$

$$\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \to \mathbf{z}_{-e_1} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -i\frac{\sqrt{2}}{2} \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \to \mathbf{z}_{e_2} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}, \quad \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \to \mathbf{z}_{-e_2} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix}$$

Therefore, given the transformation is one-to-one,[11] for any $\mathbf{v}, \mathbf{w} \in \Omega$, their transformations satisfy $\mathbf{z}_v^\dagger \mathbf{z}_w = 1$ iff $\mathbf{v} = \mathbf{w}$. This allows us to define a proper distance.

We can then work directly with the possibility space $\omega := \overline{\mathbb{C}}^2$, where

$$\overline{\mathbb{C}}^2 := \{\mathbf{z} \in \mathbb{C}^2 : \mathbf{z}^\dagger \mathbf{z} = 1\}. \tag{13}$$

Note that the phase ambiguity exists also in this case. This is due to the fact that

$$\mathbf{z}^\dagger G \mathbf{z} = (e^{i\varphi}\mathbf{z})^\dagger G (e^{i\varphi}\mathbf{z}),$$

for any $\varphi \in (0, 2\pi]$. However, we can consider $\{e^{i\varphi}\mathbf{z}\}$ as an equivalence class and choose the representative belonging to the image of the transformation (12), when it is necessary to convert it back to the original space $\mathbb{R}^3$. It is in fact easy to show that

Lemma 3. *Any $\mathbf{z} \in \Omega$ can be written as*

$$e^{i\varphi} \begin{bmatrix} \cos\left(\frac{\theta}{2}\right) \\ e^{i\phi} \sin\left(\frac{\theta}{2}\right) \end{bmatrix}.$$

PROOF. Consider a generic

$$\mathbf{z} = \frac{1}{\sqrt{a^2+b^2+c^2+d^2}} \begin{bmatrix} a + ib \\ c + id \end{bmatrix},$$

for any $a, b, c, d \in \mathbb{R}$, with either a or b different from 0, and either c or d different from 0, and which satisfies $\mathbf{z}^\dagger \mathbf{z} = 1$. Note that

$$\mathbf{z} = \begin{bmatrix} \left(\frac{a}{\sqrt{a^2+b^2}} + i\frac{b}{\sqrt{a^2+b^2}}\right) \frac{\sqrt{a^2+b^2}}{\sqrt{a^2+b^2+c^2+d^2}} \\ \left(\frac{c}{\sqrt{c^2+d^2}} + i\frac{d}{\sqrt{c^2+d^2}}\right) \frac{\sqrt{c^2+d^2}}{\sqrt{a^2+b^2+c^2+d^2}} \end{bmatrix} = \begin{bmatrix} (\cos(\varphi) + i\sin(\varphi))\cos\left(\frac{\theta}{2}\right) \\ (\cos(\phi) + i\sin(\phi))\sin\left(\frac{\theta}{2}\right) \end{bmatrix},$$

with $\varphi = \operatorname{atan2}(b, a) + \pi$, $\phi = \operatorname{atan2}(d, c) + \pi$ and $\theta = 2\arccos\left(\frac{\sqrt{a^2+b^2}}{\sqrt{a^2+b^2+c^2+d^2}}\right)$.[12] Note that $e^{i\varphi} = \cos(\varphi) + i\sin(\varphi)$ and so

$$\mathbf{z} = e^{i\varphi} \begin{bmatrix} \cos\left(\frac{\theta}{2}\right) \\ e^{i(\phi-\varphi)} \sin\left(\frac{\theta}{2}\right) \end{bmatrix} = e^{i\varphi} \begin{bmatrix} \cos\left(\frac{\theta}{2}\right) \\ e^{i\tilde{\phi}} \sin\left(\frac{\theta}{2}\right) \end{bmatrix},$$

where $\tilde{\phi} = mod(\phi - \varphi, 2\pi)$.

In short: by working with the complex numbers and using the set

$$\mathscr{F} = \{\mathbf{z}^\dagger H \mathbf{z} : H \text{ is a } 2 \times 2 \text{ Hermitian matrix}\}, \tag{14}$$

[11] Note in fact that, (12) is just a rewriting (and rescaling) of (11): it uses the imaginary unit to reduce the dimension from 3 to 2: $e^{i\phi}\sin\left(\frac{\theta}{2}\right) = cos(\phi)\sin\left(\frac{\theta}{2}\right) + i\sin(\phi)\sin\left(\frac{\theta}{2}\right)$.

[12] $\operatorname{atan2}$ returns an angle between $(-\pi, \pi]$. We add π so that the angle belongs to $(0, 2\pi]$.

of real-valued polynomial functions of the complex variable **z** instead of (2) as targets for inference,[13] Deliverex can compute the unambiguous expected distance between Bob's location and any other location on the Earth's surface.

As before, in this case too it is possible to show that an operator L satisfying **P1** and **P2** can be written as $L(\cdot) = Tr(\cdot M)$ but with:

$$\{M \text{ is a } 2 \times 2 \text{ Hermitian matrix} : M \geq 0, \;\; Tr(M) = 1\}.$$

These are the belief states (moment matrices) when we are interested in computing expectations of the functions in (14). The extension to joint inferences on Alice and Bob's location under algorithmic rationality is similar to what is done in Section 2.2.

To summarise, in this section we have shown how we can use complex numbers to get rid of the phase ambiguity. We can then work with the possibility space (13) and the functions in (14) to make computationally efficient (under algorithmic rationality) inferences on Alice and Bob's positions.

3. Algorithmic rationality and QM

In the previous section, we used an example to introduce the concept of algorithmic rationality. In finite-dimensional QM, we are interested in the directions of the spin of particles (and other quantities). The possibility space is similar to the one we used for Alice and Bob's location (in the complex case). In this section, we redefine the framework and define algorithmic rationality from a dual perspective: the theory of desirable gambles. Since the theory of desirable gambles is a logic, this allows us to show how QM can be derived from logic and computation principles in the way a subject accepts gambles (real-valued bounded functions) on the results of a QM experiment [2].

We first recall some definitions and results from [2]. Consider a systems of m particles (each one is an n_j-level system, for instance if we consider the spin of an electron $n_j = 2$: the spin can be "up" or "down"). When $m > 1$ the system is said to be composite, whereas in case $m = 1$ we are considering a single particle system. Hence, the possibility space is

$$\Omega := \times_{j=1}^{m} \overline{\mathbb{C}}^{n_j}.$$

where

$$\overline{\mathbb{C}}^{n_j} := \{\mathbf{x} \in \mathbb{C}^{n_j} : \; \mathbf{x}^{\dagger}\mathbf{x} = 1\}.$$

Next, we describe the observables, the gambles in our setting. Let us recall that in QM any real-valued observable is described by a Hermitian operator (matrix). This naturally imposes restrictions on the type of 'permitted gambles' g on a quantum experiment. For a single particle, given a Hermitian operator $G \in \mathscr{H}^{n\times n}$ (with $\mathscr{H}^{n\times n}$ being the set of Hermitian matrices of dimension $n \times n$), a gamble on $\mathbf{x} \in \overline{\mathbb{C}}^{n}$ can be

[13] The fact that the matrix is Hermitian guarantees that the polynomial is real-valued.

defined as the polynomial function:

$$g(\mathbf{x}) = \mathbf{x}^{\dagger} G \mathbf{x}.$$

Since G is Hermitian and $\mathbf{x}$ is bounded ($\mathbf{x}^{\dagger}\mathbf{x} = 1$), g is a real-valued bounded function. For a composite system of m particles, the functions are m-quadratic forms:

$$g(\mathbf{x}_1, \ldots, \mathbf{x}_m) = (\otimes_{j=1}^{m} \mathbf{x}_j)^{\dagger} G (\otimes_{j=1}^{m} \mathbf{x}_j), \tag{15}$$

with $G \in \mathscr{H}^{n\times n}$, $n = \prod_{j=1}^{m} n_j$, and where $\otimes$ denotes the Kronecker product between vectors regarded as column matrices.[14] The functions (15) are fully defined by the matrix of coefficients G. Therefore, we have that

$$\mathscr{L}_R = \{g \mid \ G \in \mathscr{H}^{n\times n}\}$$

These are the set of 'permitted gambles' in a quantum experiment.[15]

We can also define the subset of nonnegative gambles $\mathscr{L}_R^{\geq} \coloneqq \{g \in \mathscr{L}_R \mid \min g \geq 0\}$ and the subset of negative gambles $\mathscr{L}_R^{<} \coloneqq \{g \in \mathscr{L}_R \mid \max g < 0\}$.[16]

Since $\mathscr{L}_R$ is a vector space including the constant gambles ($G = cI$ with I the identity matrix),[17] we can use standard almost-desirability [12, Appendix F] to impose rationality principles (coherence) in the way a subject should accept gambles.[18]

However, this would not lead to QM. Indeed, as discussed in the Introduction, QM follows by the two principles of coherence and computation.[19]

As shown by [14, Sec.6], for $m > 1$ the problem of deciding whether a gamble is nonnegative, that is whether it belongs to $\mathscr{L}_R^{\geq}$, is **NP**-hard, thus leading to a violation of the aforementioned computation principle.[20] To fulfil the computation requirement, we therefore need to change the meaning of 'being nonnegative' by considering a subset $\Sigma^{\geq} \subsetneq \mathscr{L}^{\geq}$ for which the membership problem is in **P**. This can be done for instance

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[14] Why the Kronecker product? We have discussed this aspect in Section 2.

[15] As discussed in Section 2, this class of functions allows us to "bet" on the "directions" of the spin. Indeed, the case $n_j = 2$ is formally equivalent to Alice and Bob's location problem. For the case $n_j > 2$, $\mathbf{x}$ represents a generalised direction.

[16] Notice that, since g is a polynomial and Ω is bounded, $\min g = \inf g$ and $\max g = \sup g$.

[17] The constant functions take the form $g(\mathbf{x}_1, \ldots, \mathbf{x}_m) = c(\otimes_{j=1}^{m} \mathbf{x}_j)^{\dagger} I (\otimes_{j=1}^{m} \mathbf{x}_j) = c$.

[18] This is the approach originally proposed by De Finetti for the subjective foundation of classical probability theory [17]. The idea is that of introducing a betting scheme and defining bettors as rational if their stakes are placed so as to avoid a sure loss (this is also called a Dutch book; Economics refers to it as 'arbitrage'). De Finetti shows that avoiding sure loss is equivalent to representing a bettor's beliefs through classical probability, thus providing a solid foundation for the latter. De Finetti's bright intuition has greatly been extended in [12, 18], giving rise to the so-called *theory of desirable gambles* (TGD). This is a dual theory of probability in the sense that probability is recovered from TGD via standard mathematical duality. TGD is based on 3 core principles of coherence/consistency: (1) a rational bettor must always accept nonnegative gambles; (2) a rational bettor must always reject negative gambles; (3) bettors' utility is linear.

[19] QM is a theory of bounded (algorithmic) rationality [19, 20]. Generalised types of coherence were described in some detail in [11, 21].

[20] The infimum coincides with the minimum because gambles are bounded polynomials.

by considering the following new set of nonnegative gambles:

$$\Sigma^{\geq} \coloneqq \{g \in \mathscr{L}_R \mid G \geq 0\}.$$

That is, a gamble is 'nonnegative' whenever G is PSD. Note that $\Sigma^{\geq}$ is the so-called cone of *Hermitian sum-of-squares* polynomials .

What is described above constitutes the essence of the algorithmic rationality behind QM. In other words, the corresponding algorithmic theory of desirable gambles is based on the following redefinition of the tautologies (that is, the gambles a subject should always accept):

- $\Sigma^{\geq}$ should always be desirable,

The rest of the theory follows exactly the footprints of the standard theory of (almost) desirability. In particular, the deductive closure for a finite set of assessments $\mathscr{G}$ is defined by:[21]

- $\mathscr{C} \coloneqq \text{posi}(\Sigma^{\geq} \cup \mathscr{G})$.

And finally the coherence postulate simply states that

- A set $\mathscr{C}$ of desirable gambles is said to be *A-coherent* if and only if $-1 \notin \mathscr{C}$,

where 'A' stands for the the fact that the algorithmic bounds of the coherence problem for a finite set of assessments are established according to the choice of $\Sigma^{\geq}$.[22] Indeed, this postulate is just a 'no-Dutch book' criterion.[23]

Remark 2. There are different notions of desirability (almost, strict, real [12]); here we use the term desirability for almost desirability. A-coherence is an instance of almost desirability.[24]

Remark 3. Hermitian sum-of-squares is the definition of nonnegativity used in QM. Other set of nonnegative gambles for which the membership problem is in **P** can be defined, see for instance [2, Appendix]. Answering the question why Hermitian sum-of-squares is an open problem we intend to investigate in future work.

We can finally associate a 'probabilistic' interpretation through the dual of an A-coherent set. Let us consider the dual space $\mathscr{L}_R^*$ of all bounded linear functionals

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[21] 'posi($\mathscr{A}$)' denotes the conic hull of a set of gambles $\mathscr{A}$. It is defined as $\text{posi}(\mathscr{A}) = \{\sum_{i=0}^{\ell} \lambda_i g_i : \lambda_i \in \mathbb{R}^{\geq}, g_i \in \mathscr{A}, \ell > 0\}$, where $\mathbb{R}^{\geq}$ denotes the set of all non negative real numbers. This deductive-closure is equivalent to state that the underlying utility is linear.

[22] In classical probability coherence, the tautologies are the set of all nonnegative gambles $\mathscr{L}_R^{\geq}$. This is the only difference w.r.t. QM. The classical axioms of desirability are: (i) $\mathscr{L}_R^{\geq}$ should always be desirable; (ii) $\mathscr{K} \coloneqq \text{posi}(\mathscr{L}_R^{\geq} \cup \mathscr{G})$; (iii) $-1 \notin \mathscr{K}$. From these axioms, one can derive classical probability theory.

[23] Note in fact that if a subject accepts a negative gamble, that is a negative gamble f is included in $\mathscr{G}$, then we can find a positive gamble h and a positive constant λ such that $\lambda(f + h) = -1$. Therefore, $\mathscr{C}$ includes -1 violating the coherence postulate.

[24] We could extend this framework to real desirability, but this would not preserve the duality between moment matrices and set of desirable gambles, which is necessary to derive the QM axioms.

$\widetilde{L} : \mathscr{L}_R \to \mathbb{R}$. With the additional condition that linear functionals preserve the unit gamble, the dual cone of an A-coherent $\mathscr{C} \subset \mathscr{L}_R$ is given by

$$\mathscr{C}^\circ := \left\{ \widetilde{L} \in \mathsf{S} \mid \widetilde{L}(g) \geq 0, \ \forall g \in \mathscr{G} \right\}, \tag{16}$$

where $\mathsf{S} = \{\widetilde{L} \in \mathscr{L}_R^* \mid \widetilde{L}(1) = 1, \ \ \widetilde{L}(h) \geq 0 \ \ \forall h \in \Sigma^{\geq}\}$ is the set of states. It is not difficult to prove [2] that $\mathscr{C}^\circ$ can actually equivalently be defined as:

$$\mathscr{M} := \{\rho \in \mathcal{S} \mid Tr(G\rho) \geq 0, \ \ \forall g \in \mathscr{G}\}, \tag{17}$$

where $\mathcal{S} = \{\rho \in \mathscr{H}^{n \times n} \mid \rho \geq 0, \ \ Tr(\rho) = 1\}$ is the set of all density matrices and gambles g are defined as in (15) and are essentially specified by the Hermitian matrix G. As for Bob and Alice's location problem in Section 2, we have just shown that the valid operators $\widetilde{L}$ are those that can be written as

$$\widetilde{L}(g) = Tr(G\rho),$$

where ρ plays the same role of M that they are quasi-moment[25] matrices: $\rho := \widetilde{L}(\mathbf{z}\mathbf{z}^\dagger)$ with $\mathbf{z} = \otimes_{j=1}^m \mathbf{x}_j$.

The derivation allows us to formulate quantum weirdness (that is the disagreement between QM and classical physics) as a Dutch book (sure loss). This goes as follows. Given that QM uses a stronger notion of positivity/negativity, a set of desirable gambles can include a gamble $f \in \mathscr{L}_R^{<} \backslash \Sigma^{<}$ and still be A-coherent. When this happens, we have entanglement. In this case, the experimental results appear illogical to us (incompatible with our common understanding), because they are simply incoherent under classical desirability.

3.1. What is the relationship with Alice and Bob's location example?

When confronting QM with the case depicted by Alice and Bob's location example, we notice that the possibility space is essentially the same. We used algorithmic rationality to derive (16) via duality from A-coherence and have shown that the set of valid operators are:

$$\mathsf{S} = \{\widetilde{L} \in \mathscr{L}_R^* \mid \widetilde{L}(1) = 1, \ \ \widetilde{L}(h) \geq 0 \ \ \forall h \in \Sigma^{\geq}\}.$$

[25] In classical probability, given a (real) variable x and an expectation operator E, the n-th (non-central) moment of x is defined as $m_n := E[x^n]$ (we can also define multivariate moments, e.g., $E[x_1^n x_2^m]$). Given a sequence of moments $m_0, m_1, m_2, \ldots, m_n$, there exist infinitely many probability distributions corresponding to the same moments and they form a convex set. A sequence of scalars $m_0, m_1, m_2, \ldots, m_n$ is a valid sequence of moments provided that they satisfy certain consistency constraints. For instance, the moment matrix, obtained by organising that sequence into a matrix (in a certain way), must be positive semi-definite. This gives reason for the constraint $\rho \geq 0$ for density matrices in QT. In general, ρ is a quasi-moment matrix, that is a moment matrix computed with respect to a probability 'charge'. [2].

Any $\widetilde{L} \in \mathsf{S}$ satisfies **P1** and it is immediate to show that it also satisfies **P2***. In fact by exploiting (17), that is the fact that $\widetilde{L}(g) = Tr(G\rho)$, we can prove that

$$\underline{\lambda}_G \leq \widetilde{L}((\otimes_{j=1}^m \mathbf{x}_j)^\dagger G(\otimes_{j=1}^m \mathbf{x}_j)) \leq \overline{\lambda}_G.$$

where $\underline{\lambda}_G, \overline{\lambda}_G$ are the smallest and, respectively, largest eigenvalue of G.

4. The symmetrisation postulate

In this section, we formulate the symmetrisation postulate using QM theory [22, XIV.C-1, p. 1434]. In the next section, we will instead derive this postulate using exchangeable gambles.

Suppose we have m particles, each with single-particle state space represented by a vector space $V = \overline{\mathbb{C}}^n$ (we assume $n_j = n$, same dimension for all particles). We denote a state (a wavefunction) with $|\psi\rangle$, where $|\psi\rangle \in V$.[26]

According to QM postulates, if the particles were distinguishable the composite space of m particles would be given by $\otimes_{i=1}^m V$. Let $\left|\alpha^{(1)}\right\rangle, \ldots, \left|\alpha^{(m)}\right\rangle$ denote a basis of V, so that an element of the basis of $\otimes_{i=1}^m V$ is denoted as $|\psi\rangle = |\alpha_1\rangle \otimes \cdots \otimes |\alpha_m\rangle$ where $\alpha_i \in \{\alpha^{(1)}, \ldots, \alpha^{(m)}\}$ is the state of the i-th particle. Note that the underlying field here is $\mathbb{C}$ and, therefore, complex combination of the basis elements are allowed (these complex combinations correspond to the most general form of the so-called *superposition* of basis states in QM).

Remark 4. In section 3 we considered $\mathbf{x}_i \in V$, while in this section we use $|\alpha_i\rangle \in V$. Why? The reason is that, in Section 3, $\mathbf{x}_i$ represents an unknown "classical" variable (e.g., the direction of the spin) and we ask a subject to express her beliefs about $\mathbf{x}_i$ in terms of acceptance of gambles. Conversely, $|\alpha_i\rangle$ is a state: a proxy quantity which is used in QM to compute the probability of the results of an experiment. QM postulates are formulated in terms of $|\alpha_i\rangle$ (usually denoted as $|\psi_i\rangle$). Indeed, under the epistemic interpretation of QM, $|\alpha_i\rangle$ corresponds to a belief state and so it is different from $\mathbf{x}_i$. This difference is also evident from the fact that, for a composite system, $|\psi\rangle = |\alpha_1\rangle \otimes \cdots \otimes |\alpha_m\rangle \in \otimes_{i=1}^m V$, while $[\mathbf{x}_1, \ldots, \mathbf{x}_m] \in \times_{i=1}^m V$. To understand this difference, consider the toss of a classical coin: $\Omega = \{H, T\}$ and $p = [p_H, p_T] \in \mathbb{R}^2$ is the vector of probabilities for Heads and Tails. Now consider the toss of three coins, the composite possibility space is $\times_{i=1}^3 \Omega$, while the joint probability mass function belongs to $\otimes_{i=1}^3 \mathbb{R}^2 = \mathbb{R}^8$.

In this work, we are interested in defining the state space for indistinguishable particles.

Let π denote a permutation of the indices of the elements of the tensor product $|\alpha_1\rangle \otimes \cdots \otimes |\alpha_m\rangle$. Since such a permutation defines the product $\left|\alpha_{\pi(1)}\right\rangle \otimes \cdots \otimes \left|\alpha_{\pi(m)}\right\rangle$,

[26]Note that, $|\psi\rangle$ is a ket, it is just the notation used in QM for a column vector (Dirac notation), which is commonly denoted as $\boldsymbol{\psi}$ (bold) in linear algebra. $\boldsymbol{\psi}^T$ is instead denoted as $\langle\psi|$ in QM. The quadratic form $\boldsymbol{\psi}^T G \boldsymbol{\psi}$ is then written as $\langle\psi| G |\psi\rangle$, while the matrix $\boldsymbol{\psi}\boldsymbol{\psi}^T$ as $|\psi\rangle\langle\psi|$ (which is called a pure density matrix in QM).

by permuting the elements of the tensor products, we are basically permuting the labels of the particles. A permutation that only swaps two variables is called a *transposition*.

The *sign of a permutation* π, denoted by $\mathrm{sign}(\pi)$, equals 1 if π can be written as a product of an even number of transpositions, and equals -1 if π can be written as a product of an odd number of transpositions. Notice that the sign of π can be calculated as follows:

$$\mathrm{sign}(\pi) = \det \sum_{i=1}^{m} \mathbf{e}_i \mathbf{e}_{\pi(i)}^{\top},$$

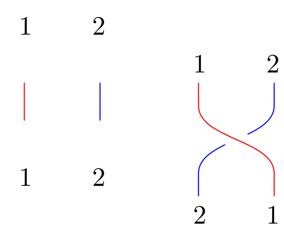
where $\mathbf{e}_i$ is an element of the canonical basis of $\mathbb{R}^m$ (see [22, XIV.B-2-c]).

Since permutations are linear operators, we can equivalently express a permutation π as a matrix operator P_π acting on the tensor product:

$$P_\pi(|\alpha_1\rangle \otimes \cdots \otimes |\alpha_m\rangle) := \left|\alpha_{\pi(1)}\right\rangle \otimes \cdots \otimes \left|\alpha_{\pi(m)}\right\rangle .$$

The matrix P_π is unitary, that is $P_\pi^\dagger P_\pi = P_\pi P_\pi^\dagger = I$, but not necessarily Hermitian [22, XIV.B-2-b].[27] In what follows, by $\mathbb{P}_m$ we both denote the collection of all permutations and of all corresponding permutation operators.

Example 3. Consider $m = 2$ particles with $|\alpha_1\rangle, |\alpha_2\rangle \in \overline{\mathbb{C}}^2$. In this case there are only two possible permutations π_a (identity) and π_b (swap) with $\mathrm{sign}(\pi_b) = -1$:



The permutation matrices are $P_{\pi_a} = I$ and:

$$P_{\pi_b} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} . \tag{18}$$

The latter acts on $|\alpha_1\rangle \otimes |\alpha_2\rangle$ as follows

$$P_{\pi_b}(|\alpha_1\rangle \otimes |\alpha_2\rangle) = P_{\pi_b} \begin{bmatrix} \alpha_{11}\alpha_{21} \\ \alpha_{11}\alpha_{22} \\ \alpha_{12}\alpha_{21} \\ \alpha_{12}\alpha_{22} \end{bmatrix} = \begin{bmatrix} \alpha_{11}\alpha_{21} \\ \alpha_{12}\alpha_{21} \\ \alpha_{11}\alpha_{22} \\ \alpha_{12}\alpha_{22} \end{bmatrix} = |\alpha_2\rangle \otimes |\alpha_1\rangle .$$

[27]A transposition is Hermitan. A generic permutation can be written as product of transpositions and, since the transpositions do not necessarily commute, their product may not be Hermitian.

where $|\alpha_1\rangle = [\alpha_{11}, \alpha_{12}]^\top$ and $|\alpha_2\rangle = [\alpha_{21}, \alpha_{22}]^\top$.

We now introduce the *symmetriser* and the *antisymmetriser*:

$$\Pi_{\text{Sym}} := \frac{1}{m!} \sum_{\pi_r \in \mathbb{P}_m} P_{\pi_r},$$
$$\Pi_{\text{Anti}} := \frac{1}{m!} \sum_{\pi_r \in \mathbb{P}_m} \text{sign}(\pi_r) P_{\pi_r}.$$

which are projectors[28] [22, XIV.B-2-c]. They project onto respectively:

$$\text{Sym}^m V = \{|\psi\rangle \in \otimes_{i=1}^m V : P_\pi |\psi\rangle = |\psi\rangle\,, \ \ \forall \pi \in \mathbb{P}_m\}$$
$$\text{Anti}^m V = \{|\psi\rangle \in \otimes_{i=1}^m V : P_\pi |\psi\rangle = \text{sign}(\pi)\, |\psi\rangle\,, \ \ \forall \pi \in \mathbb{P}_m\}.$$

We call the elements of $\text{Sym}^m V$ (respectively, $\text{Anti}^m V$) symmetric (respectively, antisymmetric) states.

Lemma 4 (Sec. XIV.B-2-c [22]). *The following equalities hold for any permutation operator $P_\pi \in \mathbb{P}_m$:*

1. $P_\pi \Pi_{\text{Sym}} = \Pi_{\text{Sym}} P_\pi = \Pi_{\text{Sym}}$*;*
2. $P_\pi \Pi_{\text{Anti}} = \Pi_{\text{Anti}} P_\pi = \text{sign}(\pi) \Pi_{\text{Anti}}$*.*

PROOF. Given two permutations $P_{\pi_i} \neq P_{\pi_j}$, we have that $P_\pi P_{\pi_i} \neq P_\pi P_{\pi_j}$. Hence we have that

$$P_\pi \Pi_{\text{Sym}} = \frac{1}{m!} \sum_{\pi_r \in \mathbb{P}_m} P_\pi P_{\pi_r} = \frac{1}{m!} \sum_{\pi'_r \in \mathbb{P}_m} P_{\pi'_r}.$$

Analogously, since $\text{sign}(\pi)\text{sign}(\pi) = 1$

$$\begin{aligned} P_\pi \Pi_{\text{Anti}} &= \frac{1}{m!} \sum_{\pi_r \in \mathbb{P}_m} \text{sign}(\pi_r) P_\pi P_{\pi_r} \\ &= \frac{\text{sign}(\pi)}{m!} \sum_{\pi_r \in \mathbb{P}_m} \text{sign}(\pi_r)\text{sign}(\pi) P_\pi P_{\pi_r} \\ &= \frac{\text{sign}(\pi)}{m!} \sum_{\pi'_r \in \mathbb{P}_m} \text{sign}(\pi'_r) P_{\pi'_r}. \end{aligned}$$

Similar results can be derived for the other sides: $\Pi_{\text{Sym}} P_\pi$ and $\Pi_{\text{Anti}} P_\pi$.

The *symmetrisation postulate* states the following:

[28] They are Hermitian $\Pi_{\text{Sym}}^\dagger = \Pi_{\text{Sym}}, \Pi_{\text{Anti}}^\dagger = \Pi_{\text{Anti}}$ and they satisfy $\Pi_{\text{Sym}}^2 = \Pi_{\text{Sym}}, \Pi_{\text{Anti}}^2 = \Pi_{\text{Anti}}$ and $\Pi_{\text{Sym}} \Pi_{\text{Anti}} = \Pi_{\text{Anti}} \Pi_{\text{Sym}} = 0$ [22, XIV.B-2-c].

> When a system includes several identical particles, only certain states of its state space can describe its physical states. Physical states are, depending on the nature of the identical particles, either completely symmetric or completely antisymmetric with respect to permutation of these particles. Those particles for which the physical states are symmetric are called bosons, and those for which they are antisymmetric, fermions. [22, XIV.C-1, p. 1434]

The postulate thus limits the state space (possibility space) for a system of identical particles. Contrary to the case of particles of different natures, this space is no longer the tensor product $\otimes_{i=1}^{m} V$ of the individual state spaces of the particles constituting the system, but rather a subspace, namely $\mathrm{Sym}^m V$ or $\mathrm{Anti}^m V$, depending on whether the particles are bosons or fermions. Only states belonging either to $\mathrm{Sym}^m V$ or to $\mathrm{Anti}^m V$ are physically possible. This is the reasons they are called *physical states*.

Given k physical states $|\psi_i\rangle$ (belonging to either $\mathrm{Sym}^m V$ or $\mathrm{Anti}^m V$), we can then define the density matrix as usual:

$$\rho = \sum_{i=1}^{k} p_i \, |\psi_i\rangle \, \langle\psi_i| \, ,$$

where p_i are probabilities, $p_i \geq 0$ and $\sum_{i=1}^{k} p_i = 1$. It can then be verified that, in the symmetric case, given that $|\psi_i\rangle = \Pi_{\mathrm{Sym}} |\psi_i\rangle$, we have that $\rho = \Pi_{\mathrm{Sym}} \rho \Pi_{\mathrm{Sym}}$. Similarly, in the antisymmetric case, $\rho = \Pi_{\mathrm{Anti}} \rho \Pi_{\mathrm{Anti}}$.

Example 4. We continue Example 3 by defining the the projectors:

$$\Pi_{\mathrm{Sym}} = \frac{I + P_{\pi_b}}{2}, \;\; \Pi_{\mathrm{Anti}} = \frac{I + \mathrm{sign}(\pi_b) P_{\pi_b}}{2} = \frac{I - P_{\pi_b}}{2}, \tag{19}$$

which act on $|\alpha_1\rangle \otimes |\alpha_2\rangle$ as follows[29]

$$\Pi_{\mathrm{Sym}}(|\alpha_1\rangle \otimes |\alpha_2\rangle) = \begin{bmatrix} \alpha_{11}\alpha_{21} \\ \frac{\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21}}{2} \\ \frac{\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21}}{2} \\ \alpha_{12}\alpha_{22} \end{bmatrix}, \tag{20}$$

$$\Pi_{\mathrm{Anti}}(|\alpha_1\rangle \otimes |\alpha_2\rangle) = \begin{bmatrix} 0 \\ \frac{\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}}{2} \\ \frac{\alpha_{12}\alpha_{21} - \alpha_{11}\alpha_{22}}{2} \\ 0 \end{bmatrix} \tag{21}$$

From the last equality, it follows that, in case $\alpha_1 = \alpha_2$, $\Pi_{\mathrm{Anti}}(|\alpha_1\rangle \otimes |\alpha_2\rangle) = 0$. This is called the *Pauli exclusion principle*: two fermions cannot have identical state.

[29] The right hand side term in (20) or (21) is a complex vector, but its norm can be different from one. In this latter case, it needs to be normalised (if it is not zero) [22, XIV.C-3-b].

5. Exchangeable gambles

In the previous section, we discussed the symmetrisation postulate. In this section, we formulate it in terms of A-coherence and exchangeability. In doing so, we extend some of the definitions and results originally presented in [11] to the quantum setting introduced in Section 3.

As discussed in Section 3, we consider gambles on $\mathbf{x}_i \in V = \overline{\mathbb{C}}^n$. Given m particles, the possibility space is $\times_{i=1}^m V$. Therefore, π denotes a permutation of the indices of the vector $(\mathbf{x}_1, \ldots, \mathbf{x}_m)$, i.e.,

$$\pi(\mathbf{x}_1, \ldots, \mathbf{x}_m) = (\mathbf{x}_{\pi(1)}, \ldots, \mathbf{x}_{\pi(m)}).$$

This is the same kind of symmetry we considered in Section 2.3, when we discussed the case of "exchangeable employees". We are simply considering an invariance to re-labelling.

A generic gamble is denoted as:

$$g(\mathbf{z}, \mathbf{z}) := \mathbf{z}^\dagger G \mathbf{z},$$

with $\mathbf{z} := \otimes_{j=1}^m \mathbf{x}_j$. Note that we are assuming that g is a function of two arguments just for convenience to define exchangeability symmetries. Let π_r, π_l be two permutations, we define

$$\begin{aligned}\pi_l g(\mathbf{z}, \mathbf{z}) \pi_r &:= \frac{1}{2} \left(g(\pi_l \mathbf{z}, \pi_r \mathbf{z}) + g(\pi_r \mathbf{z}, \pi_l \mathbf{z}) \right) \\ &= \frac{1}{2} \left(\mathbf{z}^\dagger P_{\pi_l}^\dagger G P_{\pi_r} \mathbf{z} + \mathbf{z}^\dagger P_{\pi_r}^\dagger G P_{\pi_l} \mathbf{z} \right).\end{aligned}$$

Note that (i) $\pi_l g \pi_r = \pi_r g \pi_l$, and (ii) $\pi_l g \pi_r$ is a gamble (it returns real values).[30]

Remark 5. This definition of permuted gamble is different from the one used in [11] (the permutation of $g(\omega)$ is defined as $\pi \circ g = g(\pi\omega)$). In QM, gambles are quadratic forms of complex variables and, therefore, we can define more general symmetries by exploiting the fact that $\mathbf{z}$ and its complex conjugate $\mathbf{z}^\dagger$ can be treated as two "different" variables.

Example 5. Consider $m = 2$ particles with $\mathbf{x}_1, \mathbf{x}_2 \in \overline{\mathbb{C}}^2$. We have already seen that there are only two possible permutations π_a (identity) and π_b (swap). Therefore, we have $\pi_a g \pi_a = g$ and

$$\begin{aligned}\pi_a g \pi_b &= \frac{1}{2} \left((\mathbf{x}_1 \otimes \mathbf{x}_2) G (\mathbf{x}_2 \otimes \mathbf{x}_1) + (\mathbf{x}_2 \otimes \mathbf{x}_1) G (\mathbf{x}_1 \otimes \mathbf{x}_2) \right), \\ \pi_b g \pi_b &= (\mathbf{x}_2 \otimes \mathbf{x}_1) G (\mathbf{x}_2 \otimes \mathbf{x}_1).\end{aligned}$$

[30] This holds because $P_{\pi_l}^\dagger G P_{\pi_r} + P_{\pi_r}^\dagger G P_{\pi_l}$ is Hermitian.

For $\pi_l, \pi_r \in \mathbb{P}_m$, we write

$$\delta^\star_{l,r} := \begin{cases} \text{sign}(\pi_l)\text{sign}(\pi_r) & \text{when } \star = Anti, \\ 1 & \text{when } \star = Sym. \end{cases}$$

Given this definition, in the remaining of this section, all definitions, results and corresponding proofs will be parameterised by $\star \in \{Anti, Sym\}$ and $\delta^\star_{l,r}$. They therefore apply, uniformly, to both the symmetric and the antisymmetric cases.

We now provide the definition of an A-coherent $\star$-exchangeable set of desirable gambles.

Definition 3. Consider the set

$$\mathscr{I}_\star := \{g - \delta^\star_{l,r}\pi_l g \pi_r \mid g \in \mathscr{L}_R,\ \pi_l, \pi_r \in \mathbb{P}_m\}.$$

We say that an A-coherent set of desirable gambles $\mathscr{C}$ is *$\star$-exchangeable* if $\mathscr{I}_\star \subseteq \mathscr{C}$. Intuitively, this means that a subject, whose set of desirable gambles includes $\mathscr{I}_\star$, is willing to exchange any gamble g for $\delta^\star_{l,r}\pi_l g \pi_r$.

Given Definition 3, we can prove the following result.

Proposition 4. *Let $\mathscr{C}$ be an A-coherent set of desirable gambles. If $\mathscr{C}$ is $\star$-exchangeable, then it is also $\star$-permutable, that is $\delta^\star_{l,r}\pi_l g \pi_r$ are in $\mathscr{C}$ for all $g \in \mathscr{C}$ and all $\pi_l, \pi_r \in \mathbb{P}_m$.*

PROOF. The proof is similar as the one for [11, Prop.9]. For $g \in \mathscr{C}$ and $\pi_l, \pi_r \in \mathbb{P}_m$, we have $-g - \delta^\star_{l,r}\pi_l(-g)\pi_r \in \mathscr{I}_\star \subseteq \mathscr{C}$. Given that $-g = \mathbf{z}^\dagger(-G)\mathbf{z}$, then $-g - \delta^\star_{l,r}\pi_l(-g)\pi_r = \delta^\star_{l,r}\pi_l g \pi_r - g$. Since $\delta^\star_{l,r}\pi_l g \pi_r = \delta^\star_{l,r}\pi_l g \pi_r - g + g$ and $g, \delta^\star_{l,r}\pi_l g \pi_r - g \in \mathscr{C}$, we conclude by additivity that $\delta^\star_{l,r}\pi_l g \pi_r \in \mathscr{C}$.

As in [11], but taking into account that we are working with quadratic forms, we define the linear operators

$$\text{ex}^m_\star(g) := \mathbf{z}^\dagger \Pi^\dagger_\star G \Pi_\star \mathbf{z}.$$

We verify some of their properties; in particular that they can be used to equivalently characterise symmetric and antisymmetric exchangeability (Corollary 1).

The first result follows immediately from the fact that the symmetrisers and the antisymmetriser are projectors.

Lemma 5. *Let g be a gamble, then $ex^m_\star(ex^m_\star(g)) = ex^m_\star(g)$.*

The idea behind this linear transformations $\text{ex}^m_\star(g)$ is that they render a gamble g insensitive to permutation by replacing it with the uniform average $\text{ex}^m_\star(g)$ of all its permutations $\pi_l g \pi_r$, as shown hereafter.

Proposition 5. *Let g be a gamble, then*

$$ex^m_\star(g) = \frac{1}{m!m!} \sum_{\pi_r, \pi_l \in \mathbb{P}_m} \delta^\star_{l,r}\pi_l g \pi_r.$$

PROOF. It is immediate to verify that $\text{ex}_\star^m(g) = \frac{1}{m!m!}\sum_{\pi_r,\pi_l\in\mathbb{P}_m}\delta^\star_{l,r} g(\pi_l\mathbf{z},\pi_r\mathbf{z})$. To conclude, note that:

$$\begin{aligned}
&\sum_{\pi_l,\pi_r\in\mathbb{P}_m}\delta^\star_{l,r}\pi_l g\pi_r = \\
&= \sum_{\pi_l,\pi_r\in\mathbb{P}_m}\frac{\delta^\star_{l,r}}{2}\left(g(\pi_l\mathbf{z},\pi_r\mathbf{z}) + g(\pi_r\mathbf{z},\pi_l\mathbf{z})\right)\\
&= \frac{1}{2}\sum_{\pi_l,\pi_r\in\mathbb{P}_m}\delta^\star_{l,r} g(\pi_l\mathbf{z},\pi_r\mathbf{z}) + \frac{1}{2}\sum_{\pi_l,\pi_r\in\mathbb{P}_m}\delta^\star_{r,l} g(\pi_r\mathbf{z},\pi_l\mathbf{z})\\
&= \sum_{\pi_l,\pi_r\in\mathbb{P}_m}\delta^\star_{l,r} g(\pi_l\mathbf{z},\pi_r\mathbf{z}).
\end{aligned}$$

Clearly, the linear transformations $\text{ex}_\star^m$ assume the same value for all gambles that can be related to each other through some permutation.

Proposition 6. *Let g be a gamble, and $\pi_l,\pi_r\in\mathbb{P}_m$. Then*

$$ex_\star^m(\delta^\star_{l,r}\pi_l g\pi_r) = ex_\star^m(g).$$

PROOF. By exploiting linearity

$$\begin{aligned}
&\text{ex}_\star^m(\delta^\star_{l,r}\pi_l g\pi_r) = \delta^\star_{l,r}\text{ex}_\star^m(\pi_l g\pi_r) = \\
&= \delta^\star_{l,r}\left(\mathbf{z}^\dagger\Pi_\star^\dagger\left(\frac{1}{2}\left(P^\dagger_{\pi_l}GP_{\pi_r} + P^\dagger_{\pi_r}GP_{\pi_l}\right)\right)\Pi_\star\mathbf{z}\right)\\
&= \frac{\delta^\star_{l,r}}{2}\mathbf{z}^\dagger\Pi_\star^\dagger P^\dagger_{\pi_l}GP_{\pi_r}\Pi_\star\mathbf{z} + \frac{\delta^\star_{l,r}}{2}\mathbf{z}^\dagger\Pi_\star^\dagger P^\dagger_{\pi_r}GP_{\pi_l}\Pi_\star\mathbf{z}\\
&= \frac{\delta^\star_{l,r}}{2}\mathbf{z}^\dagger(P_{\pi_l}\Pi_\star)^\dagger G(P_{\pi_r}\Pi_\star)\mathbf{z} + \frac{\delta^\star_{l,r}}{2}\mathbf{z}^\dagger(P_{\pi_r}\Pi_\star)^\dagger G(P_{\pi_l}\Pi_\star)\mathbf{z}
\end{aligned}$$

By Lemma 4 and the fact that $\delta^\star_{l,r}\delta^\star_{l,r} = 1$, we finally obtain

$$\begin{aligned}
&\frac{\delta^\star_{l,r}}{2}\mathbf{z}^\dagger(P_{\pi_l}\Pi_\star)^\dagger G(P_{\pi_r}\Pi_\star)\mathbf{z} + \frac{\delta^\star_{l,r}}{2}\mathbf{z}^\dagger(P_{\pi_r}\Pi_\star)^\dagger G(P_{\pi_l}\Pi_\star)\mathbf{z} = \\
&= \frac{\delta^\star_{l,r}\delta^\star_{l,r}}{2}\mathbf{z}^\dagger\Pi_\star^\dagger G\Pi_\star\mathbf{z} + \frac{\delta^\star_{l,r}\delta^\star_{l,r}}{2}\mathbf{z}^\dagger\Pi_\star^\dagger G\Pi_\star\mathbf{z}\\
&= \text{ex}_\star^m(g).
\end{aligned}$$

Similarly to what was done by [11], we can prove the following.

Corollary 1. *Let $\mathscr{C}$ be an A-coherent set of desirable gambles. Given*

$$\mathscr{V}_\star := \{g - ex_\star^m(g) \mid g\in\mathscr{L}_R\}$$

the following claims are equivalent,

(1) *$\mathscr{C}$ is $\star$-exchangeable;*

(2) $\mathscr{V}_\star \subseteq \mathscr{C}$.

PROOF. For $(1 \Rightarrow 2)$, by Proposition 5, we can write $g - \text{ex}_\star^m(g) = \frac{1}{m!m!}\sum_{\pi_l\pi_r}(g - \delta^\star_{l,r}\pi_l g\pi_r)$. Since $\mathscr{C}$ satisfies additivity and given $\mathscr{I}_\star \subseteq \mathscr{C}$, then $g - \text{ex}_\star^m(g) \in \mathscr{C}$.

For $(2 \Rightarrow 1)$, by linearity of $\text{ex}_\star^m$ and Proposition 6

$$g - \delta^\star_{l,r}\pi_l g\pi_r - \text{ex}_\star^m(g - \delta^\star_{l,r}\pi_l g\pi_r) = g - \delta^\star_{l,r}\pi_l g\pi_r,$$

which shows that $g - \delta^\star_{l,r}\pi_l g\pi_r \in \mathscr{C}$.

The following result also holds.

Proposition 7. *Let $\mathscr{C}$ be an A-coherent set of desirable gambles. Then, assuming $\mathscr{C}$ is $\star$-exchangeable, the following claims hold for all gambles g, g':*

1. *$g \in \mathscr{C}$ iff $ex_\star^m(g) \in \mathscr{C}$;*
2. *if $ex_\star^m(g) = ex_\star^m(g')$ then $g \in \mathscr{C}$ iff $g' \in \mathscr{C}$.*

PROOF. The proof is the same as for [11, Prop.10]. First notice that the first claim follows from the second, by taking $g' := \text{ex}_\star^m(g)$ and applying Lemma 5. For the second claim, assume $\text{ex}_\star^m(g) = \text{ex}_\star^m(g')$ and $g \in \mathscr{C}$. Notice that $g' - \text{ex}_\star^m(g') = g' - \text{ex}_\star^m(g), -g - \text{ex}_\star^m(-g) = \text{ex}_\star^m(g) - g \in \mathscr{V}_\star$. By Corollary 1 and additivity, we obtain $(g' - \text{ex}_\star^m(g)) + (\text{ex}_\star^m(g) - g) + g = g' \in \mathscr{C}$.

We now consider the dual of an A-coherent $\star$-exchangeable set of gambles.

From Section 3, to define the dual, we consider the dual space $\mathscr{L}_R^*$ of all bounded linear functionals $\widetilde{L} : \mathscr{L}_R \to \mathbb{R}$. With the additional condition that linear functionals preserve the unit gamble, the dual cone of an A-coherent $\mathscr{C} \subset \mathscr{L}_R$ is given by

$$\mathscr{C}^\circ = \left\{\widetilde{L} \in \mathsf{S} \mid \widetilde{L}(g) \geq 0,\ \forall g \in \mathscr{C}\right\}, \tag{22}$$

where $\mathsf{S} = \{\widetilde{L} \in \mathscr{L}_R^* \mid \widetilde{L}(1) = 1,\ \ \widetilde{L}(h) \geq 0\ \ \forall h \in \Sigma^{\geq}\}$ is the set of states.

Definition 4. Let $\widetilde{L} \in \mathsf{S}$. We say that $\widetilde{L}$ is *$\star$-exchangeable* if it belongs to the dual $\mathscr{C}^\circ$ of an A-coherent $\star$-exchangeable set of gambles $\mathscr{C}$.

Proposition 8. *Assume $\widetilde{L} \in \mathsf{S}$. The following statements are equivalent:*

1. *$\widetilde{L}$ is $\star$-exchangeable;*
2. *$\widetilde{L}(f) = 0$ for all $f \in \mathscr{I}_\star$.*
3. *$\widetilde{L}(f) = 0$ for all $f \in \mathscr{V}_\star$.*

PROOF. We verify $(1 \Leftrightarrow 2)$. If $\widetilde{L}$ is $\star$-exchangeable, we know that $g - \delta^\star_{l,r}\pi_l g\pi_r$, $\delta^\star_{l,r}\pi_l g\pi_r - g \in \mathscr{C}$, meaning that $\widetilde{L}(g - \delta^\star_{l,r}\pi_l g\pi_r) \geq 0$ and $-\widetilde{L}(g - \delta^\star_{l,r}\pi_l g\pi_r) \geq 0$. Therefore $\widetilde{L}(f) = \widetilde{L}(g - \delta^\star_{l,r}\pi_l g\pi_r) = 0$. For the other direction, assume that

$L(f) = 0$ for all $f \in \mathscr{I}_\star$, From L, by duality, we can define the set of desirable gambles $\{g \in \mathscr{L}_R : L(g) \geq 0\}$. We have proven in [2] that this is an A-coherent set of desirable gambles and, moreover, it includes $\mathscr{I}_\star$ by hypothesis. By Corollary 1, the equivalence (1⇔3) can be proven in a similar way.

We recall the following well-know result (see e.g. [23]).

Proposition 9. *Let G be a Hermitian matrix; then $G \geq 0$ if and only if $Tr(SG) \geq 0$ for all Hermitian $S \geq 0$.*

We use the previous result to prove the following.

Proposition 10. *Assume $\widetilde{L} \in \mathsf{S}$. The following statements are equivalent:*

1. *$\widetilde{L}$ is $\star$-exchangeable;*
2. *$\widetilde{L}\left(\mathbf{z}\mathbf{z}^\dagger - \frac{\delta^\star_{l,r}}{2} P_{\pi_r}\mathbf{z}\mathbf{z}^\dagger P^\dagger_{\pi_l} - \frac{\delta^\star_{l,r}}{2} P_{\pi_l}\mathbf{z}\mathbf{z}^\dagger P^\dagger_{\pi_r}\right) = 0$ for all $\pi_l, \pi_r \in \mathbb{P}_m$;*
3. *$\widetilde{L}\left(\mathbf{z}\mathbf{z}^\dagger - \Pi_\star\mathbf{z}\mathbf{z}^\dagger\Pi^\dagger_\star\right) = 0$.*

PROOF. As before, we only verify the equivalence (1⇔2). Assume $\widetilde{L} \in \mathsf{S}$ is $\star$-exchangeable and consider the set of gambles $\mathscr{A} = \{g - \delta^\star_{l,r}\pi_l g \pi_r : \pi_l, \pi_r \in \mathbb{P}_m, G \geq 0\}$ and $\mathscr{B} = \{\delta^\star_{l,r}\pi_l g\pi_r - g : \pi_l, \pi_r \in \mathbb{P}_m,\ G \geq 0\}$. Since $\widetilde{L}$ is $\star$-exchangeable, it follows that $\widetilde{L}(f), \widetilde{L}(f') \geq 0$ for each $f \in \mathscr{A}, f' \in \mathscr{B}$. This implies that

$$\begin{aligned}
0 &\leq \widetilde{L}(g - \delta^\star_{l,r}\pi_l g\pi_r) \\
&= \widetilde{L}(\mathbf{z}^\dagger G\mathbf{z}) - \frac{\delta^\star_{l,r}}{2}\widetilde{L}\left(\mathbf{z}^\dagger P^\dagger_{\pi_l} G P_{\pi_r}\mathbf{z} + \mathbf{z}^\dagger P^\dagger_{\pi_r} G P_{\pi_l}\mathbf{z}\right) \\
&= Tr\left(G\widetilde{L}\left(\mathbf{z}\mathbf{z}^\dagger - \frac{\delta^\star_{l,r}}{2}P_{\pi_r}\mathbf{z}\mathbf{z}^\dagger P^\dagger_{\pi_l} - \frac{\delta^\star_{l,r}}{2}P_{\pi_l}\mathbf{z}\mathbf{z}^\dagger P^\dagger_{\pi_r}\right)\right)
\end{aligned}$$

for each $\pi_l, \pi_r \in \mathbb{P}_m,\ G \geq 0$. To derive the above equalities, we have exploited that: $\widetilde{L}(\mathbf{z}^\dagger G\mathbf{z}) = \widetilde{L}(Tr(G\mathbf{z}\mathbf{z}^\dagger)) = Tr(G\widetilde{L}(\mathbf{z}\mathbf{z}^\dagger))$. By applying Proposition 9, we can conclude that $\widetilde{L}\left(\mathbf{z}\mathbf{z}^\dagger - \frac{\delta^\star_{l,r}}{2}P_{\pi_r}\mathbf{z}\mathbf{z}^\dagger P^\dagger_{\pi_l} - \frac{\delta^\star_{l,r}}{2}P_{\pi_l}\mathbf{z}\mathbf{z}^\dagger P^\dagger_{\pi_r}\right) \geq 0$. Similarly, we have that $0 \leq \widetilde{L}(-g + \delta^\star_{l,r}\pi_l g\pi_r) = -\widetilde{L}(g - \delta^\star_{l,r}\pi_l g\pi_r)$. Together these inequalities imply that $\widetilde{L}\left(\mathbf{z}\mathbf{z}^\dagger - \frac{\delta^\star_{l,r}}{2}P_{\pi_r}\mathbf{z}\mathbf{z}^\dagger P^\dagger_{\pi_l} - \frac{\delta^\star_{l,r}}{2}P_{\pi_l}\mathbf{z}\mathbf{z}^\dagger P^\dagger_{\pi_r}\right) = 0$. To prove the other direction, simply note that the second claim implies that $0 = \widetilde{L}(-g + \delta^\star_{l,r}\pi_l g\pi_r) = -\widetilde{L}(g - \delta^\star_{l,r}\pi_l g\pi_r)$.

This result simply means that $\widetilde{L}$ is invariant to exchangeability (particles re-labelling). This is the same kind of symmetry we considered in Section 2.3 for Bob and Alice.

In order to derive the main result of the paper, we introduce the following definition.

Definition 5. A density matrix $\rho \in \mathcal{S} = \{\rho \in \mathscr{H}^{n\times n} \mid \rho \geq 0,\ Tr(\rho) = 1\}$ satisfying $\rho = \Pi_\star\rho\Pi^\dagger_\star$ is called symmetric if $\star = \mathrm{Sym}$ and antisymmetric if $\star = \mathrm{Anti}$.

The *symmetrisation postulate* discussed in Section 4 is equivalent to the claim that, for identical particles, the only valid density matrices are those satisfying $\rho = \Pi_\star \rho \Pi_\star^\dagger$ [22, XIV.C-1, p. 1434].

Corollary 2. *Exchangeability implies the symmetrization postulate and vice versa.*

PROOF. We exploit that $\rho = \widetilde{L}(\mathbf{z}\mathbf{z}^\dagger)$ and so $\widetilde{L}\left(\mathbf{z}\mathbf{z}^\dagger - \Pi_\star \mathbf{z}\mathbf{z}^\dagger \Pi_\star^\dagger\right) = 0$ in Proposition 10 can be rewritten as $\rho = \Pi_\star \rho \Pi_\star$. From Proposition 10, we can then conclude that $\widetilde{L}$ is $\star$-exchangeable iff $\rho = \Pi_\star \rho \Pi_\star$.

Corollary 2 therefore derives the *symmetrisation postulate* via duality from an A-coherent exchangeable set of gambles. This means we can equivalently call the symmetric and antisymmetric density matrices as $\star$-exchangeable (for $\star = \text{Sym}$ and, respectively $\star = \text{Anti}$).

Example 6. Consider the density matrix

$$\begin{aligned}\rho := \widetilde{L}(\mathbf{z}\mathbf{z}^\dagger) &= \widetilde{L}\left(\begin{bmatrix} x_{11}x_{11}^\dagger x_{21}x_{21}^\dagger & x_{11}^\dagger x_{12}x_{21}x_{21}^\dagger & x_{11}x_{11}^\dagger x_{21}^\dagger x_{22} & x_{11}^\dagger x_{12}x_{21}^\dagger x_{22} \\ x_{11}x_{12}^\dagger x_{21}x_{21}^\dagger & x_{12}x_{12}^\dagger x_{21}x_{21}^\dagger & x_{11}x_{12}^\dagger x_{21}^\dagger x_{22} & x_{12}x_{12}^\dagger x_{21}^\dagger x_{22} \\ x_{11}x_{11}^\dagger x_{21}x_{22}^\dagger & x_{11}^\dagger x_{12}x_{21}x_{22}^\dagger & x_{11}x_{11}^\dagger x_{22}x_{22}^\dagger & x_{11}^\dagger x_{12}x_{22}x_{22}^\dagger \\ x_{11}x_{12}^\dagger x_{21}x_{22}^\dagger & x_{12}x_{12}^\dagger x_{21}x_{22}^\dagger & x_{11}x_{12}^\dagger x_{22}x_{22}^\dagger & x_{12}x_{12}^\dagger x_{22}x_{22}^\dagger \end{bmatrix}\right) \\ &= \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}\end{aligned} \tag{23}$$

where we have assumed that $\mathbf{z} = \mathbf{x}_1 \otimes \mathbf{x}_2$ with $\mathbf{x}_1 = [x_{11}, x_{12}]^\top$ and $\mathbf{x}_2 = [x_{21}, x_{22}]^\top$. For $P_{\pi_a} = I$ and P_{π_b} as in (18), we have

$$\rho = P_{\pi_a}^\dagger \rho P_{\pi_b} = P_{\pi_b}^\dagger \rho P_{\pi_a} = P_{\pi_b}^\dagger \rho P_{\pi_b}.$$

Therefore, ρ is symmetrically exchangeable (it also satisfies $\Pi_{\text{Sym}} \rho \Pi_{\text{Sym}} = \rho$.) Instead the matrix

$$\rho = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{24}$$

is antisymmetrically exchangeable. It satisfies $\Pi_{\text{Anti}} \rho \Pi_{\text{Anti}} = \rho$ as well as $\rho = -0.5(P_{\pi_a}^\dagger \rho P_{\pi_b} + P_{\pi_b}^\dagger \rho P_{\pi_a}) = P_{\pi_b}^\dagger \rho P_{\pi_b}$.

6. Updating

Let us assume we measure a subset of particles $\mathbf{x}_1, \ldots, \mathbf{x}_{\check{m}}$ with $\check{m} \leq m$. Quantum projection measurements can then be described by a collection $\{\Pi_i\}_{i=1}^{n\check{m}}$, with $\Pi_i \in \mathscr{H}^{n\check{m} \times n\check{m}}$, of projection operators that satisfy the completeness equation $\sum_{i=1}^{n\check{m}} \Pi_i = I$.

We recall the following definition from [2].

Definition 6. Let $\mathscr{C}$ be an A-coherent $\star$-exchangeable coherent set of desirable gambles. Then the set obtained as

$$\mathscr{C}_{\Pi_i} = \left\{ \mathbf{z}^\dagger G \mathbf{z} \mid \mathbf{z}^\dagger (\Pi_i \otimes I_{m-\check{m}})^\dagger G (\Pi_i \otimes I_{m-\check{m}}) \mathbf{z} \in \mathscr{C} \right\} \tag{25}$$

is called the **set of desirable gambles conditional** on Π_i.

Intuitively, as in classical probability, updating/conditioning in QM can be seen as a projection on the subspace of observed variables. Indeed, $\Pi_i \otimes I_{m-\check{m}}$ projects $\mathbf{z}$ on the subspace of observed variables and leaves the unobserved variables unchanged. We already know [1] that updating in QM preserves coherence.[31] We now see that it also preserves exchangeability.[32]

Proposition 11. *Let $\mathscr{C}$ be an A-coherent $\star$-exchangeable coherent set of desirable gambles. Then $\mathscr{C}_{\Pi_i}$ is an A-coherent $\star$-exchangeable coherent set of desirable gambles on the variables $x_{\check{m}+1}, \dots, x_m$ and its dual is*

$$\mathscr{M}_{\Pi_i} = \left\{ \frac{(\Pi_i \otimes I_{m-\check{m}})^\dagger \rho (\Pi_i \otimes I_{m-\check{m}})}{Tr((\Pi_i \otimes I_{m-\check{m}})^\dagger \rho (\Pi_i \otimes I_{m-\check{m}}))} \middle| \rho \in \mathscr{M} \right\}, \tag{26}$$

where $\mathscr{M}$ is the dual of $\mathscr{C}$.

PROOF. In [2] we have already proved that $\mathscr{C}_{\Pi_i}$ is coherent and that $\mathscr{M}_{\Pi_i}$ is the dual of $\mathscr{C}_{\Pi_i}$. Therefore, we only need to prove that $\mathscr{C}_{\Pi_i}$ is a $\star$-exchangeable coherent set of desirable gambles on the variables $\mathbf{x}_{\check{m}+1}, \dots, \mathbf{x}_m$. This means we need to prove that $\mathbf{z}^\dagger G \mathbf{z} - \left(\frac{\delta^\star_{l,r}}{2} \mathbf{z}^\dagger (I_{\check{m}} \otimes P^{m-\check{m}}_{\pi_l})^\dagger G (I_{\check{m}} \otimes P^{m-\check{m}}_{\pi_r}) \mathbf{z} + \frac{\delta^\star_{l,r}}{2} \mathbf{z}^\dagger (I_{\check{m}} \otimes P^{m-\check{m}}_{\pi_l})^\dagger G (I_{\check{m}} \otimes P^{m-\check{m}}_{\pi_r}) \mathbf{z} \right) \in \mathscr{C}_{\Pi_i}$ for each gamble $\mathbf{z}^\dagger G \mathbf{z}$. This gamble is in $\mathscr{C}_{\Pi_i}$ provided that:

$$\begin{aligned} &\mathbf{z}^\dagger (\Pi_i \otimes I_{m-\check{m}})^\dagger \Big[G - \tfrac{\delta^\star_{l,r}}{2} (I_{\check{m}} \otimes P^{m-\check{m}}_{\pi_l})^\dagger G (I_{\check{m}} \otimes P^{m-\check{m}}_{\pi_r}) \\ &- \tfrac{\delta^\star_{l,r}}{2} (I_{\check{m}} \otimes P^{m-\check{m}}_{\pi_r})^\dagger G (I_{\check{m}} \otimes P^{m-\check{m}}_{\pi_l}) \Big] (\Pi_i \otimes I_{m-\check{m}}) \mathbf{z}, \end{aligned}$$

is in $\mathscr{C}$. By exploiting the following well-know property of the Kronecker product

$$(I_d \otimes B)(A \otimes I_{d'}) = (A \otimes I_{d'})(I_d \otimes B),$$

[31] More precisely, coherence is preserved provided that $\widetilde{L}((\Pi_i \otimes I_{m-\check{m}})^\dagger \mathbf{z}\mathbf{z}^\dagger (\Pi_i \otimes I_{m-\check{m}})) > 0$. The denominator of the conditioning formula is different from zero.

[32] The fact that ("in the usual setting") updating preserves exchangeability for sets of desirable gambles has been shown in [11, Prop.15].

we need to verify that

$$\mathbf{z}^\dagger \Big[(\Pi_i \otimes I_{m-\check{m}})^\dagger G(\Pi_i \otimes I_{m-\check{m}})$$
$$- \tfrac{\delta^\star_{l,r}}{2}(I_{\check{m}} \otimes P^{m-\check{m}}_{\pi_l})^\dagger(\Pi_i \otimes I_{m-\check{m}})^\dagger G(\Pi_i \otimes I_{m-\check{m}})(I_{\check{m}} \otimes P^{m-\check{m}}_{\pi_r})$$
$$- \tfrac{\delta^\star_{l,r}}{2}(I_{\check{m}} \otimes P^{m-\check{m}}_{\pi_r})^\dagger(\Pi_i \otimes I_{m-\check{m}})^\dagger G(\Pi_i \otimes I_{m-\check{m}})(I_{\check{m}} \otimes P^{m-\check{m}}_{\pi_l})\Big]\mathbf{z},$$

is in $\mathscr{C}$. This is true because $\mathscr{C}$ is $\star$-exchangeable.

7. Entanglement

Unlike systems consisting of distinguishable particles, in the case of identical particles the notion of entanglement is still under debate (see e.g. [24]). The reason being that, for instance, the two matrices in Example 6 are entangled density matrices for distinguishable particles[33] and, at the same time, they also satisfy the symmetry and anti-symmetry relationship of identical particles. Are those density matrices entangled in the (anti-)symmetric case?

For distinguishable particles, our gambling formulation of QM allows us to formulate and detect entangled density matrices through a Dutch book (sure loss) [2]. This goes as follows. Given a density matrix ρ, we can first compute its dual (an A-coherent set of desirable gambles):

$$\mathscr{C} := \{g(\mathbf{x}\otimes\mathbf{y}, \mathbf{x}\otimes\mathbf{y}) = (\mathbf{x}\otimes\mathbf{y})^\dagger G(\mathbf{x}\otimes\mathbf{y}) : \widetilde{L}(g) = Tr(G\rho) \geq 0\}$$

and then consider its "classical" extension

$$\mathscr{K} := \text{posi}(\mathscr{C} \cup \mathscr{L}^{\geq}_R).$$

Hence, $\mathscr{K}$ is coherent (under the standard axioms of desirability) provided that $\mathscr{K} \cap \mathscr{L}^{<}_R = \emptyset$.

As done in [2], we thus state the following definition.

Definition 7. Let ρ be a density matrix. Then ρ is entangled if $\mathscr{K} \cap \mathscr{L}^{<}_R \neq \emptyset$ ($\mathscr{K}$ does not avoid sure loss).

If ρ is not entangled, it can be written as an integral with respect to a probability charge [2]:[34]

$$\rho = \int_\Omega (\mathbf{x}\otimes\mathbf{y})(\mathbf{x}\otimes\mathbf{y})^\dagger d\mu(\mathbf{x},\mathbf{y}). \tag{27}$$

As an immediate consequence of Definition 7 and Equation (27) we get:

Proposition 12. *Let ρ be a density matrix, then ρ is not entangled iff it is a truncated moment matrix (with respect to a probability charge $\mu(\mathbf{x},\mathbf{y})$).*

[33] They cannot be written as moment matrices w.r.t. a probability.

[34] To do that, we need to perform another natural extension to the space of all gambles: posi($\mathscr{K} \cup \mathscr{L}^{\geq}$). Also note that a probability charge is not a probability measure because it is only finitely additive.

The question is therefore how we can extend this result to the case of indistinguishable particles. Here we follow the approach proposed in [25], which uses a *Dutch book* based approach to define entanglement.[35] We need to consider the constraint that *not all Dutch books can be constructed*. In a system of indistinguishable particles, *physical observables* (that is, gambles which can be evaluated through an experiment or, equivalently, physically realisable gambles) must be invariant under all permutations of the m identical particles [22, XIV.C-4-a]:

$$g((\mathbf{x}\otimes\mathbf{y}),(\mathbf{x}\otimes\mathbf{y})) = (\mathbf{x}\otimes\mathbf{y})^{\dagger}G(\mathbf{x}\otimes\mathbf{y}) = (\mathbf{x}\otimes\mathbf{y})^{\dagger}\Pi_{\star}G\Pi_{\star}(\mathbf{x}\otimes\mathbf{y}) \;\; \forall\, \mathbf{x},\mathbf{y}. \quad (28)$$

Based on this constraint, we can thus obtain the following analogous of Proposition 12 for indistinguishable particles.

Proposition 13. *Let ρ be a $\star$-exchangeable density matrix, then the following two statements are equivalent:*

1. *there exists a physical observable $g(\mathbf{x}\otimes\mathbf{y},\mathbf{x}\otimes\mathbf{y})$ which belongs to $\mathscr{L}_R^{<}$ such that $Tr(G\rho)\geq 0$;*
2. *ρ cannot be written as*

$$\int_{\Omega}\frac{\Pi_{\star}(\mathbf{x}\otimes\mathbf{y})(\mathbf{x}\otimes\mathbf{y})^{\dagger}\Pi_{\star}^{\dagger}}{\int_{\Omega}Tr(\Pi_{\star}(\mathbf{x}\otimes\mathbf{y})(\mathbf{x}\otimes\mathbf{y})^{\dagger}\Pi_{\star}^{\dagger})d\mu(\mathbf{x},\mathbf{y})}d\mu(\mathbf{x},\mathbf{y}), \quad (29)$$

for any probability charge μ (here we are assuming that the denominator in (29) is different from zero).

PROOF. We first prove the direction $(1)\Rightarrow(2)$. Denote $\mathbf{z}=\mathbf{x}\otimes\mathbf{y}$ and $\mu(\mathbf{z})=\mu(\mathbf{x},\mathbf{y})$ Assume that $g(\mathbf{z},\mathbf{z})<0$. Towards a contradiction, let us assume that there is a probability measure μ such that ρ can be written as in Equation (29). However it holds that

$$Tr\left(G\int_{\Omega}\frac{\Pi_{\star}\mathbf{z}\mathbf{z}^{\dagger}\Pi_{\star}^{\dagger}}{\int_{\Omega}Tr(\Pi_{\star}\mathbf{z}\mathbf{z}^{\dagger}\Pi_{\star}^{\dagger})d\mu(\mathbf{z})}d\mu(\mathbf{z})\right)<0.$$

In fact, $\int_{\Omega}Tr(\Pi_{\star}\mathbf{z}\mathbf{z}^{\dagger}\Pi_{\star}^{\dagger})d\mu(\mathbf{z})>0$ (because the matrix $\Pi_{\star}\mathbf{z}\mathbf{z}^{\dagger}\Pi_{\star}^{\dagger}$ is positive semidefinite and we have assumed the integral to be different from zero) and

[35] For indistinguishable particles, there are different viewpoints about physical meaning and assessment of entanglement. Many of these approaches are summarized in a recent review [26], which reviews 5 different definitions of entanglement (including the one in [25]) for indistinguishable particles. We leave to future work the comparison of these definitions in the context of the theory developed in the present paper.

$$
\begin{aligned}
0 &> \int_\Omega \mathbf{z}^\dagger G \mathbf{z} d\mu(\mathbf{z}) \\
&= \int_\Omega \mathbf{z}^\dagger \Pi_\star G \Pi_\star \mathbf{z} d\mu(\mathbf{z}) \\
&= \int_\Omega Tr(G \Pi_\star \mathbf{z}\mathbf{z}^\dagger \Pi_\star) d\mu(\mathbf{z}) \\
&= Tr\left(G \int_\Omega \Pi_\star \mathbf{z}\mathbf{z}^\dagger \Pi_\star d\mu(\mathbf{z})\right),
\end{aligned}
$$

where the second equality follows by the assumption that g is a *physical observable* and thus Equation (28).

For the direction $(2) \Rightarrow (1)$, we reason as follows. Let A be the set of all density matrices ρ' that can be written as in Equation (29) for some probability charge μ. This set is a closed convex set.[36] Since $\rho \notin A$, by the hyperplane separation theorem, there is a gamble $g(\mathbf{x} \otimes \mathbf{y}, \mathbf{x} \otimes \mathbf{y})$ such that $Tr(G\rho) \geq 0$ but $Tr(G\rho') < 0$, for each $\rho' \in A$. Remember that ρ is a $\star$-exchangeable density matrix. Therefore $Tr(G\rho) = Tr(\Pi_\star G \Pi_\star \rho)$. Thus, let $g(\mathbf{x} \otimes \mathbf{y}, \mathbf{x} \otimes \mathbf{y}) := (\mathbf{x} \otimes \mathbf{y})^\dagger \Pi_\star G \Pi_\star (\mathbf{x} \otimes \mathbf{y})$. Since $\Pi_\star \Pi_\star = \Pi_\star$, gamble g satisfies Equation (28) and it is thence a *physical observable* such that $Tr(G\rho) \geq 0$ (but $Tr(G\rho') < 0$, for each $\rho' \in A$). Finally, notice that $\sup_{\mathbf{x},\mathbf{y}} (\mathbf{x} \otimes \mathbf{y})^\dagger \Pi_\star G \Pi_\star (\mathbf{x} \otimes \mathbf{y}) = \sup_{\mathbf{x},\mathbf{y}} Tr(\Pi_\star G \Pi_\star (\mathbf{x} \otimes \mathbf{y})(\mathbf{x} \otimes \mathbf{y})^\dagger) = \sup_\mu Tr(G \int_\Omega \Pi_\star (\mathbf{x} \otimes \mathbf{y})(\mathbf{x} \otimes \mathbf{y})^\dagger \Pi_\star \mu(\mathbf{x}, \mathbf{y})) < 0$. This implies that $g(\mathbf{x} \otimes \mathbf{y}, \mathbf{x} \otimes \mathbf{y})$ belongs to $\mathscr{L}_R^<$, which ends the proof.

The above result tells us that, in the case of indistinguishable particles, the disagreement between quantum A-coherence with classical probability coherence (in the sense that a state cannot be represented by a truncated moment matrix) is actually witnessed by a Dutch book that is a physical observable.

Notice that Proposition 13 is in agreement with definitions of entanglement, and ways to detect it, discussed in [25]. In particular, [25, Eq. 12] states that a $\star$-exchangeable density matrix is not entangled if it can be written as:

$$
\int_\Omega \frac{\Pi_\star (\mathbf{x} \otimes \mathbf{y})(\mathbf{x} \otimes \mathbf{y})^\dagger \Pi_\star^\dagger}{Tr(\Pi_\star (\mathbf{x} \otimes \mathbf{y})(\mathbf{x} \otimes \mathbf{y})^\dagger \Pi_\star^\dagger)} d\mu(\mathbf{x}, \mathbf{y}), \tag{30}
$$

for some probability measure μ. The two definitions are equivalent in the sense that they define the same set of symmetric, respectively antisymmetric, (under exchangeability) density matrices.

Proposition 14. *The definitions* (29) *and* (30) *are equivalent.*

PROOF. Note that, if μ is an atomic probability measure, that is a Dirac's delta, then (29) and (30) coincide.

[36] Here 'closed' is with respect to the weak*-topology.

More in general, we can focus on m-finitely supported probability distributions:

$$\mu_1(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{m} p_i \delta_{\mathbf{v}_i}(\mathbf{x}, \mathbf{y}), \quad \mu_2(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{m} q_i \delta_{\mathbf{v}_i}(\mathbf{x}, \mathbf{y}).$$

where $m \leq dim(\mathbf{x} \otimes \mathbf{y})^2$ and $\mathbf{v}_i = \mathbf{x}_{a_i} \otimes \mathbf{y}_{b_j}$ for $\mathbf{x}_{a_i}, \mathbf{y}_{b_i} \in V$. This is due to the Carathèodory's theorem [27]. Note that this decomposition is not unique in general and we consider the decomposition with the smallest m. Note also that, if the separable density matrix has rank-one, then $m = 1$.

The expectation (30) w.r.t. μ_1 and, respectively, (29) w.r.t. μ_2 give:

$$\sum_{i=1}^{m} p_i \frac{\Pi_\star \mathbf{v}_i \mathbf{v}_i^\dagger \Pi_\star^\dagger}{Tr(\Pi_\star \mathbf{v}_i \mathbf{v}_i^\dagger \Pi_\star^\dagger)}, \quad \sum_{i=1}^{m} q_i \frac{\Pi_\star \mathbf{v}_i \mathbf{v}_i^\dagger \Pi_\star^\dagger}{\sum_{j=1}^{m} q_j Tr(\Pi_\star \mathbf{v}_j \mathbf{v}_j^\dagger \Pi_\star^\dagger)}.$$

These two are equal whenever

$$p_i = \frac{q_i Tr(\Pi_\star \mathbf{v}_i \mathbf{v}_i^\dagger \Pi_\star^\dagger)}{\sum_{j=1}^{m} q_j Tr(\Pi_\star \mathbf{v}_j \mathbf{v}_j^\dagger \Pi_\star^\dagger)}.$$

For the other implication, we need to solve the following system of equations $B\mathbf{q} = \mathbf{b}$, which is

$$\begin{bmatrix} a_1 p_1 - a_1 & a_2 p_1 & \dots & a_{m-1} p_1 & a_m p_1 \\ a_1 p_2 & a_2 p_2 - a_2 & \dots & a_{m-1} p_1 & a_m p_2 \\ \dots & \dots & \dots & \dots & \dots \\ a_1 p_m & a_2 p_m & \dots & a_{m-1} p_m & a_m p_m - a_m \\ 1 & 1 & \dots & 1 & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_{m-1} \\ q_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

where $a_i = Tr(\Pi_\star \mathbf{v}_i \mathbf{v}_i^\dagger \Pi_\star^\dagger) > 0$ ($a_i > 0$ because m is the minimal dimension). Let us rewrite the system of equations as

$$\begin{bmatrix} A \\ \mathbf{1}_m^\top \end{bmatrix} \mathbf{q} = \begin{bmatrix} \mathbf{0}_{m-1} \\ 1 \end{bmatrix},$$

and denote the matrix in the left hand side as B and the vector in the r.h.s. as $\mathbf{b}$.

By Farkas' lemma (see for instance [28, Corollary 1]), either $B\mathbf{q} = \mathbf{b}$ has a nonnegative solution $\mathbf{q} \geq 0$ or there exists a vector $\mathbf{d}$ such that $\mathbf{d}^\top B \geq 0$ and $\mathbf{d}^\top \mathbf{b} < 0$.

Suppose a nonnegative solution of $B\mathbf{q} = \mathbf{b}$ does not exist. By the Farkas' lemma, there exists a vector $\mathbf{d} = [z_1, \dots, z_m, -\gamma]^\top$ such that $\mathbf{d}^\top B \geq 0$ and $\mathbf{d}^\top \mathbf{b} < 0$. Note that,

$$\mathbf{d}^\top B = \mathbf{z}^\top A - \gamma \geq 0,$$

and

$$\mathbf{d}^\top \mathbf{b} = -\gamma < 0 \quad \rightarrow \quad \gamma > 0.$$

Therefore, for all $j \in \{1, \ldots, m\}$, we have that

$$a_j \left(\sum_{i=1}^{m} z_i p_i - z_j \right) \geq \gamma > 0.$$

Consider the element j corresponding to the maximum z_j and denote it with k. Observe that $z_k \geq \sum_{i=1}^{m} z_i p_i$. Then we have that

$$0 = a_k(z_k - z_k) \geq a_k \left(\sum_{i=1}^{m} z_i p_i - z_k \right) \geq \lambda > 0,$$

a contradiction. Hence there must be a probability vector $\mathbf{q}$ such that $B\mathbf{q} = \mathbf{b}$.

Example 7. We apply Proposition 13 to the previous two particles in Example 6.

Fermions: consider the atomic charge (Dirac's delta) $\mu = \delta_{\tilde{\mathbf{z}}}$ with $\tilde{\mathbf{z}} = [1, 0]^\top \otimes [0, 1]^\top = [0, 1, 0, 0]^\top$. Note that,

$$\int_\Omega \frac{\Pi_{\text{Anti}}(\mathbf{x} \otimes \mathbf{y})(\mathbf{x} \otimes \mathbf{y})^\dagger \Pi_{\text{Anti}}^\dagger}{\int_\Omega Tr(\Pi_{\text{Anti}}(\mathbf{x} \otimes \mathbf{y})(\mathbf{x} \otimes \mathbf{y})^\dagger \Pi_{\text{Anti}}^\dagger) \delta_{\tilde{\mathbf{z}}}(\mathbf{x}, \mathbf{y})} \delta_{\tilde{\mathbf{z}}}(\mathbf{x}, \mathbf{y}). = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which is the density matrix in (24). From Proposition 13, we conclude that the density matrix is not entangled.

Bosons: consider the atomic charge (Dirac's delta) $\delta_{\tilde{\mathbf{z}}}$ with $\tilde{\mathbf{z}} = \frac{1}{2}[-\iota, 1]^\top \otimes [\iota, 1]^\top = [0.5, -0.5\iota, 0.5\iota, 0.5]^\top$, where ι is the complex unit. Note that,

$$\int_\Omega \frac{\Pi_{\text{Sym}}(\mathbf{x} \otimes \mathbf{y})(\mathbf{x} \otimes \mathbf{y})^\dagger \Pi_{\text{Sym}}^\dagger}{\int_\Omega Tr(\Pi_{\text{Sym}}(\mathbf{x} \otimes \mathbf{y})(\mathbf{x} \otimes \mathbf{y})^\dagger \Pi_{\text{Sym}}^\dagger) \delta_{\tilde{\mathbf{z}}}(\mathbf{x}, \mathbf{y})} \delta_{\tilde{\mathbf{z}}}(\mathbf{x}, \mathbf{y}). = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix},$$

which is the density matrix in (23). From Proposition 13, we conclude that the density matrix is not entangled.

8. Partial exchangeability

In many situations, it is not reasonable to assume that all the variables are exchangeable. More precisely, it may be possible to divide the variables into b blocks that are considered to be block-exchangeable. In that case, the corresponding variables are called *b-fold partially exchangeable* For $b = 1$, it reduces to regular exchangeability. Partial exchangeability can be defined in other ways, for instance by considering the variables to be exchangeable within their block [29].

Example 8. In the previous sections, we only considered one degree of freedom (unknown) for each particle (e.g., the spin). Here, we consider the case where the particle has two degrees of freedom. For instance, we denote the space degree of freedom by $\mathbf{x} \in \overline{\mathbb{C}}^{n_x}$ and spin degree of freedom by $\mathbf{y} \in \overline{\mathbb{C}}^{n_y}$. Suppose we have 2 such particles, whose corresponding variables are denoted by $\mathbf{x}_1, \mathbf{y}_1$ and, respectively, $\mathbf{x}_2, \mathbf{y}_2$. Consider the gamble

$$g(\mathbf{z}, \mathbf{z}) = (\mathbf{x}_1 \otimes \mathbf{y}_1 \otimes \mathbf{x}_2 \otimes \mathbf{y}_2)^\dagger G(\mathbf{x}_1 \otimes \mathbf{y}_1 \otimes \mathbf{x}_2 \otimes \mathbf{y}_2).$$

If the particles are indistinguishable, then g should be block-exchangeable w.r.t. the blocks $\mathbf{x}_1, \mathbf{y}_1$ and $\mathbf{x}_2, \mathbf{y}_2$. This means that only $g(\mathbf{z}, \mathbf{z})$,

$$\begin{aligned} g'(\mathbf{z}, \mathbf{z}) = \frac{1}{2}\Big(& (\mathbf{x}_2 \otimes \mathbf{y}_2 \otimes \mathbf{x}_1 \otimes \mathbf{y}_1)^\dagger G(\mathbf{x}_1 \otimes \mathbf{y}_1 \otimes \mathbf{x}_2 \otimes \mathbf{y}_2) \\ & + (\mathbf{x}_1 \otimes \mathbf{y}_1 \otimes \mathbf{x}_2 \otimes \mathbf{y}_2)^\dagger G(\mathbf{x}_2 \otimes \mathbf{y}_2 \otimes \mathbf{x}_1 \otimes \mathbf{y}_1)\Big) \end{aligned}$$

and

$$g''(\mathbf{z}, \mathbf{z}) = (\mathbf{x}_2 \otimes \mathbf{y}_2 \otimes \mathbf{x}_1 \otimes \mathbf{y}_1)^\dagger G(\mathbf{x}_2 \otimes \mathbf{y}_2 \otimes \mathbf{x}_1 \otimes \mathbf{y}_1)$$

are considered to be exchangeable.

To define partial exchangeability, we assume we have ℓ variables $(\mathbf{x}_1, \ldots, \mathbf{x}_\ell)$, which are grouped in b-blocks $(\mathbf{x}_{J_1}, \ldots, \mathbf{x}_{J_b})$ with $\{J_i\}_{i=1}^b$ being a partition of $\{1, 2, \ldots, \ell\}$.

Let π^B denote a block-permutation, that is $\pi^B(\mathbf{x}_{J_1}, \ldots, \mathbf{x}_{J_b}) = (\mathbf{x}_{\pi^B(J_1)}, \ldots, \mathbf{x}_{\pi^B(J_b)})$. Similarly to the previous sections, we can then define the set:

$$\mathscr{I}_\star^B := \{g - \delta^\star_{l,r} \pi_l^B g \pi_r^B \mid g \in \mathscr{L}_R,\ \pi_l^B, \pi_r^B \in \mathbb{P}_\ell^B\}.$$

where $\mathbb{P}_\ell^B$ denotes the collection of all b-block permutations. The results of the previous case can be extended to partial exchangeability by simply working with the restricted set of allowed permutations $\mathbb{P}_\ell^B$.

9. Conclusions

In this paper we showed that we can derive the *symmetrization postulate* for indistinguishable particles in QM using the framework of exchangeable desirable gambles. Therefore, once again, we proved that QM is a theory of probability: it can be derived by the principles of coherence and computation plus structural assessments of exchangeability. Moreover, we showed that, also in the case of indistinguishable particles, entanglement can be defined (and detected) as a Dutch book: the clash between the QM notion of rationality (which accounts for the principle of computation) and the classical notion of rationality (which assumes infinite computational resources).

We obtained these results by exploiting symmetrization procedures to model structural assessments of indistinguishability. This approach, which is called "first quantization" in QM, has a main drawback: it includes redundant information. More specifically, it potentially allows us to gamble on the state of a single particle which is not a physical observable (it is impossible in the first place to tell which particle is which). This constitutes a well-known limit in QM. As an example, we had to impose the condition on physical observables given by Equation (28) in order to obtain Proposition 13.

In QM, there is another formalism to work with indistinguishable particles, called *second quantization*. Its language allows one to ask the following question "How many particles are there in each state?". Since this formalism does not refer to the labelling of particles, it contains no redundant information. As future work, we plan to provide a gambling formulation of QM for the *second quantization*, exploring the connection with the count vectors formalism developed by [11].

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